A HOPF ALGEBRA FRAMEWORK FOR SET SYSTEM COLOURINGS WITH A GROUP ACTION

EXTENDED ABSTRACT

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ABSTRACT. Binomial Hopf algebras and their duals, the algebras of formal divided power series (or Hurwitz series), have been the object of study of umbral calculus. Closely related objects are divided power Hopf algebras and their duals, the algebras of formal power series. All these algebras have also arisen in algebraic topology, and have connections with the theory of formal groups. Since the methods used to study these algebras have a distinct combinatorial flavour, it has long been a challenge to develop methods providing explanations in terms of certain discrete structures. We have initiated such a programme in [5] by defining a combinatorial model for a binomial Hopf algebra based on set systems and their colourings. By a set system colouring, we mean a colouring of the vertices such that each monochromatic block belongs to the set system. We have considered the binomial Hopf algebra $\Phi_*[x]$ over the ring $\Phi_* = \mathbb{Z}[\phi_1, \phi_2, \ldots]$ because this is the universal one, in a sense which will be made precise in the paper. As an application of our earlier constructions, we present here a bijective proof of a familiar formal group law identity. The main aim of this paper is to extend the constructions in [5] to a combinatorial model for a divided power Hopf algebra and its dual. Passing from a binomial to a divided power Hopf algebra corresponds, in the combinatorial setup, to associating with a set system a subgroup of its full automorphism group. Such a pair will be called a set system with an automorphism group (often abbreviated to SSWAG). We define the umbral chromatic polynomial of a SSWAG as an element of a certain divided power Hopf algebra; upon umbral substitution, this polynomial enumerates the orbits of the automorphism group acting on the set system colourings, by a certain weight. We define a Hopf algebra structure on SSWAGs, and show that the map taking a SSWAG to its umbral chromatic polynomial is a Hopf algebra map. We present similar results for other polynomials associated with a SSWAG. Then, we define combinatorial analogues of delta operators from umbral calculus. We succeed in lifting to our combinatorial setup two algebraic identities concerning the interaction of a delta operator with the product and the antipode. The obtained identities encode deep combinatorial relations.

RÉSUMÉ. Les algèbres de Hopf binomiales et leurs duales, les algèbres de séries formelles divisées (ou séries de Hurwitz), ont été étudiées dans le cadre du calcul ombral. Des objets très proches sont les algèbres de puissances divisées et leurs duales, les algèbres de séries formelles. Toutes ces algèbres sont apparues en topologie algébrique et sont reliées à la théorie des groupes formels. Comme les méthodes utilisées pour étudier ces algèbres ont un aspect combinatoire marqué, la question se posait depuis longtemps de développer des méthodes fournissant des explications en termes de certaines structures combinatoires. Nous avons commencé un tel programme dans [5], en définissant un modèle combinatoire pour une algèbre de Hopf binomiale, besé sur des systèmes d'ensembles et leurs coloriages. Par coloriage d'un système d'ensembles, on entend un coloriage des sommets tel que tout bloc monochromatique appartient au système. Nous avons considéré l'algèbre de Hopf binomiale $\Phi_*[x]$ sur l'anneau $\Phi_* = \mathbb{Z}[\phi_1, \phi_2, \ldots]$ car elle est universelle dans un sens qui sera rendu précis dans la suite de l'article. Comme application de nos constructions précédentes, nous présentons une preuve bijective d'une identité appartenant à la théorie des groupes formels. Le principal objet de cet article est d'étendre les constructions de [5] à un modèle combinatoire pour une algèbre de Hopf de puissances divisées et sa duale. Le passage de l'algèbre binomiale aux puissances divisées correspond, d'un point de vue combinatoire, à associer à un système d'ensembles un sous-groupe de son groupe d'automorphismes (SSWAG). Nous définissons le polynôme chromatique ombral d'un SSWAG comme un élément d'une certaine algèbre de Hopf de puissances divisées; par substitution ombrale, ce polynôme énumère, par un certain poids, les orbites du groupe d'automorphismes agissant sur l'ensemble des coloriages. Nous définissons une structure d'algèbre de Hopf sur les SSWAG, et nous montrons que l'application qui envoie un SSWAG sur son polynôme chromatique ombral est un morphisme d'algèbres de Hopf. Nous présentons des resultats analogues pour autres polynômes associés à un SSWAG. Ensuite, nous définissons des analogues combinatoires des opérateurs delta du calcul ombral. Nous parvenons à relever à notre formalisme combinatoire deux identités algébriques concernant l'interaction d'un opérateur delta avec le produit et l'antipode. Les identités obtenues codent de profondes relations combinatoires.

1. Binomial and Divided Power Hopf Algebras

Let R be a commutative ring with identity. Consider a coalgebra C over R, with coproduct δ and augmentation ε. There exists an algebra isomorphism between the dual C^* of C, with the dual algebra structure, and the algebra, under composition, of those linear operators Γ on C which satisfy

$$\delta \circ \Gamma = (I \otimes \Gamma) \circ \delta \tag{1.1}$$

(here I denotes the identity on C). Such operators are known as *left-invariant operators* (or shift-invariant operators if $C = R[x]$). This isomorphism associates with $f \in C^*$ the following composite, denoted by Γ_f:

$$C \xrightarrow{\ \delta\ } C \otimes C \xrightarrow{\ I \otimes f\ } C \otimes R \cong C\,. \tag{1.2}$$

Conversely, to a linear operator Γ satisfying 1.1 corresponds the linear functional $\langle \Gamma \mid \cdot \rangle \in C^*$ defined by $\langle \Gamma \mid x \rangle := \varepsilon(\Gamma x)$, for all $x \in C$. Throughout, we identify C^* and its subalgebras with their images in the algebra of left-invariant operators. If C and R are evenly graded, $f \colon C \to R$ is a homomorphism of degree -2, and $f(C_2)$ contains the identity of R, then Γ_f will be called a *delta operator*. We can define the category of coalgebras with delta operator, its morphisms being those coalgebra maps which commute with the delta operators.

Let Φ_* be the graded ring $\mathbb{Z}[\phi_1, \phi_2, \ldots]$, where ϕ_i has degree $2i$. Consider the graded polynomial algebra $\Phi_*[x]$, with x having degree 2. Let D be the linear operator d/dx acting on $\Phi_*[x]$. It is well-known that $\Phi_*[x]$ is a cocommutative Hopf algebra over Φ_* (called a *binomial Hopf algebra*), with coproduct, augmentation, and antipode specified respectively by

$$\delta(x) = x \otimes 1 + 1 \otimes x\,, \qquad \varepsilon(x^i) = \delta_{i0}\,, \qquad S(x) = -x\,. \tag{1.3}$$

Let $\Phi^{2n}((D))$ denote the $\mathbb{Z}$-module of formal divided power series (or *Hurwitz series*) of the form

$$\alpha_0 \frac{D^n}{n!} + \alpha_1 \frac{D^{n+1}}{(n+1)!} + \ldots + \alpha_i \frac{D^{n+i}}{(n+i)!} + \ldots\,,$$

with $\alpha_i \in \Phi_{2i}$. The graded dual of the coalgebra $\Phi_*[x]$ is the algebra $\Phi^*((D)) := \bigoplus_{n \geq 0} \Phi^{2n}((D))$. The duality is expressed by $\langle D^m/m! \mid x^n \rangle = [(D^m/m!)x^n]_{x=0} = \delta_{mn}$.

