MINOR SUMMATION FORMULA OF PFAFFIANS AND SCHUR FUNCTION IDENTITIES

MASAO ISHIKAWA

ABSTRACT. We give a minor summation formula of Pfaffians and give Schur function identities as applications of this formula. These formulae (which are regarded as a kind of Littlewood formulae) involve Schur functions and Čebyšev polynomials. These new types of formulae are stated in the last section. The simplest cases of these formulae are interpreted as character formulae of certain representations. As corollaries of these formulae we obtain certain new identities on the generating functions of partitions.

Nous donnons une formule sommatoire pour les pfaffiens, et comme application, des identités sur les fonctions de Schur. Ces formules font intervenir les fonctions de Schur et les polynômes de Čebyšev, et peuvent être vues comme des identités de type Littlewood. Ces nouvelles formules sont énoncées dans la dernière section. Les cas les plus simples sont interprétés comme des formules de caractères pour certaines représentations. Comme corollaires, nous obtenons de nouvelles identités satisfaites par des fonctions génératrices de partitions.

1. INTRODUCTION

In the paper [IW], we exploited a minor summation formula of Pfaffians which is an extension of the formula given in [Ok]. In this talk I would like to state this minor summation formula and certain Schur function identities which are obtained as applications of this formula. Some of these identities can be interpreted as character identities of certain representations. (See [IOW].)

2. MINOR SUMMATION FORMULA OF PFAFFIANS

Let r, m, n be positive integers such that $r \leq m, n$. Let T be an arbitrary m by n matrix. For two sequences $\boldsymbol{i} = (i_1, \dots, i_r)$ and $\boldsymbol{k} = (k_1, \dots, k_r)$, let $T^{\boldsymbol{i}}_{\boldsymbol{k}} = T^{i_1 \dots i_r}_{k_1 \dots k_r}$ denote the sub-matrix of T obtained by picking up the rows and columns indexed by $\boldsymbol{i}$ and $\boldsymbol{k}$, respectively.

Assume $m \leq n$ and let B be an arbitrary n by n antisymmetric matrix, that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. As long as B is a square antisymmetric matrix, we write $B_{\boldsymbol{i}} = B_{i_1 \dots i_r}$ for $B^{\boldsymbol{i}}_{\boldsymbol{i}} = B^{i_1 \dots i_r}_{i_1 \dots i_r}$ in abbreviation. One of the main result in [IW] is the following theorem. (See p.6, Theorem 1 of [IW].)

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Theorem 2.1. *Let $m \le n$ and $T = (t_{ik})$ be an arbitrary m by n matrix. Let m be even and $B = (b_{ik})$ be any n by n antisymmetric matrix with entries b_{ik}. Then*

$$\sum_{1\le k_1<\cdots<k_m\le n} \operatorname{pf}(B_{k_1\ldots k_m})\det(T^{1\ldots m}_{k_1\ldots k_m}) = \operatorname{pf}(Q), \tag{2.1}$$

where Q is the m by m antisymmetric matrix defined by $Q = TB^tT$, i.e.

$$Q_{ij} = \sum_{1\le k<l\le n} b_{kl}\det(T^{ij}_{kl}), \qquad (1\le i,j\le m). \tag{2.2}$$

We regard the Pfaffian $\operatorname{pf}(B_{\boldsymbol{k}})$ as certain "weights" of the subdeterminants $\det(T^{1\ldots m}_{k_1\ldots k_m})$. By changing this antisymmetric matrix we obtain a considerably wide variation of the minor summation formula. Set $S_n = (s_{ij})$ to be the antisymmetric matrix defined by

$$s_{ij} = \begin{cases} 1 & \text{if } 1\le i<j\le n, \\ 0 & \text{if } 1\le i=j\le n, \\ -1 & \text{if } 1\le j<i\le n. \end{cases}$$

It is easy to see that $\operatorname{pf}(S_{k_1\ldots k_m}) = 1$ for any sequence $\boldsymbol{k}$ when m is even. So we have

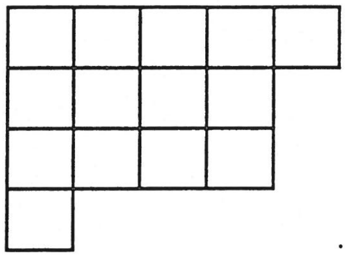
$$\sum_{1\le k_1<\cdots<k_m\le n} \det(T^{1\ldots m}_{k_1\ldots k_m}) = \operatorname{pf}(Q), \tag{2.3}$$

with $Q = TS^tT$. This special case appeared in [Ok] and applied to calculate the generating functions of the totally symmetric plane partitions. The lattice path interpretation of this formula and very fruitful applications of this formula are studied in [Ste].

