ON STIRLING PARTITIONS OF THE SYMMETRIC GROUP

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Abstract

We construct partitions of the symmetric group $\mathfrak{S}_N$ into intervals with respect to the Bruhat order such that every interval is a Boolean set and the number of intervals with 2^k elements is the signless Stirling number of the first kind $c(N-1,k)$. A projection $\mathfrak{S}_N \to \mathfrak{S}_{N-1}$ whose fibers form a partition of $\mathfrak{S}_N$ with these properties is studied. Several constructions of Laplace operators for the orthogonal and symplectic Lie algebras, which involve this projection are reviewed.

Résumé

On construit des partitions du groupe symétrique $\mathfrak{S}_N$ en intervalles par rapport à l'ordre de Bruhat de sorte que chaque intervalle soit un ensemble booléen, et que le nombre d'intervalles à 2^k éléments soit la valeur absolue du nombre de Stirling de première espèce $c(N-1,k)$. On étudie une projection de $\mathfrak{S}_N \to \mathfrak{S}_{N-1}$ dont les fibres forment une partition de $\mathfrak{S}_N$ possédant ces propriétés. On décrit aussi plusieurs constructions d'opérateurs de Laplace pour les algèbres de Lie orthogonales et symplectiques qui font intervenir cette projection.

0. Introduction

Let $A = (A_{ij})$ be an $N \times N$ numerical matrix and let $\det A$ be its determinant

$$\det A = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p) A_{p(1),1} \cdots A_{p(N),N}. \tag{0.1}$$

For any fixed map $p \mapsto p'$ of the symmetric group $\mathfrak{S}_N$ into itself such that the map $p \mapsto p(p')^{-1}$ is a bijection, formula (0.1) can be also rewritten as

$$\det A = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(pp') A_{p(1),p'(1)} \cdots A_{p(N),p'(N)}. \tag{0.2}$$

However, if the entries of the matrix A belong to a noncommutative ring, the right hand sides of formulae (0.1) and (0.2) are different in general, and each of them can be regarded as a noncommutative analogue of the determinant of the matrix A. Noncommutative determinants of the form (0.1) were used in [HU] for constructing central elements in the universal enveloping algebras for the general linear and orthogonal Lie algebras. In the case of the orthogonal and symplectic Lie algebras central elements were constructed in [M1] by using a determinant of the form (0.2) with a special projection $p \mapsto p'$, $\mathfrak{S}_N \to \mathfrak{S}_{N-1}$, where $\mathfrak{S}_{N-1}$ is regarded as a natural subgroup of $\mathfrak{S}_N$ (these constructions are reviewed in Section 4). We prove here (Section 3) that the fibers of this projection form a partition of $\mathfrak{S}_N$ with the following properties. The fibers are intervals with respect to the Bruhat order on $\mathfrak{S}_N$, isomorphic to the Boolean sets (as partially ordered sets). In particular, each fiber contains 2^k elements for some $k \in \{1, \ldots, N-1\}$. Moreover, the number of fibers containing 2^k elements coincides with the signless Stirling number of the first kind $c(N-1, k)$. In Section 2 we construct a simpler partition of $\mathfrak{S}_N$ which admits the same properties as the one formed by the fibers of this projection.

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1. Signless Stirling numbers of the first kind

Let n and k be positive integers. The *signless Stirling number of the first kind* $c(n,k)$ is defined as the number of permutations $p \in \mathfrak{S}_n$ with exactly k cycles (see, e.g., [S]). One has the following formula which can be regarded as an equivalent definition of $c(n,k)$ [S]: for a formal variable x

$$\sum_{k=1}^{n} c(n,k)\, x^k = x(x+1)\cdots(x+n-1). \qquad (1.1)$$

We shall also use the following property of the numbers $c(n,k)$ below [S]. If $p = (p_1, \ldots, p_n)$ is a sequence of distinct positive integers, an element p_i is called a *left-to-right maximum* of p, if $p_j < p_i$ for every $j < i$. Then the number of permutations $p \in \mathfrak{S}_n$ with k left-to-right maxima is $c(n,k)$.