To each sequence $\alpha = (1, \alpha_1, \alpha_2, \ldots)$ with $\alpha_i \in \Phi_{2i}$ corresponds a delta operator

$$\alpha(D) := D + \alpha_1 \frac{D^2}{2!} + \ldots + \alpha_{i-1} \frac{D^i}{i!} + \ldots \in \Phi^2((D))\,. \tag{1.4}$$

A sequence of the above type is called an *umbra*. Let ϕ be the umbra $(1, \phi_1, \phi_2, \ldots)$. We know from [7] that $(\Phi_*[x], \phi(D))$ is the universal binomial Hopf algebra with delta operator. The action of $\alpha(D)$ on the polynomial $p(x)$ is given by *umbral difference*:

$$\alpha(D)\, p(x) = p(x+\alpha) - p(x)\,, \qquad \alpha^i \equiv \alpha_{i-1}\,. \tag{1.5}$$

The effect of the linear functional corresponding to $e^{\alpha D} := I + \alpha(D)$ on the polynomial $p(x)$ is given by *umbral substitution*:

$$\langle e^{\alpha D} \mid p(x) \rangle = p(\alpha)\,, \qquad \alpha^i \equiv \alpha_{i-1}\,. \tag{1.6}$$

The operator $e^{\alpha D}$ can be applied repeatedly n times; we denote by $e^{n\alpha D}$ the iterated operator, and by $p(n\alpha)$ the corresponding umbral substitution. The set of delta operators of type 1.4 is a group under substitution (as formal power series), with identity D. We denote by $\overline{\alpha}$ the umbra corresponding to the inverse of $\alpha(D)$, called *conjugate delta operator*.

Given the delta operator $\alpha(D)$, there exists a unique sequence $B^\alpha = (B_0^\alpha(x), B_1^\alpha(x), \ldots)$ of polynomials in $\Phi_*[x]$, satisfying the following conditions:

$$B_0^\alpha(x) = 1\,; \qquad B_n^\alpha(0) = 0\,,\ n > 0\,; \qquad \alpha(D)\, B_n^\alpha(x) = nB_{n-1}^\alpha(x)\,,\ n > 0\,. \tag{1.7}$$

The sequence B^α is called the *(unnormalised) associated sequence* of $\alpha(D)$. A direct consequence of the conditions above is the fact that

$$B_n^\alpha(m\alpha) = [m]_n\,, \tag{1.8}$$

where $[m]_n := m(m-1)\ldots(m-n+1)$ is the falling factorial. The associated sequence of the delta operator D is just $(1, x, x^2, \ldots)$. The associated sequences of $\phi(D)$ and $\overline{\phi}(D)$ consist of the *conjugate Bell polynomials* and the *Bell polynomials*, respectively.

We now consider the graded ring $(H\cdot\Phi)_* := \mathbb{Z}[b_1, b_2, \ldots]$, where b_i has degree $2i$. We identify $(H\cdot\Phi)_*$ with a subring of $(\Phi \otimes \mathbb{Q})_*$ via the inclusion $b_i \mapsto \phi_i/(i+1)!$. Thus, Φ_* can be regarded as a subring of $(H\cdot\Phi)_*$. Let $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ be the divided power algebra over $(H\cdot\Phi)_*$, and let $x^i/i!$ have degree $2i$. The above identifications allow us to identify $\Phi_*[x]$ with a subring of $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$, and $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ with a subring of $(\Phi \otimes \mathbb{Q})_*[x]$. We can define a Hopf algebra structure on $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ by taking the coproduct

$$\frac{x^n}{n!} \mapsto \sum_{i=0}^{n} \frac{x^i}{i!} \otimes \frac{x^{n-i}}{(n-i)!}\,. \tag{1.9}$$

The antipode S is given by $S(x^n/n!) = (-x)^n/n!$. Let $(H\cdot\Phi)^{2n}[[D]]$ denote the $\mathbb{Z}$-module of formal power series of the form

$$a_0 D^n + a_1 D^{n+1} + \ldots + a_i D^{n+i} + \ldots\,,$$

with $a_i \in (H\cdot\Phi)_{2i}$. The graded dual of the coalgebra $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ is the algebra $(H\cdot\Phi)^*[[D]] := \bigoplus_{n\geq 0} (H\cdot\Phi)^{2n}[[D]]$.

To each sequence $a = (1, a_1, a_2, \ldots)$ with $a_i \in (H\cdot\Phi)_{2i}$ corresponds a delta operator

$$a(D) := D + a_1 D^2 + \ldots + a_{i-1} D^i + \ldots \in (H\cdot\Phi)^2[[D]]\,. \tag{1.10}$$

A sequence of the above type is also called an umbra. Let b be the umbra $(1, b_1, b_2, \ldots)$. We make the following convention: the formal power series associated with an umbra denoted by a Greek letter or a capital Roman letter will always be a Hurwitz series, whereas if a small Roman letter is used, the corresponding formal power series is a usual one. The set of delta operators of type 1.10 is a group under substitution, with identity

D. As before, we denote by $\bar{a}$ the umbra corresponding to the inverse of $a(D)$, called *conjugate delta operator.*

Given the delta operator $a(D)$, there exists a unique sequence $\beta^a = (\beta_0^a(x), \beta_1^a(x), \ldots)$ of elements in $(H \cdot \Phi)_* \left\{\frac{x^i}{i!}\right\}$, satisfying the following conditions:

$$\beta_0^a(x) = 1\,; \qquad \beta_n^a(0) = 0\,,\ n > 0\,; \qquad a(D)\,\beta_n^a(x) = \beta_{n-1}^a(x)\,,\ n > 0\,. \tag{1.11}$$

The sequence β^a is called the *normalised associated sequence* of $a(D)$. Let us note that $B_n^\phi(x) = n!\beta_n^b(x)$.