In fact, we gave a quantum version of this formula in [IW], and this theorem is the $q = 1$ case of Theorem 1 in [IW]. We also gave a summation formula in which some column indices are fixed, and a summation formula in which not only column indices but also row indices move. The $q = 1$ case of these formulae can be proved not only by an algebraic method but also by a lattice path method. The quantum version was proved only by an algebraic method; here we state applications in which only the above formula will be used.

3. Notation and Preliminaries

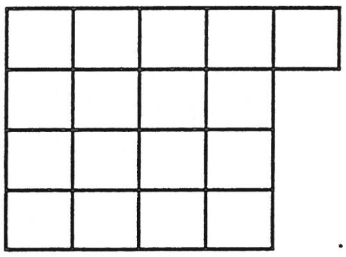
Now we review some basic notation which may be found in [Ma]. A weakly decreasing sequence of nonnegative integers $\lambda := (\lambda_1,\cdots,\lambda_m)$ with $\lambda_1 \ge \cdots \ge \lambda_m \ge 0$ is called a *partition* of $|\lambda| = \lambda_1 + \cdots + \lambda_m$. The partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ defined by $\lambda'_i = \sharp\{j : \lambda_j \ge i\}$ is called the conjugate partition of λ. Let $n(\lambda) = \sum_{i\ge1}(i-1)\lambda_i = \sum_{i\ge1}\binom{\lambda'_i}{2}$. For each cell $x = (i,j)$ in λ, the *hook-length* of λ at x is defined to be $h(x) = \lambda_i - j + \lambda'_j - i + 1$. Suppose that the main diagonal of λ consists of $r = p(\lambda)$ nodes. Let $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $1 \le i \le r$. We sometimes denote the partition λ by $\lambda = (\alpha_1,\ldots,\alpha_r|\beta_1,\ldots,\beta_r) = (\alpha|\beta)$, which is called the Frobenius notation. If a is a nonnegative integer which doesn't coincide with any of α_i, then let $q(\alpha, a)$ denote the number of α_i which are bigger than a. For example, $\lambda = (5441)$ is the partition of 14 and $p(\lambda) = 3$. This partition is denoted by $\lambda = (421|310)$ in the Frobenius notation. If $\alpha = (310)$ then $q(\alpha, 2) = 1$ and $(\alpha + 1|\alpha) = (421|310)$. This partition $\lambda = (421|310)$ is visualized by the Young diagram:



Let $\lambda = (\alpha_1, \dots, \alpha_r|\beta_1, \dots, \beta_r)$ be a partition expressed in the Frobenius notation. Let a and b be nonnegative integers such that $a \neq \alpha_1, \dots, \alpha_r$ and $b \neq \beta_1, \dots, \beta_r$. There are some k and l such that $\alpha_k > a > \alpha_{k+1}$ and $\beta_l > b > \beta_{l+1}$. The partition $\lambda \Cup (a|b)$ is defined by

$$\lambda \Cup (a|b) = (\alpha_1, \dots, \alpha_k, a, \alpha_{k+1}, \alpha_r|\beta_1, \dots, \beta_l, b, \beta_{l+1}, \dots, \beta_r). \tag{3.1}$$

For example, $(421|310)\Cup(0|2) = (4210|3210)$. Thus the Young diagram of $(421|310)\Cup(0|2)$ is given by



We now review the definition of Čebyšev's polynomials. The polynomials defined by $T_n(x) = \cos(n \arccos x)$ are called Čebyšev's polynomials of the first kind, and, on the other hand, the polynomials $U_n(x) = \sin(n \arccos x)/\sqrt{1-x^2}$ are called Čebyšev's polynomials of the second kind. Both are known to satisfy the same recurrence formula:

$$P_{n+1}(x) - 2xP_n(x) + P_{n-1}(x) = 0. \tag{3.2}$$

The first few polynomials are easily calculated from the following recursive formula.

$$\begin{gathered} T_0(x) = 1, \qquad U_0(x) = 0, \\ \begin{cases} T_{n+1}(x) = xT_n(x) + (x^2-1)U_n(x) \\ U_{n+1}(x) = T_n(x) + xU_n(x). \end{cases} \end{gathered} \tag{3.3}$$

The Weyl character formula tells us how to calculate the characters of the finite dimensional irreducible representations of highest weight $\lambda = \sum_i \lambda_i \epsilon_i$ of classical groups.