A combinatorial proof of formula (1.1) for positive integers x can be found in [S]. The partitions of $\mathfrak{S}_N$ constructed in Sections 2 and 3 provide an interpretation of (1.1) for $x = 2$.

2. Stirling partitions of $\mathfrak{S}_N$

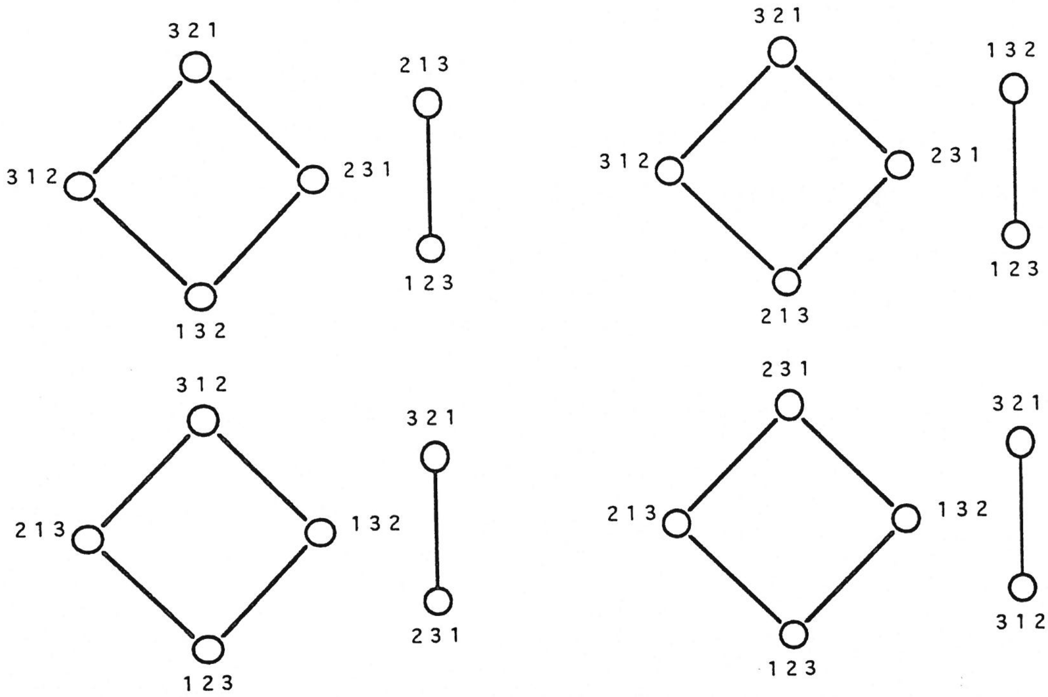
We shall consider the following two examples of partially ordered sets. The first is the *Boolean set* B_n consisting of 2^n subsets of the set $\{1, 2, \ldots, n\}$. One defines $S \leq T$, if $S \subseteq T$ as sets.

The second example is the symmetric group $\mathfrak{S}_N$ with the *Bruhat order* which is defined as follows (see, e.g., [S]). If $q = (q_1, \ldots, q_N) \in \mathfrak{S}_N$, then a *reduction* of q is a permutation obtained from q by interchanging two elements q_i and q_j, where $i < j$ and $q_i > q_j$. One says that $p \leq q$ with respect to the Bruhat order, if p can be obtained from q by a sequence of reductions.