We now consider the following formal power series in two variables:

$$F^\phi(X,Y) := \phi\,(\,\bar{\phi}(X) + \bar{\phi}(Y)\,) \;\in \Phi^2((X,Y))\,,$$

$$f^b(X,Y) := b\,(\,\bar{b}(X) + \bar{b}(Y)\,) \;\in (H\cdot\Phi)^2[[X,Y]]\,. \tag{1.12}$$

The formal power series $f^b(X,Y)$ is a *formal group law* over $(H\cdot\Phi)_*$, while $b(X)$ and $\bar{b}(X)$ are its *exp* and *log-series*, respectively (see [3] for an encyclopaedic description of the theory of formal groups). There exists a unique formal power series $i^b(X) \in (H\cdot\Phi)^2[[X]]$ such that $f^b(X, i^b(X)) = 0$; it is called the *formal inverse.* The coefficients of X^nY^m in $f^b(X,Y)$, of $X^n/n!\; Y^m/m!$ in $F^\phi(X,Y)$, and of X^k in $i^b(X)$ are denoted by f_{nm}^b, F_{nm}^ϕ, and i_k^b, respectively. Note that $F_{nm}^\phi = n!\,m!f_{nm}^b$. Let ${}^L\Phi_*$ denote the subring of $(H\cdot\Phi)_*$ generated by the elements f_{nm}^b. It is known from [3] that $f^b(X,Y)$, as a formal group law over ${}^L\Phi_*$, is the universal formal group law. It is also known that the free ${}^L\Phi_*$-module ${}^L\Phi_*\{\beta_i^b(x)\}$ generated by the elements $\beta_i^b(x)$ is a sub-Hopf algebra of $(H\cdot\Phi)_* \left\{\frac{x^i}{i!}\right\}$, thought of as a Hopf algebra over ${}^L\Phi_*$ (i.e. the polynomials $\beta_i^b(x)$ are closed under multiplication with respect to the subring ${}^L\Phi_*$ of $(H\cdot\Phi)_*$).

We conclude this section by presenting two identities which will subsequently be lifted to a combinatorial framework.

Proposition 1.13. *The following identities hold for arbitrary* $p(x)$, $q(x) \in (H\cdot\Phi)_* \left\{\frac{x^i}{i!}\right\}$*:*

$$b(D)\,(p(x)q(x)) = \sum_{n,m\geq 0} f_{nm}^b\,(b(D)^n\,p(x))\,(b(D)^m\,q(x)) \qquad \textit{(Leibniz rule)} \tag{1.14}$$

$$b(D)\,S(p(x)) = \sum_{k\geq 1} i_k^b\,S(b(D)^k\,p(x))\,. \tag{1.15}$$

2. Set Systems, Weights, and Polynomials

In order to construct combinatorial models for the algebraic structures in the previous section, we need the following concepts: set systems, the Möbius type function, partition type polynomials, umbral chromatic polynomials, and characteristic type polynomials. In this section we define these concepts, and related ones. Then, we recall some results obtained in [5], and give a bijective proof of a familiar formal group law identity.

Let V be a finite set, possibly empty. A collection of subsets $\mathcal{S} \subseteq 2^V$ is called a *set system* with vertices V if $\emptyset \in \mathcal{S}$ and $V = \bigcup_{W\in\mathcal{S}} W$; the set $\bigcup_{W\in\mathcal{S}} W$ is usually denoted by $V(\mathcal{S})$. Partitions of V are defined in the usual way, except for the case $V = \emptyset$, when $\{\emptyset\}$ is considered the only partition. A *preferential arrangement* of V is a partition of V together with a linear order on its blocks; we denote by σ_f the preferential arrangement with underlying partition σ, and linear order on σ specified by the bijection $f\colon \sigma \to [|\sigma|]$ (throughout, $[n] := \{1, 2, \ldots, n\}$). Let π be a partition of V, and $\mathrm{Bool}(\pi)$ the Boolean algebra of subsets of V consisting of arbitrary unions of blocks of π. A set system $\mathcal{P}$

satisfying $\pi \subseteq \mathcal{P} \subseteq \mathrm{Bool}(\pi)$ will be called a *partition system*. If $V \neq \emptyset$, then the blocks of π are the atoms of the poset $(\mathcal{P}, \subseteq)$; we will call them, simply, the *atoms* of $\mathcal{P}$. Since π is uniquely determined by $\mathcal{P}$, it makes sense to denote π by $\mathrm{At}(\mathcal{P})$, and $\mathrm{Bool}(\pi)$ by $\mathrm{Bool}(\mathcal{P})$. The sets belonging to $\mathrm{Non}(\mathcal{P}) := \mathcal{P} \setminus \{\emptyset\} \setminus \mathrm{At}(\mathcal{P})$ will be called *non-atoms*. Instead of considering arbitrary set sytems, in this paper we restrict ourselves to partition systems, as they provide a nice framework for our constructions. The following are a few examples:

$$\mathcal{N}_V := \{\{x\} : x \in V\} \cup \{\emptyset\}, \qquad \mathcal{K}_V := 2^V, \qquad \mathcal{K}_\pi := \bigcup_{B \in \pi} 2^B,$$

where π is a partition of V. If $V = [n]$, we denote $\mathcal{N}_V$ by $\mathcal{N}_n$, and $\mathcal{K}_V$ by $\mathcal{K}_n$; furthermore, if $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{Z}_+^r$, and $\pi = \{[n_1], n_1+[n_2], \ldots, \sum_{i=1}^{r-1} n_i+[n_r]\}$, where $n+[m] := \{n+1, n+2, \ldots n+m\}$, we denote $\mathcal{K}_\pi$ by $\mathcal{K}_{n_1,n_2,\ldots,n_r}$ or $\mathcal{K}_{\mathbf{n}}$.

The partition systems $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic if there exists a bijection $f\colon V(\mathcal{P}) \to V(\mathcal{P}')$ such that $\{f(U) : U \in \mathcal{P}\} = \mathcal{P}'$. We denote by $\mathbb{P}$ and $\mathbb{S}$ the sets of isomorphism classes of all partition systems, and of those with singleton atoms, respectively. In what follows, it will be clear from the context whether we mean a partition system or its isomorphism class, so no other notation is used for isomorphism classes. A partition σ of $V(\mathcal{P})$ satisfying $\sigma \subseteq \mathcal{P}$ will be called a *division* by $\mathcal{P}$; we denote by $\Pi(\mathcal{P})$ the set of divisions by $\mathcal{P}$. Similarly, we denote by $A(\mathcal{P})$ the set of preferential arrangements of $V(\mathcal{P})$ with all blocks belonging to $\mathcal{P}$.

