Natural Basis for ε-Shuffle Algebras

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Abstract

We present a super version of D. E. Radford's theorem that the free associative algebra as a shuffle algebra is the free commutative associative algebra with a set of free generators consisting of Lyndon words.

Shuffle algebras were introduced by R. Ree in [3], [4]. For applications of shuffle algebras in free Lie algebras see, for example, [5].

Let K be a commutative associative ring with the unity element, K^* the group of invertible elements of K, G an Abelian semigroup. A mapping $\varepsilon: G \times G \to K^*$ is called a commutation factor on G if:

$$\varepsilon(g,h)\,\varepsilon(h,g) = 1;$$
$$\varepsilon(g,g) = \pm 1;$$
$$\varepsilon(g,h+f) = \varepsilon(g,h)\,\varepsilon(g,f);$$
$$\varepsilon(g+h,f) = \varepsilon(g,f)\,\varepsilon(h,f)$$

for all $f, g, h \in G$.

Let also

$$G_+ = \{g \in G \mid \varepsilon(g,g) = 1\}, \qquad G_- = \{g \in G \mid \varepsilon(g,g) = -1\}.$$

A K-algebra A is said to be G-graded if $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subset A_{g+h}$ for all $g, h \in G$. The grading function d is given by $d(a) = g$ for $a \in A_g$.

A G-graded K-algebra A is said to be ε-commutative if $ab = \varepsilon(d(a), d(b))\, ba$ for all G-homogeneous elements $g, h \in G$.

Let X be a G-graded set, i.e.

$$X = \bigcup_{g \in G} X_g, \qquad X_f \cap X_h = \emptyset$$

for all $f \neq h$, ε a commutation factor on G. If $x \in X_g$, then $d(x) = g$. Let also

$$X_+ = \bigcup_{g \in G_+} X_g, \qquad X_- = \bigcup_{g \in G_-} X_g.$$

We consider the associative K-algebra $K[X]$ given by generators X and defining relations $xy - \varepsilon(d(x), d(y))\, yx = 0$ with $x, y \in X$.

It is clear that 1 and the monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} y_{j_1} \cdots y_{j_l}$ with $x_{i_j} \in X_+$, $y_{j_t} \in X_-$, $x_{i_1} < \cdots < x_{i_k}$; $y_{j_1} < \cdots < y_{j_l}$ form a basis of the free K-module $K[X]$.

For a monomial $z_1 \cdots z_m$, $z_j \in X$, we set

$$d(z_1 \cdots z_m) = \sum_{j=1}^{m} d(z_j).$$

In addition, we assume that $d(1) = 0$.

The algebra $K[X]$ is the free ε-commutative associative K-algebra on X, i.e. if A is an ε-commutative associative K-algebra and $\varphi\colon X \to A$ is a mapping of degree zero (for each homogeneous $x \in X$ the element $\varphi(x)$ is also a homogeneous element of A and $d(\varphi(x)) = d(x)$), then there exists the unique homomorphism of zero degree of ε-commutative associative K-algebras $\psi\colon K[X] \to A$ such that $\psi|_X = \varphi$.

Let $S(X)$ be the free semigroup on X and $S^+(X)$ the free semigroup without unity element, $A(X)$ the free G-graded associative K-algebra with the set X of free generators. Consider the Abelian semigroup $H = (\mathbf{N} \cup 0)^{|X|}$ with the standard set of generators $\{x \mid x \in X\}$. The multidegree function $m\colon S(X) \to H$ is the semigroup homomorphism given by $m(x) = x$.

Suppose that the set X is totally ordered, and the set $S^+(X)$ is ordered lexicographically, i.e. for $u = x_1 \cdots x_r$ and $v = y_1 \cdots y_m$ where $x_i, y_j \in X$ we have $u > v$ if either $x_i = y_i$ for $i = 0, 1, \ldots, t-1$ and $x_t > y_t$ or $x_i = y_i$ for $i = 1, 2, \ldots, r$ and $r < m$. Then $S^+(X)$ is linearly ordered. A word $u \in S^+(X)$ is said to be regular if for any decomposition $u = ab$, where

$a, b \in S^+(X)$, we have $u > ba$ (equivalent condition: it follows from $u = ab$ that $u > b$). A word $w \in S^+(X)$ is said to be s-regular if either w is a regular word or $w = vv$ with v a regular word and $d(v) \in G_-$.