An interval $[p, q]$ in $\mathfrak{S}_N$ will be called *Boolean* if it is isomorphic to B_k for some k. Let us call a partition of $\mathfrak{S}_N$ into Boolean intervals *Stirling* if for any k the number of intervals isomorphic to B_k equals the signless Stirling number of the first kind $c(N-1,k)$. In particular, the total number of intervals equals

$$c(N-1,1) + \cdots + c(N-1,N-1) = (N-1)!$$

One has 4 Stirling partitions of $\mathfrak{S}_3$:

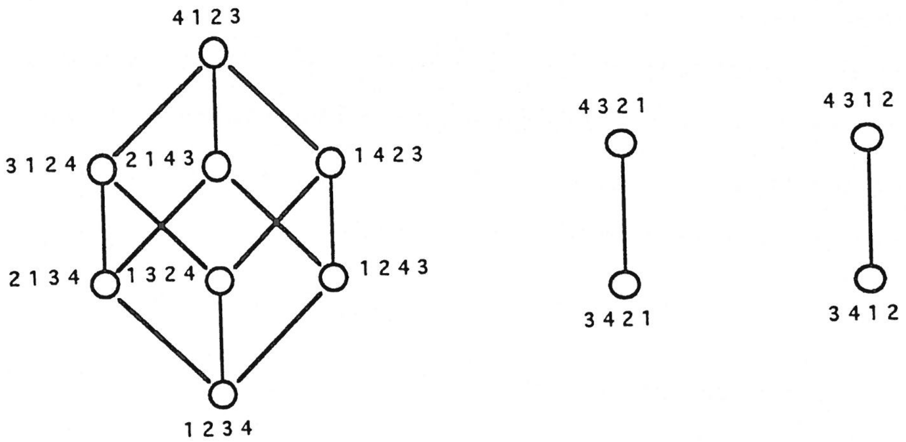


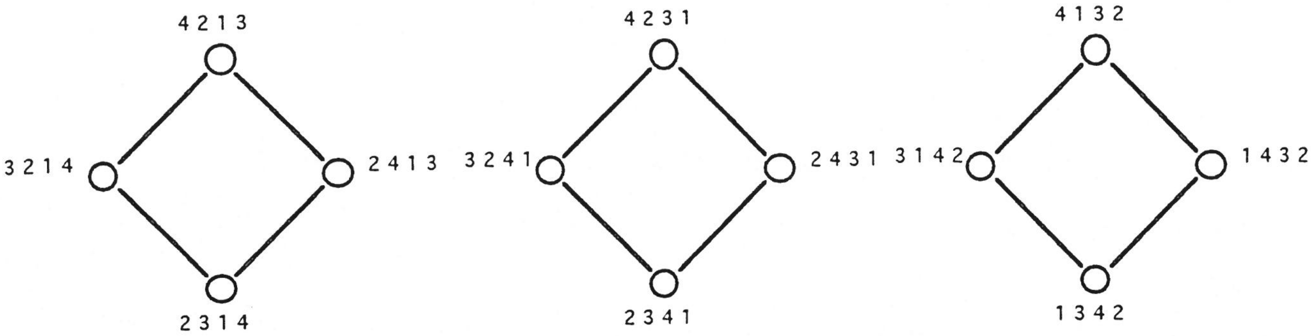
We construct now a Stirling partition of $\mathfrak{S}_N$ for arbitrary N.

Let us fix a permutation $q = (q_1, \dots, q_{N-1}) \in \mathfrak{S}_{N-1}$ and let $q_{i_1} < q_{i_2} < \dots < q_{i_k}$ be the left-to-right maxima of q. Consider the permutation $p_{\max} = (N, q_1, \dots, q_{N-1}) \in \mathfrak{S}_N$ and denote by $p_{\min}$ the permutation which is obtained from $p_{\max}$ by replacing the subsequence $(N, q_{i_1}, \dots, q_{i_k})$ with the subsequence $(q_{i_1}, \dots, q_{i_k}, N)$ and leaving the remaining entries of $p_{\max}$ unchanged.