Let $\sigma \subseteq \mathrm{Bool}(\mathcal{P})$ be a partition of a set $U \subseteq V(\mathcal{P})$ (necessarily $U \in \mathrm{Bool}(\mathcal{P})$). We define the partition systems

$$\mathcal{P}|\sigma := \{X \in \mathcal{P} : X \subseteq W \text{ for some } W \in \sigma\}, \tag{2.1}$$

$$\mathcal{P}/\sigma := \{X \in \mathcal{P} : W \subseteq X \text{ or } W \cap X = \emptyset, \text{ for all } W \in \sigma\} \cup \sigma, \tag{2.2}$$

and call them the *restriction* of $\mathcal{P}$ to σ, and the *contraction* of $\mathcal{P}$ through σ, respectively. The partition systems $\mathcal{P}|\{U\}$ and $\mathcal{P}/\{U\}$ will be written simply as $\mathcal{P}|U$ and $\mathcal{P}/U$. Let us also define

$$\tilde{\Pi}(\mathcal{P}) := \bigcup_{U \in \mathrm{Bool}(\mathcal{P})} \Pi(\mathcal{P}|U). \tag{2.3}$$

Given two partition systems $\mathcal{P}$ and $\mathcal{P}'$ with $\mathrm{At}(\mathcal{P}') = \mathrm{At}(\mathcal{P})$ and $\mathcal{P}' \subseteq \mathcal{P}$, we define the *complement* of $\mathcal{P}'$ in $\mathcal{P}$ to be the partition system

$$\complement_{\mathcal{P}}\mathcal{P}' := \mathcal{P} \setminus \mathrm{Non}(\mathcal{P}'). \tag{2.4}$$

The complement of $\mathcal{P}$ in $\mathrm{Bool}(\mathcal{P})$ will be denoted by $\overline{\mathcal{P}}$, and called, simply, the complement of $\mathcal{P}$. Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two arbitrary partition systems. Their *disjoint union* will be denoted by $\mathcal{P}_1 \cdot \mathcal{P}_2$. We define a complementary operation by

$$\mathcal{P}_1 \odot \mathcal{P}_2 := \overline{\overline{\mathcal{P}_1} \cdot \overline{\mathcal{P}_2}}. \tag{2.5}$$

Since isomorphism of partition systems is a congruence with respect to the operations above, they can be defined on isomorphism classes by taking representatives.

A *colouring* of the partition system $\mathcal{P}$ with at most n colours is a map $c\colon V(\mathcal{P}) \to [n]$, for which the coimage is a division by $\mathcal{P}$; we denote by $\Gamma_n(\mathcal{P})$ the collection of such maps. A map $w\colon \tilde{\Pi}(\mathcal{P}) \to R$, where R is a commutative ring, is called a *weight*; the pair $(\mathcal{P}, w)$ is called a *weighted partition system*. The weight w is called multiplicative if

$w(\sigma_1 \cup \sigma_2) = w(\sigma_1)\, w(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \widetilde{\Pi}(\mathcal{P})$ with $V(\sigma_1) \cap V(\sigma_2) = \emptyset$. Given a partition $\sigma \subseteq \mathrm{Bool}(\mathcal{P})$ of a subset of $V(\mathcal{P})$, we define the weight $w/\sigma \colon \widetilde{\Pi}(\mathcal{P}/\sigma) \to R$ by

$$(2.6) \qquad (w/\sigma)(\pi) = \begin{cases} w(\pi) & \text{if } \pi \in \widetilde{\Pi}(\mathcal{P}) \\ 0 & \text{otherwise}. \end{cases}$$

By the weight of a colouring of a partition system, we understand the weight of its coimage.

Let V be a finite set, and P a subposet of $\Pi(\mathcal{K}_V)$ with the usual order by refinement. The *incidence algebra* $\Phi_*(P)$ of P over Φ_* is defined to be the Φ_*-algebra of all functions from the collection of intervals in P to the ring Φ_*, with the usual module structure over Φ_*, and multiplication (or *convolution*) specified by

$$(2.7) \qquad (f_1 * f_2)(\sigma, \pi) := \sum_{\sigma \le \omega \le \pi} f_1(\sigma, \omega)\, f_2(\omega, \pi).$$

The identity of $\Phi_*(P)$ is the function δ given by $\delta(\sigma,\pi) = \delta_{\sigma\pi}$. The function ζ^ϕ in $\Phi_*(P)$ is defined by $\zeta^\phi(\sigma,\pi) = \phi_1^{m_1}\phi_2^{m_2}\ldots$, where m_i is the number of blocks of π that are unions of exactly $i+1$ blocks of σ. The function ζ^ϕ has a convolution inverse, which is denoted by μ^ϕ (or μ_P^ϕ if there is possible ambiguity); it is called the *Möbius type function* of P.

Given the partition systems $\mathcal{P}$ and $\mathcal{P}'$ such that $V(\mathcal{P}) = V(\mathcal{P}')$ and $\mathrm{At}(\mathcal{P}') \le \mathrm{At}(\mathcal{P})$, we define the weights τ^ϕ, $\nu^\phi_{\mathcal{P}'} \colon \widetilde{\Pi}(\mathcal{P}) \to \Phi_*$ as follows:

$$(2.8) \qquad \tau^\phi(\sigma) := \zeta^\phi(\widehat{0}_{\Pi(\mathcal{K}_{V(\sigma)})}, \sigma), \qquad \nu^\phi_{\mathcal{P}'}(\sigma) := \sum_{\pi \in \Pi(\mathcal{P}'|\sigma)} \mu^\phi_{\Pi(\mathcal{P}'|\sigma)}(\widehat{0}, \pi)\, \zeta^\phi(\pi, \sigma).$$

Let us note that if $\sigma \notin \widetilde{\Pi}(\mathcal{P}')$ then $\nu^\phi_{\mathcal{P}'}(\sigma) = -\mu^\phi_{\Pi(\mathcal{P}'|\sigma)\cup\{\sigma\}}(\widehat{0}, \sigma)$; otherwise, $\nu^\phi_{\mathcal{P}'}(\sigma)$ is 1 if $\sigma \subseteq \mathrm{At}(\mathcal{P}')$, and 0 if not. We call $\tau^\phi(\sigma)$ the type of σ, and $\nu^\phi_{\mathcal{P}'}(\sigma)$ the *Möbius type* of σ with respect to $\mathcal{P}'$. We proved in [5] that τ^ϕ and $\nu^\phi_{\mathcal{P}'}$ are multiplicative weights.

We now define several polynomials in $\Phi_*[x]$ associated with the partition system $\mathcal{P}$. These polynomials will have the following form:

$$(2.9) \qquad \theta(\mathcal{P}, w\,; x) := \sum_{\sigma \in \Pi(\mathcal{P})} w(\sigma)\, p_{|\sigma|}(x),$$

where $w \colon \widetilde{\Pi}(\mathcal{P}) \to \Phi_*$ is a weight, and $(p_n(x))_{n \ge 1}$ is a sequence of polynomials in $\Phi_*[x]$. For each polynomial defined, we indicate the corresponding choice of w and $p_n(x)$, its name, notation, and degree:

- $w = \tau^\phi$, $p_n(x) = x^n$: the partition type polynomial $\rho^\phi(\mathcal{P}\,; x)$, degree $2|V(\mathcal{P})|$;
- $w = \tau^\phi$, $p_n(x) = B_n^\phi(x)$: the umbral chromatic polynomial $\gamma^\phi(\mathcal{P}\,; x)$, degree $2|V(\mathcal{P})|$;
- $w = \nu^\phi_{\overline{\mathcal{P}}}$, $p_n(x) = B_n^\phi(x)$: the modified umbral chromatic polynomial $\widetilde{\gamma}^\phi(\mathcal{P}\,; x)$, degree $2|\mathrm{At}(\mathcal{P})|$;
- $w(\sigma) = \mu^\phi_{\Pi(\mathcal{P})}(\widehat{0}, \sigma)$, $p_n(x) = x^n$: the characteristic type polynomial $\chi^\phi(\mathcal{P}\,; x)$, degree $2|\mathrm{At}(\mathcal{P})|$.