Let Γ_λ be the irreducible representation of highest weight λ. Then the character is given by

$$\mathrm{ch}(\Gamma_\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\delta)}}{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\delta)}},$$

where W is the Weyl group and δ is half the sum of positive roots. These characters can be written as fractions of certain determinants for each classical group. In the case of $GL(m,\mathbb{C})$, the finite dimensional irreducible characters are the well-known symmetric functions called the Schur functions, which are given by

$$s_\lambda(x_1,\dots,x_m) = \frac{|x_i^{\lambda_j+m-j}|}{|x_i^{m-j}|}, \tag{3.4}$$

where $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and all λ_i are integers. In the case of $O(2m+1,\mathbb{C})$, the finite-dimensional irreducible characters are parametrized by highest weights $\lambda = \sum_i \lambda_i\epsilon_i$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, and written as

$$so_\lambda(x_1^{\pm1},\dots,x_m^{\pm1},1) = \frac{|x_i^{\lambda_j+m-j+1/2} - x_i^{-(\lambda_j+m-j+1/2)}|}{|x_i^{m-j+1/2} - x_i^{-(m-j+1/2)}|}, \tag{3.5}$$

where λ_i are all integers or all half integers. In the case of $Sp(2m,\mathbb{C})$, the finite-dimensional irreducible characters are parametrized by highest weights $\lambda = \sum_i \lambda_i\epsilon_i$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, and written as

$$sp_\lambda(x_1^{\pm1},\dots,x_m^{\pm1}) = \frac{|x_i^{\lambda_j+m-j+1} - x_i^{-(\lambda_j+m-j+1)}|}{|x_i^{m-j+1} - x_i^{-(m-j+1)}|}, \tag{3.6}$$

where λ_i are all integers. In the case of $O(2m,\mathbb{C})$, the finite-dimensional irreducible characters are parametrized by highest weights $\lambda = \sum_i \lambda_i\epsilon_i$ with $\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m| \geq 0$, and written as

$$so_\lambda(x_1^{\pm1},\dots,x_m^{\pm1}) = \frac{|x_i^{\lambda_j+m-j} + x_i^{-(\lambda_j+m-j)}| + |x_i^{\lambda_j+m-j} - x_i^{-(\lambda_j+m-j)}|}{|x_i^{m-j} + x_i^{-(m-j)}|}. \tag{3.7}$$

This tells us how we should take T in the above theorem for each case.

$$\begin{aligned}
&T = (x_i^{n-j})\\
&T = (x_i^{n-j+1/2} - x_i^{-(n-j+1/2)})\\
&T = (x_i^{n-j+1} - x_i^{-(n-j+1)})\\
&T = (x_i^{n-j} + x_i^{-(n-j)}) \text{ and } T = (x_i^{n-j} - x_i^{-(n-j)})
\end{aligned}$$

We take the above T for each case of $GL(m,\mathbb{C})$, $O(2m+1,\mathbb{C})$, $Sp(2m,\mathbb{C})$, $O(2m,\mathbb{C})$, respectively.

4. Schur function identities

Typical examples of Littlewood's formulae are the followings.

$$\sum_{\lambda} t^{c(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m}(1-tx_i)^{-1} \prod_{1\le i<j\le m}(1-x_ix_j)^{-1}, \tag{4.1}$$

$$\sum_{\lambda} t^{r(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m}\frac{1+tx_i}{1-x_i^2} \prod_{1\le i<j\le m}\frac{1}{1-x_ix_j}, \tag{4.2}$$

where the summation is over all partitions λ, and $c(\lambda)$ (resp. $r(\lambda)$) stands for the number of columns (resp. rows) of odd length. (See [Ma].)

The following formulae are also called the Littlewood formulae.

$$\sum_{\lambda=(\alpha|\alpha+1)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m}(1-x_ix_j), \tag{4.3}$$

$$\sum_{\lambda=(\alpha|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m}(1-x_i)\prod_{1\le i<j\le m}(1-x_ix_j), \tag{4.4}$$

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i\le j\le m}(1-x_ix_j). \tag{4.5}$$

(See [KT].)

We should mention that our minor summation formula gives a proof of all the above formulae. Moreover a deformation of this proof gives us several Schur function identities of this kind. First we enumerate these identities.