Theorem 2.1. *The set of intervals $[p_{\min}, p_{\max}]$, where q runs over the set $\mathfrak{S}_{N-1}$, forms a Stirling partition of $\mathfrak{S}_N$.*

For $N = 4$ this partition has the form:





3. Projection $\mathfrak{S}_N \to \mathfrak{S}_{N-1}$

Let us define now the projection

$$\pi_N : \mathfrak{S}_N \to \mathfrak{S}_{N-1}, \tag{3.1}$$

(see Introduction). It will be convenient to realize $\mathfrak{S}_N$ as the group of permutations of the indices $c_1, \ldots, c_N$, where the c_i are some positive integers and $c_1 < \cdots < c_N$. For $N = 2$ we take as the projection π_2 the only map $\mathfrak{S}_2 \to \mathfrak{S}_1$. For $N > 2$ we define π_N inductively. First define a map from the set of all ordered pairs (c_k, c_l), $k \neq l$ into itself by the following rule:

$$\begin{aligned}
(c_k, c_l) &\mapsto (c_l, c_k), \quad k, l < N,\\
(c_k, c_N) &\mapsto (c_{N-1}, c_k), \quad k < N-1,\\
(c_N, c_k) &\mapsto (c_k, c_{N-1}), \quad k < N-1,\\
(c_{N-1}, c_N) &\mapsto (c_{N-1}, c_{N-2}),\\
(c_N, c_{N-1}) &\mapsto (c_{N-1}, c_{N-2}).
\end{aligned} \tag{3.2}$$

Further, if $p = (p_1, \ldots, p_N)$ is a permutation of the indices $c_1, \ldots, c_N$, its image $q = \pi_N(p)$ is defined as follows. We take as (q_1, q_{N-1}) the image of the ordered pair (p_1, p_N) under the map (3.2). Assuming that the projection π_{N-2} has been already defined, we take as $(q_2, \ldots, q_{N-2})$ the image of $(p_2, \ldots, p_{N-1})$ with respect to this projection, where $(p_2, \ldots, p_{N-1})$ is regarded as a permutation of the family of the indices obtained from $\{c_1, \ldots, c_N\}$ by removing p_1 and p_N.

Let us describe now the fibers of the projection π_N. First we suppose that N is odd, $N = 2n + 1$. Let $q = (q_1, \ldots, q_{2n})$ be an element of $\mathfrak{S}_{2n}$. Consider another permutation

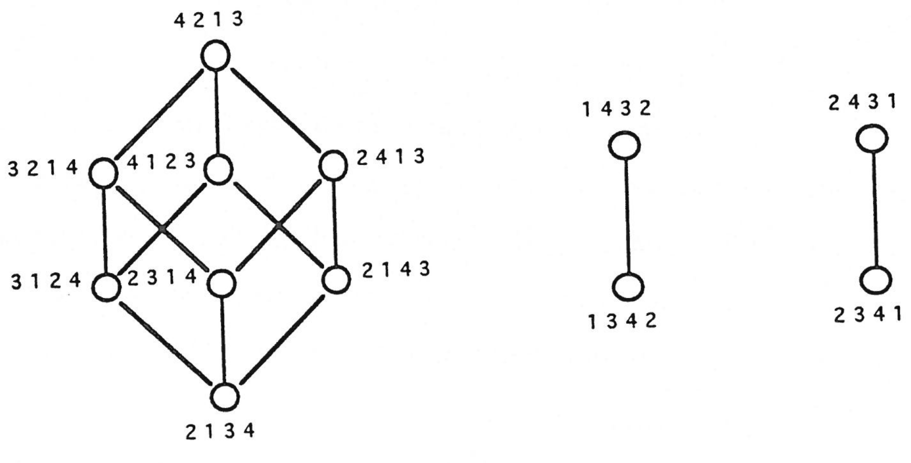
$\widetilde{q} = (q_{n+1}, q_n, q_{n+2}, q_{n-1}, \ldots, q_{2n}, q_1)$ and denote by A the set of left-to-right maxima in $\widetilde{q}$. It follows from the definition of π_N that the element $p_0 = (q_{2n}, \ldots, q_{n+1}, c_{2n+1}, q_n, \ldots, q_1)$ is contained in the fiber over q. Introduce now the elements $p_{\min}$ and $p_{\max}$ in the following way. Consider the subsequence of p_0 which has the form $(c_{2n+1}, q_{i_1}, \ldots, q_{i_m})$, where $q_{i_1}, \ldots, q_{i_m}$ are those elements among $\{q_n, \ldots, q_1\}$ which are contained in the set A. Then $p_{\min}$ is obtained from p_0 by replacing this subsequence with the subsequence $(q_{i_1}, \ldots, q_{i_m}, c_{2n+1})$, while the rest of p_0 remains unchanged.