As discussed in [8], we have:

$$\rho^\phi(\mathcal{N}_n\,; x) = \gamma^\phi(\mathcal{K}_n\,; x) = \chi^\phi(\mathcal{N}_n\,; x) = x^n, \qquad \rho^\phi(\mathcal{K}_n\,; x) = B_n^{\overline{\phi}}(x),$$

$$(2.10) \qquad \gamma^\phi(\mathcal{N}_n\,; x) = \chi^\phi(\mathcal{K}_n\,; x) = B_n^\phi(x), \qquad n > 0.$$

According to 1.8, upon umbral substitution at $n\phi$, the polynomials $\gamma^\phi(\mathcal{P}\,;x)$ and $\tilde{\gamma}^\phi(\mathcal{P}\,;x)$ enumerate the colourings of $\mathcal{P}$ with at most n colours by their type and Möbius type, respectively.

Special cases of the polynomials defined above are well-known. For instance, the partition polynomial of $\mathcal{P}$ investigated in [17] can be retrieved by substituting ϕ_i with 1 in $\rho^\phi(\mathcal{P}\,;x)$. The umbral chromatic polynomial of a set system was first defined in [11]. If $\mathcal{P}$ is a simplicial complex (i.e. $U \in \mathcal{P}$ and $W \subseteq U$ imply $W \in \mathcal{P}$), then $\nu^\phi_{\overline{\mathcal{P}}}(\sigma) = \tau^\phi(\sigma)$ for all $\sigma \in \Pi(\mathcal{P})$; hence $\gamma^\phi(\mathcal{P}\,;x) = \tilde{\gamma}^\phi(\mathcal{P}\,;x)$. If $\mathcal{P}$ is the independence complex $\mathcal{I}(G)$ of a graph G, then $\gamma^\phi(\mathcal{I}(G)\,;x)$ is the umbral chromatic polynomial of the graph, first defined in [12]. The classical chromatic polynomial of the graph can be retrieved by substituting ϕ_i with 1. The characteristic type polynomial of a subposet of $\Pi(\mathcal{K}_V)$ appears in [8], [10], and [11]. If $\mathcal{P}$ is a simplicial complex, and all the maximal partitions of $\Pi(\mathcal{P})$ have the same number of blocks m, then the substitution $\phi_i \mapsto 1$ maps $\chi^\phi(\mathcal{P}\,;x)$ to the characteristic polynomial of the poset $\Pi(\mathcal{P})$ (in the variable x) multiplied by x^m.

In [5], we proved the following general complementation formula:

Theorem 2.11. *Consider two partition systems $\mathcal{P}$ and $\mathcal{P}'$ such that* $\mathrm{At}(\mathcal{P}) = \mathrm{At}(\mathcal{P}')$ *and $\mathcal{P}' \subseteq \mathcal{P}$. The following identity holds:*

$$\chi^\phi(\mathcal{P}'\,;x) = \sum_{\sigma\in\Pi(\complement_{\mathcal{P}}\mathcal{P}')} \nu^\phi_{\mathcal{P}'}(\sigma)\,\chi^\phi(\mathcal{P}/\sigma\,;x)\,. \tag{2.12}$$

A corollary of this result tells us that

$$\tilde{\gamma}^\phi(\mathcal{P}\,;x) = \chi^\phi(\overline{\mathcal{P}}\,;x)\,. \tag{2.13}$$

Another corollary gives the following combinatorial interpretation of the elements $F^\phi_{nm} \in \Phi_*$ defined in Section 1:

$$F^\phi_{nm} = \nu^\phi_{\mathcal{K}_{n,m}}(\{[n+m]\})\,. \tag{2.14}$$

We use 2.14 to give a bijective proof of a familiar formal group law identity, which is usually proved by formal power series manipulations (see e.g. [6]).

Proposition 2.15. *Let us denote by $\overline{\phi}'(X)$ and $\frac{\partial F^\phi}{\partial Y}(X,Y)$ the formal derivatives of the corresponding formal power series. We have*

$$\overline{\phi}'(X) = \left(\frac{\partial F^\phi}{\partial Y}(X,0)\right)^{-1}, \tag{2.16}$$

where $(\,\cdot\,)^{-1}$ denotes, as expected, the multiplicative inverse in the ring $\Phi^*((X))$.

3. Set Systems with an Automorphism Group

In this section, we consider only partition systems with isomorphism classes belonging to $\mathbb{S}$. Given a group G acting on a set X, we denote, as usual, the orbit of $x \in X$ by $G(x)$, and the stabilizer of x by G_x. A pair $(\mathcal{S}, G)$ consisting of a set system (of the type mentioned above), and a group G of automorphisms of $\mathcal{S}$ (i.e. a subgroup of the stabilizer of $\mathcal{S}$ under the action of the symmetric group on $V(\mathcal{S})$ on set systems with vertices $V(\mathcal{S})$) will be called a *set system with an automorphism group* (often abbreviated to SSWAG). Two SSWAGs $(\mathcal{S}_1, G_1)$ and $(\mathcal{S}_2, G_2)$ are called isomorphic if there exists an isomorphism $f\colon V(\mathcal{S}_1) \to V(\mathcal{S}_2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ such that G_2 is the homomorphic image of G_1 under the map $g \mapsto f\circ g\circ f^{-1}$ (thus, this map is an isomorphism of G_1 and G_2). Given a SSWAG

$(\mathcal{S}, G)$ and $\sigma \in \widetilde{\Pi}(\mathcal{K}_{V(\mathcal{S})})$, we define the restriction $G|\sigma$ of G to σ to be the group induced on $V(\sigma)$ by $\bigcap_{W \in \sigma} G_W$. Again, $G|\{U\}$ will be written $G|U$. Given two SSWAGs, we define

(3.1)
$$(\mathcal{S}_1, G_1) \cdot (\mathcal{S}_2, G_2) := (\mathcal{S}_1 \cdot \mathcal{S}_2, G_1 \times G_2), \quad (\mathcal{S}_1, G_1) \odot (\mathcal{S}_2, G_2) := (\mathcal{S}_1 \odot \mathcal{S}_2, G_1 \times G_2).$$