Theorem 4.1. *Let m be a positive integer.*

$$\sum_{k=0}^{\infty} U_{k+1}(a) \sum_{r=0}^{\infty} s_{(r+k,k)}(x_1,\dots,x_m) = \prod_{i=1}^{m}(1-2ax_i+x_i^2)^{-1}. \tag{4.6}$$

$$\sum_{k=0}^{m} U_{k+1}(a) \sum_{r=0}^{m-k} s_{(2^r1^k)}(x_1,\dots,x_m) = \prod_{i=1}^{m}(1+2ax_i+x_i^2). \tag{4.7}$$

In the former identity there appear the Schur functions indexed by the partitions with $(r+k)$ cells in the first row and k cells in the second row. On the other hand, in the latter identity there appear the Schur functions indexed by the partitions with $(r+k)$ cells in the first column and k cells in the second column. These identities have clear meaning in representation theory as follows. The products appearing in these formulae are nothing but the characters of $SU(2)\otimes GL(n,\mathbb{C})$ acting on the space of polynomial rings of $2n$ variables and of the n-fold tensor product of Grassmann algebra of 2 variables, respectively. Hence our formulae are interpreted as the character identities which describe the irreducible decompositions of such spaces under the joint actions of groups $SU(2)$ and $GL_n(\mathbb{C})$.

If we put $x_i = q^{2i}$ in the latter formula and put $m \to \infty$, then we obtain a (combinatorial) proof of the q-expansion formula of Jacobi theta functions ϑ_1 and ϑ_2, for example,

(4.8)
$$\vartheta_1(u,\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi u = 2Q_0 q^{\frac{1}{4}}\sin\pi u\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi u + q^{4n}),$$

where $q = e^{i\pi\tau}$ ($\mathrm{Im}\tau > 0$) and $Q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$.

Theorem 4.2. *Let m be a positive integer. Then*

$$\begin{aligned}(4.9)\qquad &\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+p(\lambda)}s_\lambda(x_1,\dots,x_m)\\ &+2\sum_{k=1}^{m}T_k(a)\sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k-1}}(-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)}s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)\\ &=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{1\le i<j\le m}(1-x_ix_j).\end{aligned}$$

If we put $x_i = q^{2i}$ in this formula and we use the q-expansion formula of Jacobi's theta function ϑ_3, we obtain the following corollary.

Corollary 4.1.

$$(4.10)\qquad \sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+p(\lambda)}q^{\frac{|\lambda|}{2}+n(\lambda)}\prod_{x\in\lambda}\frac{1}{1-q^{h(x)}} = \frac{\prod_{r=2}^{\infty}(1-q^r)^{[\frac{r}{2}]}}{\prod_{r=1}^{\infty}(1-q^r)}.$$

Let m be a nonnegative integer.

(4.11)
$$\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+q(\alpha,m)}q^{\frac{|\lambda|}{2}+n(\lambda\uplus(0|m))}\prod_{x\in\lambda\uplus(0|m)}\frac{1}{1-q^{h(x)}} = q^{\frac{m(m+1)}{2}}\frac{\prod_{r=2}^{\infty}(1-q^r)^{[\frac{r}{2}]}}{\prod_{r=1}^{\infty}(1-q^r)}.$$

The following theorems also have the same specialization, but we omit it here.

Theorem 4.3. *Let m be a positive integer. Then*

$$\begin{aligned}(4.12)\qquad &\sum_{\lambda=(\alpha+2|\alpha)}(-1)^{\frac{|\lambda|-p(\lambda)}{2}}s_\lambda(x_1,\dots,x_m)\\ &+\sum_{k=1}^{m}\{T_k(a)+(a-1)U_k(a)\}\sum_{\substack{\lambda=(\alpha+2|\alpha)\\ \alpha\not\ni k-1}}(-1)^{\frac{|\lambda|+p(\lambda)}{2}+q(\lambda,k-1)}\\ &\qquad\times\left\{s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)-s_{\lambda\uplus(1|k-1)}(x_1,\dots,x_m)\right\}\\ &=\prod_{i=1}^{m}(1+2ax_i+x_i^2)(1-x_i)\prod_{1\le i<j\le m}(1-x_ix_j).\end{aligned}$$

Theorem 4.4. *Let m be a positive integer. Then*

$$\begin{aligned}
&\sum_{\lambda=(\alpha+3|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) \\
&+\sum_{k=1}^{m} U_{k+1}(a) \sum_{\substack{\lambda=(\alpha+3|\alpha)\\ \alpha\not\ni k-1}} (-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)} \\
&\qquad\times\left\{s_{\lambda\Cup(0|k-1)}(x_1,\dots,x_m)-s_{\lambda\Cup(2|k-1)}(x_1,\dots,x_m)\right\} \\
&=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{1\le i\le j\le m}(1-x_ix_j).
\end{aligned}\tag{4.13}$$

In the rest of this section we briefly describe the key ideas of the proof of these identities. The following lemma is the key lemma to evaluate the Pfaffian we treat.