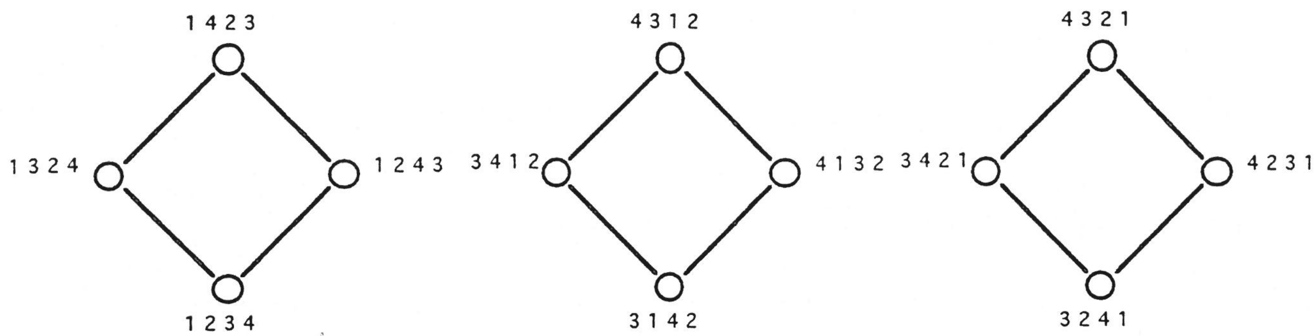
Similarly, to get $p_{\max}$, we consider the subsequence of p_0 of the form $(q_{j_1}, \ldots, q_{j_r}, c_{2n+1})$, where $q_{j_1}, \ldots, q_{j_r}$ are those elements among $\{q_{2n}, \ldots, q_{n+1}\}$ which are contained in the set A, and replace it with the subsequence $(N, q_{j_1}, \ldots, q_{j_r})$, leaving the rest of p_0 unchanged.

If $N = 2n$ we define for $q = (q_1, \ldots, q_{2n-1}) \in \mathfrak{S}_{2n-1}$ the permutation $\widetilde{q} \in \mathfrak{S}_{2n-1}$ by $\widetilde{q} = (q_n, q_{n+1}, q_{n-1}, q_{n+2}, q_{n-2}, \ldots, q_{2n-1}, q_1)$ and denote by A the set of the left-to-right maxima in $\widetilde{q}$. As in the previous case, it can be easily seen that the element $p_0 = (q_{2n-1}, \ldots, q_n, c_{2n}, q_{n-1}, \ldots, q_1)$ is contained in the fiber over q. The permutations $p_{\min}$ and $p_{\max}$ are defined in the same way as in the case of $N = 2n + 1$.

Theorem 3.1. *The fiber of the projection π_N over a permutation $q \in \mathfrak{S}_{N-1}$ is the interval $[p_{\min}, p_{\max}]$ in $\mathfrak{S}_N$ with respect to the Bruhat order. Moreover, these intervals form a Stirling partition of $\mathfrak{S}_N$.*

Here is the partition of $\mathfrak{S}_4$ formed by the fibers of the projection π_4.





4. Laplace operators for classical Lie algebras

Here we review several constructions of Laplace operators for the orthogonal and symplectic Lie algebras which use the properties of the Capelli-type determinant whose definition involves the projection (3.1).