These operation can be defined on isomorphism classes of SSWAGs. Given a SSWAG $(\mathcal{S}, G)$, the group G acts in an obvious way on $\mathcal{S}$, $\Pi(\mathcal{S})$, $\widetilde{\Pi}(\mathcal{S})$, and $A(\mathcal{S})$. It also acts on $\Gamma_n(\mathcal{S})$ by $(g, c) \mapsto c \circ g^{-1}$. A triple $(\mathcal{S}, G, w)$ is called a *weighted SSWAG* if the weight w is constant on the orbits of G on $\widetilde{\Pi}(\mathcal{S})$. In this paper we need the following technical condition on the automorphism groups G considered: for each permutation in G, all its cycles are also in G. According to [1], such a group is the direct product of its transitive constituents, each of which is a symmetric group or a cyclic group of prime order. Let us denote by $\mathbb{SG}$ and $\mathbb{SG}'$ the set of isomorphism classes of all SSWAGs, and of those for which the automorphism group satisfies the technical condition, respectively.

Consider a SSWAG $(\mathcal{S}, G)$. The set $A(\mathcal{S})$ is partially ordered by the refinement relation of preferential arrangements: $\sigma_f \leq \pi_{f'}$ iff $\pi_{f'}$ is obtained from σ_f by amalgamating adjacent blocks, while keeping the position of the blocks fixed. As G acts on the poset $A(\mathcal{S})$, we have an induced poset $A(\mathcal{S})/G$ on the set of orbits. We adjoin a least element $\widehat{0}$ to this poset, and define $z^b \in (H \cdot \Phi)_*((A(\mathcal{S})/G) \amalg \{\widehat{0}\})$ in a similar way to ζ^ϕ. As before, z^b has a convolution inverse, which will be denoted by $m^b_{(A(\mathcal{S})/G) \amalg \widehat{0}}$. We define the weight $\tau^b_G \colon \widetilde{\Pi}(\mathcal{S}) \to (H \cdot \Phi)_*$, called *divided type*, by $\tau^b_G(\sigma) := \tau^\phi(\sigma)/|G|\sigma|$. Given a partition system $\mathcal{S}'$ such that $\mathrm{At}(\mathcal{S}) = \mathrm{At}(\mathcal{S}')$ and $(\mathcal{S}', G)$ is a SSWAG, we use $m^b_{(A(\mathcal{S})/G) \amalg \widehat{0}}$ to define the weight $\nu^b_{\mathcal{S}', G}$, which will be called the *divided Möbius type* with respect to $\mathcal{S}'$. Our definitions imply that $(\mathcal{S}, G, \tau^b_G)$ and $(\mathcal{S}, G, \nu^b_{\mathcal{S}', G})$ are weighted SSWAGs. Let us note that if G is the symmetric group on $V(\mathcal{S})$ and $\sigma \in \widetilde{\Pi}(\mathcal{S})$ is arbitrary, then $\tau^b_G(\sigma)$ is obtained from $\tau^\phi(\sigma)$ via the substitution $\phi_i \mapsto b_i$.

Proposition 3.2. *Given the above setup, let $\sigma \in \widetilde{\Pi}(\mathcal{S})$, and let $f \colon \sigma \to [|\sigma|]$ be an arbitrary bijection.*

a) $m^b_{(A(\mathcal{S})/G) \amalg \widehat{0}}(\widehat{0}, [\sigma_f]) = \mu^\phi_{\widetilde{\Pi}(\mathcal{S})}(\widehat{0}, \sigma)/|G|\sigma|$.

b) $\nu^b_{\mathcal{S}', G}(\sigma) = \nu^\phi_{\mathcal{S}'}(\sigma)/|G|\sigma|$.

c) If $(\mathcal{S}', G) \in \mathbb{SG}'$, then $\nu^b_{\mathcal{S}', G}$ is a multiplicative weight.

We now define several polynomials in $(H \cdot \Phi)_*\left\{\frac{x^i}{i!}\right\}$ associated with the SSWAG $(\mathcal{S}, G)$; all of them have degree $2|V(\mathcal{S})|$. These polynomials will have the following form:

(3.3)
$$\theta(\mathcal{S}, G, w; x) := \sum_{\sigma_f \in \mathcal{T}} w(\sigma)\, q_{|\sigma|}(x),$$

where $w \colon \widetilde{\Pi}(\mathcal{S}) \to (H \cdot \Phi)_*$ is a weight such that $(\mathcal{S}, G, w)$ is a weighted SSWAG, $(q_n(x))_{n \geq 1}$ is a sequence of polynomials in $(H \cdot \Phi)_*\left\{\frac{x^i}{i!}\right\}$, and $\mathcal{T}$ is an arbitrary transversal of $A(\mathcal{S})/G$. For each polynomial defined, we indicate the corresponding choice of w and $q_n(x)$, its name, and notation:

- $w = \tau^b_G$, $q_n(x) = x^n/n!$: the partition type polynomial $\rho^b(\mathcal{S}, G; x)$;
- $w = \tau^b_G$, $q_n(x) = \beta^b_n(x)$: the umbral chromatic polynomial $\gamma^b(\mathcal{S}, G; x)$;
- $w = \nu^b_{\mathcal{S}, G}$, $q_n(x) = \beta^b_n(x)$: the modified umbral chromatic polynomial $\widetilde{\gamma}^b(\mathcal{S}, G; x)$;
- $w(\sigma) = m^b_{(A(\mathcal{S})/G) \amalg \widehat{0}}(\widehat{0}, [\sigma_f])$, ($f \colon \sigma \to [|\sigma|]$ an arbitrary bijection), $q_n(x) = x^n/n!$: the characteristic type polynomial $\chi^b(\mathcal{S}, G; x)$.

We intend to relate the polynomials defined above to the corresponding ones defined in the previous section. We do this by proving a general result about the relation between the polynomials $\theta(\mathcal{P}, w; x)$ and $\theta(\mathcal{S}, G, w; x)$ defined in 2.9 and 3.3. The proof is based on the weighted form of Burnside's lemma.