Lemma 4.1. *Let m be a positive integer and put*

$$Q_m(x,y)=\frac{(x^m-y^m)^2}{x-y}\frac{(1-t^m x^m y^m)^2}{1-txy}. \tag{4.14}$$

Then

$$\operatorname{pf}\left[Q_m(x_i,x_j)\right]_{1\le i,j\le 2m}=t^{m(m-2)/4}\prod_{1\le i<j\le 2m}(x_i-x_j)(1-tx_ix_j). \tag{4.15}$$

If we apply Theorem 2.1 to B and Q given by

$$\sum_{0\le k<l\le 4m+d-2}\beta_{kl}\begin{vmatrix} x^k & x^l\\ y^k & y^l\end{vmatrix}=-(1+2ax+x^2)(1+2ay+y^2)\frac{(x^m-y^m)^2}{x-y}, \tag{4.16}$$

then we obtain Theorem 4.1 from Lemma 4.1 with $t=0$. We should notice that Theorem 4.1 can be generalized by replacing the above $1+2ax+x^2$ by a general polynomial of degree d.

Let m be a positive integer and let $B=(\beta_{kl})_{0\le k,l\le m-1}$ be an antisymmetric matrix of size m, that is to say, its entries satisfy the restraints $\beta_{lk}=-\beta_{kl}$. Set $\mathbf{b}_i$ to be the i-th row vector of B for $0\le i\le m-1$. The matrix B is said to be *(row-)symmetrically proportional* if the $(m-1-k)$-th row is proportional to the k-th. That is to say, there is some c_k such that $\mathbf{b}_{m-1-k}=c_k\mathbf{b}_k$ or $\mathbf{b}_k=c_k\mathbf{b}_{m-1-k}$ for each $0\le k\le[\frac{m}{2}]-1$. Further B is called *row-symmetric* if the $\mathbf{b}_{m-1-k}=\mathbf{b}_k$ for $0\le i\le[\frac{m}{2}]-1$, and B is called *row-antisymmetric* if the $\mathbf{b}_{m-1-k}=-\mathbf{b}_k$ for $0\le k\le[\frac{m+1}{2}]-1$. This notion is important since it makes it possible to find all the subpfaffians $\operatorname{pf}(B_{j_1\dots j_m})$ of B. From now on we assume that B is always supposed to be antisymmetric matrix.

Let $P(x)=a_0+a_1x+\cdots+a_dx^d$ be a polynomial of degree d. $P(x)$ is said to be *symmetric* if $a_i=a_{n-i}$ for $0\le i\le[\frac{d}{2}]$, and $P(x)$ is said to be *antisymmetric* if $a_i=-a_{n-i}$ for $0\le i\le[\frac{d+1}{2}]$,

Lemma 4.2. *Let $P(x)$ be a polynomial of degree d. Let $B = (\beta_{kl})_{0\le k,l\le 4m+d-2}$ be the (diagonal-)antisymmetric matrix of size $(4m+d-1)$ which satisfy*

$$\sum_{0\le k<l\le 4m+d-2} \beta_{kl} \begin{vmatrix} x^k & x^l \\ y^k & y^l \end{vmatrix} = -P(x)P(y)Q(x,y). \tag{4.17}$$

The matrix B becomes (row-)symmetrically proportional for all m if and only if $P(x)$ is symmetric or antisymmetric. Further, if the polynomial $P(x)$ is symmetric then B becomes row-symmetric, on the other hand, if $P(x)$ is antisymmetric then B becomes row-antisymmetric.

From now we apply Theorem 2.1 to this T and B given by (4.17). Basically it is possible to find some sort of formula for each antisymmetric matrix of the form (4.17) if it is row-symmetric or row-antisymmetric. Here we investigate each formula for small d. When $d = 0$, we obtain the formula (4.3) from this argument. If $d = 1$ and $P(x)$ is antisymmetric, we obtain the formula (4.4). It is easy to see that the case of $d = 1$ and $P(x)$ being symmetric reduces to this case. If $d = 2$ and $P(x)$ is antisymmetric, then we obtain the formula (4.5). These are the known Littlewood type formulae. If we assume $d = 2$ and $P(x)$ is symmetric, then we obtain Theorem 4.2.

If $d = 3$ and $P(x)$ is antisymmetric, we obtain Theorem 4.3. The case of $d = 3$ and $P(x)$ being symmetric essentially reduces to this case.

If $d = 4$ and $P(x)$ is antisymmetric, we obtain Theorem 4.4.

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Masao Ishikawa, Department of Mathematics, Faculty of Education, Tottori University, Koyama Tottori 680, Japan

E-mail address: m-ishika@tansei.cc.u-tokyo.ac.jp