Let V and Z be G-graded sets, $V \cap Z = \emptyset$, $v_1, \ldots, v_k$ pairwise distinct elements of V, $z_1 \ldots, z_l$ pairwise distinct elements of Z. We say that a word $w \in S^+(V \cup Z)$ is a ε-shuffle word of the words $v_1 \cdots v_k$ and $z_1 \cdots z_l$ if

$$\begin{aligned} m(w) &= v_1 + \cdots + v_k + z_1 + \cdots + z_l; \\ w\big|_{\substack{z_1=1 \\ \cdots \\ z_l=1}} &= v_1 \cdots v_k; \\ w\big|_{\substack{v_1=1 \\ \cdots \\ v_k=1}} &= z_1 \cdots z_l. \end{aligned}$$

The parity $\sigma(w)$ of the ε-shuffle word w of words $v_1 \cdots v_k$ and $z_1 \cdots z_l$ is the sum of all $\varepsilon(z_i, v_j)$ such that z_i is situated before v_j in w.

Let X be a G-graded set. Consider now G-graded sets V and Z, such that $|V| = |Z| = |X|$ $d(v_i) = d(z_i) = d(x_i)$. For $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_l} \in X$ we define the ε-shuffle product $(x_{i_1} \cdots x_{i_k}) * (x_{j_1} \cdots x_{j_l})$ as the following linear combination of words of the multidegree $x_{i_1} + \cdots + x_{i_k} + x_{j_1} + \cdots + x_{j_l}$:

$$(x_{i_1} \cdots x_{i_k}) * (x_{j_1} \cdots x_{j_l}) = \sum_w \sigma(w) w\big|_{\substack{v_{i_s} = x_{i_s},\ s=1,\ldots,k; \\ z_{j_t} = x_{j_t},\ t=1,\ldots,l}}$$

where w is running through all ε-shuffle words of the words $v_{i_1} \cdots v_{i_k}$ and $z_{j_1} \cdots z_{j_l}$. Taking $1 * u = u * 1$ for all $u \in S^+(X)$ and extending $*$ on the free G-graded associative algebra $A(X)$ on X over a commutative associative ring K with the unity element by linearity, we define the ε-shuffle product $*$ on $A(X)$. Then $A(X)$ with this product is an ε-commutative and associative algebra.

One can define the ε-shuffle product $*$ on $A(X)$ in the following way:

$$\begin{aligned} 1 * u &= u * 1 = u; \\ (xu) * (yv) &= x(u * (yv)) + \varepsilon(y, x)\varepsilon(y, u)\, y((xu) * v) \end{aligned}$$

for all $x, y \in X$, $u, v \in S(X)$ (with the extension $*$ on $A(X)$ by linearity). Note that $A(X)$ is the universal enveloping algebra of the free color Lie superalgebra $L(X)$ (see [1]), and the ε-shuffle product is the adjoint operation to the coproduct in $A(X)$.

D. Radford in [2] proved that in the case of trivial grading group the free associative algebra as a shuffle algebra is the free commutative associative algebra with a set of free generators consisting of Lyndon words. For ε-shuffle algebras we get the following result.

Theorem 1 *If K is a Q-algebra, then $A(X)$ with the shuffle product $*$ is a free ε-commutative algebra with a set of free generators consisting of s-regular words.*

The proof is based on the canonical factorization of a word into the product of s-regular words: if $u \in S^+(X)$, then u has the unique presentation in the form $w_1 \cdots w_m$ where w_i is an s-regular word for all i, $w_1 \leq w_2 \leq \cdots \leq w_m$, $w_i \neq w_j$ with $d(w_i) \in G_-$.

References

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