Theorem 3.4. *Let $(\mathcal{S}, G, w)$ be a weighted SSWAG with w taking values in an arbitrary commutative and torsion free ring R. For each $n \geq 1$, consider $q_n(x) \in R[x]$, and let $p_n(x) := n!\, q_n(x)$.*

a) We have

$$\theta(\mathcal{S}, G, w; x) = \left(\sum_{g \in G} \theta(\mathcal{S}/\mathrm{cyc}(g), w/\mathrm{cyc}(g); x) \right) / |G|, \tag{3.5}$$

where $\mathrm{cyc}(g)$ *denotes the partition of* $V(\mathcal{S})$ *determined by the cycles of* g.

b) Let $w' \colon \widetilde{\Pi}(\mathcal{S}) \to R$ *be the weight defined by* $w'(\sigma) := |G|\sigma|\, w(\sigma)$. *We have*

$$\theta(\mathcal{S}, G, w; x) = \theta(\mathcal{S}, w'; x)/|G|. \tag{3.6}$$

Note that if we take $R = \mathbb{Q}$, $q_n(x) = x(x-1)\ldots(x-n+1)/n!$, $\mathcal{S}$ the independence complex of a graph, and G the automorphism group of the graph, then $\theta(\mathcal{S}, G, w; x)$ is the chromatic polynomial of the corresponding unlabeled graph, as defined in [2]. In this case, 3.5 is just a restatement of Theorem 3.1 in [2].

Theorem 3.4 and Proposition 3.2 imply that each of the four polynomials associated with the weighted SSWAG $(\mathcal{S}, G)$ can be related to the corresponding polynomial associated with $\mathcal{S}$ via 3.6. Thus, 2.13 extends naturally to SSWAGs:

$$\tilde{\gamma}^b(\mathcal{S}, G; x) = \chi^b(\overline{\mathcal{S}}, G; x). \tag{3.7}$$

We conclude this section by noting that $A(\mathcal{K}_n)/S_n$ is isomorphic to the Boolean sublattice of $\Pi(\mathcal{K}_n)$ containing those partitions for which the elements of each block, written in increasing order, are consecutive. According to the previous results, the normalised conjugate Bell polynomials $\beta_n^b(x) = B_n^\phi(x)/n!$ can be computed using this lattice, rather than $\Pi(\mathcal{K}_n)$ – used for the computation of $B_n^\phi(x)$ (see 2.10) – which is much larger. Similarly, we can compare the way in which F_{nm}^ϕ and $f_{nm}^b = F_{nm}^\phi/(n!\,m!)$ can be computed combinatorially. According to 2.14, F_{nm}^ϕ can be computed using the poset obtained from $\Pi(\mathcal{K}_{n,m})$ by adjoining the partition of $[n+m]$ into only one block. Given a linearly ordered set $V = \{v_1 < v_2 < \ldots < v_k\}$, we are now interested in those preferential arrangements in $A(\mathcal{K}_V)$ which are comparable to $\{\{v_1\} < \{v_2\} < \ldots < \{v_k\}\}$. Consider the subposet of $A(\mathcal{K}_{n,m})$ consisting of all "shuffles" of two preferential arrangements of the previous type, one from $A(\mathcal{K}_n)$, and the other from $A(\mathcal{K}_{n+[m]})$. According to Proposition 3.2, f_{nm}^b can be computed using the poset obtained from the previous one by adjoining the preferential arrangement of $[n+m]$ with only one block. This time, the first computation is more efficient.

4. A Hopf Algebra for Set Systems with an Automorphism Group

In [5] we defined several Hopf algebra structures on the free Φ_*-module $\Phi_*\{\mathbb{P}\}$ generated by the set $\mathbb{P}$; in each case, $\Phi_*\{\mathbb{S}\}$ is a sub-Hopf algebra. Our constructions represent an extension of the Tutte algebra of graphs defined in [9]; the extended framework allowed us to define the product operation in a natural way (as disjoint union of set systems), whereas for graphs, the product had a complicated expression. We now consider the the free $(H\cdot\Phi)_*$-module $(H\cdot\Phi)_*\{\mathbb{SG}\}$ generated by the set $\mathbb{SG}$, and graded by setting the degree of a SSWAG $(\mathcal{S}, G)$ to $2|V(\mathcal{S})|$. The operations: disjoint union, $\odot$, and complementation can be extended by linearity to $(H\cdot\Phi)_*\{\mathbb{SG}\}$. In this section, we extend the constructions in [5] even further, by defining a Hopf algebra structure on $(H\cdot\Phi)_*\{\mathbb{SG}\}$. As before, we use the general method – presented in [14] and [15] – for constructing the *incidence Hopf algebra* of a *hereditary family of posets* (intervals) with a *Hopf relation*.

Let $\delta\colon (H\cdot\Phi)_*\{\mathbb{SG}\} \to (H\cdot\Phi)_*\{\mathbb{SG}\} \otimes (H\cdot\Phi)_*\{\mathbb{SG}\}$ be the linear map specified by

$$\delta(S,G) := \sum_{U\in\mathcal{T}} (S|U, G|U) \otimes (S|\overline{U}, G|\overline{U}), \tag{4.1}$$

where $\mathcal{T}$ is a transversal of the orbits of G on $\mathcal{K}_{V(S)}$ and $\overline{U} = V(S)\backslash U$. Let $\varepsilon\colon (H\cdot\Phi)_*\{\mathbb{SG}\} \to (H\cdot\Phi)_*$ and $\eta\colon (H\cdot\Phi)_* \to (H\cdot\Phi)_*\{\mathbb{SG}\}$ be the linear maps specified by

$$\varepsilon(S,G) := \begin{cases} 1 & \text{if } (S,G) = (\{\emptyset\},\{1\}) \\ 0 & \text{otherwise}. \end{cases} \qquad \eta(1) = (\{\emptyset\},\{1\}). \tag{4.2}$$

Theorem 4.3. *$((H\cdot\Phi)_*\{\mathbb{SG}\},\cdot,\eta,\delta,\varepsilon)$ is a commutative and cocommutative graded Hopf algebra. The antipode $S\colon (H\cdot\Phi)_*\{\mathbb{SG}\} \to (H\cdot\Phi)_*\{\mathbb{SG}\}$ is specified by*

$$S(S,G) = \sum_{\sigma_f\in\mathcal{T}} (-1)^{|\sigma|} \prod_{W\in\sigma} (S|W, G|W), \tag{4.4}$$

where $\mathcal{T}$ is a transversal of the orbits of G on $A(\mathcal{K}_{V(S)})$. The free $(H\cdot\Phi)_*$-module $(H\cdot\Phi)_*\{\mathbb{SG}'\}$ generated by the set $\mathbb{SG}'$ is a sub-Hopf algebra.

We obtain a similar Hopf algebra structure on $(H\cdot\Phi)_*\{\mathbb{SG}\}$ by replacing the disjoint union product with $\odot$. Complementation of SSWAGs induces an isomorphism between these Hopf algebras. Let us denote by ρ^b, γ^b, $\tilde{\gamma}^b$, and χ^b the $(H\cdot\Phi)_*$-module homomorphisms of degree 0 from $(H\cdot\Phi)_*\{\mathbb{SG}\}$ to $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ mapping a SSWAG to the corresponding polynomial associated with it. Using results from [5] and 3.6 we prove the following theorem.

Theorem 4.5. *The maps $\rho^b,\chi^b\colon ((H\cdot\Phi)_*\{\mathbb{SG}'\},\cdot) \to (H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ are Hopf algebra maps. The map $\tilde{\gamma}^b\colon ((H\cdot\Phi)_*\{\mathbb{SG}'\},\odot) \to (H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ is also a Hopf algebra map. The map $\gamma^b\colon (H\cdot\Phi)_*\{\mathbb{SG}'\} \to (H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ is a coalgebra map.*

We note that the technical condition on the automorphism groups considered is essential for the maps above to be coalgebra maps. The following theorem addresses the problem of finding a combinatorial model for the Hopf algebra ${}^L\Phi_*\{\beta_i^b(x)\}$ over ${}^L\Phi_*$. Let $\mathbb{SG}''$ denote the subset of $\mathbb{SG}'$ consisting of isomorphism classes of SSWAGs (S,G) for which there exists a partition $\pi\in\Pi(S)$ such that $\mathcal{K}_\pi \subseteq S \subseteq \mathcal{K}_\pi \cup \mathrm{Bool}(\pi)$, and G is a subgroup of the restriction of the symmetric group on $V(S)$ to π. It is easy to see that the free ${}^L\Phi_*$-module ${}^L\Phi_*\{\mathbb{SG}''\}$ generated by the set $\mathbb{SG}''$ is a sub-Hopf algebra of $(H\cdot\Phi)_*\{\mathbb{SG}'\}$ (considered as a Hopf algebra over ${}^L\Phi_*$).

Theorem 4.6. *The map χ^b maps the set $\mathbb{SG}''$ to ${}^L\Phi_*\{\beta_i^b(x)\}$. Hence, we have a Hopf algebra map $\chi^b\colon {}^L\Phi_*\{\mathbb{SG}''\} \to {}^L\Phi_*\{\beta_i^b(x)\}$.*

Let $\theta\colon (H\cdot\Phi)_*\{\mathbb{SG}'\} \to (H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ be a surjective coalgebra map. The adjoint map θ^* of θ is an injective algebra map; it maps $(H\cdot\Phi)^*[[D]]$ isomorphically to a sub-algebra of the graded dual of $(H\cdot\Phi)_*\{\mathbb{SG}'\}$. We use θ^* and 1.2 to associate a delta operator on $(H\cdot\Phi)_*\{\mathbb{SG}\}$ with each delta operator $a(D)\in(H\cdot\Phi)^2[[D]]$; it makes sense to denote this delta operator by $a(D_\theta)$, where D_θ is the delta operator corresponding to D. It is not difficult to check that the map $\theta\colon ((H\cdot\Phi)_*\{\mathbb{SG}'\}, a(D_\theta)) \to ((H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}, a(D))$ becomes a map of coalgebras with delta operator. We can use this result to obtain a stronger version of Theorem 4.5. Let us note that

$$\flat(D_\gamma)(\mathcal{N}_n, \mathcal{S}_n) = (\mathcal{N}_{n-1}, \mathcal{S}_{n-1}), \qquad D_\gamma(\mathcal{K}_n, \mathcal{S}_n) = (\mathcal{K}_{n-1}, \mathcal{S}_{n-1}),$$
$$D_\chi(\mathcal{N}_n, \mathcal{S}_n) = (\mathcal{N}_{n-1}, \mathcal{S}_{n-1}), \qquad \flat(D_\chi)(\mathcal{K}_n, \mathcal{S}_n) = (\mathcal{K}_{n-1}, \mathcal{S}_{n-1}).$$

In [5] we proved that the identities 1.14 can be lifted from $\Phi_*[x]$ to $\Phi_*\{\mathbb{P}\}$. We prove here that they can be lifted from $(H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ to $(H\cdot\Phi)_*\{\mathbb{SG}'\}$.

Theorem 4.7. *Let $\theta\colon ((H\cdot\Phi)_*\{\mathbb{SG}'\}, \cdot) \to (H\cdot\Phi)_*\left\{\frac{x^i}{i!}\right\}$ be a Hopf algebra map (in particular $\rho^\flat$ or $\chi^\flat$). The following identities hold in $(H\cdot\Phi)_*\{\mathbb{SG}'\}$:*

$$(4.8)\qquad \flat(D_\theta)((\mathcal{S}_1, G_1)\cdot(\mathcal{S}_2, G_2)) = \sum_{n,m\geq 0} f^\flat_{nm}\,(\flat(D_\theta)^n(\mathcal{S}_1, G_1))\cdot(\flat(D_\theta)^m(\mathcal{S}_2, G_2))$$

$$(4.9)\qquad \flat(D_\theta)\,S(\mathcal{S}, G) = \sum_{k\geq 1} i^\flat_k\, S(\flat(D_\theta)^k(\mathcal{S}, G)).$$

These identities still hold in the Hopf algebra $((H\cdot\Phi)_*\{\mathbb{SG}'\}, \odot)$, and hence for $\tilde{\gamma}^\flat$.

References

1. P. Cameron. Private communication, October 1994.
2. P. Hanlon. The chromatic polynomial of an unlabeled graph. *J. Combin. Theory Ser. B*, 38:226–239, 1985.
3. M. Hazewinkel. *Formal Groups and Applications.* Academic Press, New York, 1978.
4. A. Kerber. *Algebraic Combinatorics via Finite Group Actions.* Wissenschaftsverlag, Mannheim - Wien - Zürich, 1991.
5. C. Lenart and N. Ray. A Hopf algebra framework for set system colourings. *submitted.*
6. D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.
7. N. Ray. Extensions of umbral calculus; penumbral coalgebras and generalised Bernoulli numbers. *Adv. Math.*, 61:49–100, 1986.
8. N. Ray. Umbral calculus, binomial enumeration and chromatic polynomials. *Trans. Amer. Math. Soc.*, 309:191–213, 1988.
9. N. Ray. Tutte algebras of graphs and formal group theory. *Proc. London Math. Soc.*, 65:23–45, 1992.
10. N. Ray and W. Schmitt. Coclosure operators and chromatic polynomials. *Proc. Natl. Acad. Sci. USA*, 87:4685–4687, 1990.
11. N. Ray, W. Schmitt, and C. Wright. Colouring simplicial complexes. *Ars Combin.*, 29A:161–169, 1990.
12. N. Ray and C. Wright. Colourings and partition types: a generalised chromatic polynomial. *Ars Combin.*, 25B:277–286, 1988.
13. S. Roman. *The Umbral Calculus.* Academic Press, Orlando, FL, 1984.
14. W. Schmitt. Antipodes and incidence coalgebras. *J. Combin. Theory Ser. A*, 46:264–290, 1987.
15. W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96:299, 1994.
16. M. E. Sweedler. *Hopf Algebras.* W. A. Benjamin, New York, 1969.
17. D. Wagner. The partition polynomial of a finite set system. *J. Combin. Theory Ser. A*, 56:138–159, 1991.

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