Capelli-type determinant. Consider a nondegenerated symmetric or alternating form on the space $\mathbb{C}^N$ (in the alternating case N has to be even), and let G be its matrix in the canonical basis of $\mathbb{C}^N$. Let $\{E_{ij}\}$ be the standard basis of the general linear algebra $\mathfrak{gl}(N)$ and let $E = (E_{ij})$ be the $N \times N$-matrix with the entries E_{ij}. Introduce the matrix $F = (F_{ij})$ by setting

$$F_{ij} := \begin{cases} (GE)_{ij} - (GE)_{ji} & \text{in the symmetric case,} \\ (GE)_{ij} + (GE)_{ji} & \text{in the alternating case.} \end{cases}$$

Then the orthogonal and symplectic Lie algebras $\mathfrak{o}(N)$ and $\mathfrak{sp}(N)$ can be realized as the Lie subalgebras in $\mathfrak{gl}(N)$ spanned by the elements F_{ij} in the symmetric and alternating case, respectively. Let $n := [N/2]$. We set for $i = 1, \dots, n$:

$$\rho_i = \begin{cases} N/2 - i, & \text{in the case of } \ \mathfrak{o}(N), \\ N/2 - i + 1, & \text{in the case of } \ \mathfrak{sp}(N) \end{cases}$$

and for $i = n+1, \dots, N$:

$$\rho_i = \begin{cases} N/2 - i + 1, & \text{in the case of } \ \mathfrak{o}(N), \\ N/2 - i, & \text{in the case of } \ \mathfrak{sp}(N). \end{cases}$$

The *Capelli-type determinant* is a formal power series in u^{-1} with coefficients from the universal enveloping algebra $\mathrm{U}(\mathfrak{o}(N))$ or $\mathrm{U}(\mathfrak{sp}(N))$, given by the formula:

$$c(u) = \det G^{-1} \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(pp')(G + \frac{F}{u+\rho_1})_{p(1),p'(1)} \cdots (G + \frac{F}{u+\rho_N})_{p(N),p'(N)}, \tag{4.1}$$

where $p \mapsto p'$ is the projection (3.1). Then

$$C(u) := (u^2 - \rho_1^2) \cdots (u^2 - \rho_n^2)\, c(u)$$

is an even monic polynomial in u,

$$C(u) = u^{2n} + u^{2n-2} C_2 + \cdots + C_{2n},$$

and all the coefficients C_{2k} are contained in the center of the universal enveloping algebra. Furthermore, the eigenvalue of C_{2k} in a highest weight representation $L(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the homogeneous elementary symmetric function of degree k in the variables $-l_1^2, \ldots, -l_n^2$, where $l_i = \lambda_i + \rho_i$ (see [M1] for details).

The element $c(u)$ can be regarded as an analogue of the *Capelli determinant* for the Lie algebra $\mathfrak{gl}(N)$ (see [HU], [N]). The invariance of the Capelli determinant and the Capelli-type determinant follows from the invariance of the *quantum determinant* in the Yangian for $\mathfrak{gl}(N)$ and the *Sklyanin determinant* in the twisted Yangian for $\mathfrak{o}(N)$ and $\mathfrak{sp}(N)$ (see [O], [MNO]).

Further we shall only consider the canonical realization ($G = 1$) of the orthogonal Lie algebra $\mathfrak{o}(N)$. For analogues of these results in the symplectic case see [M2], [M3], [MN].

Gelfand invariants. The following formula connects the well-known Gelfand invariants $\operatorname{tr} F^k$ and the element $C(u)$ in the case of $N = 2n$: in the algebra $\mathrm{U}(\mathfrak{o}(N))[[u^{-1}]]$

$$1 - \frac{u - 1/2}{u} \sum_{k=0}^{\infty} \frac{\operatorname{tr} F^k}{(u + \rho_1)^{k+1}} = \frac{C(u-1)}{C(u)}. \tag{4.2}$$

To get the corresponding formula for the case $N = 2n + 1$, one should multiply the right hand side of (4.2) by the factor $(1 - u^{-1})$ and leave the left hand side unchanged. The eigenvalues of the elements $\operatorname{tr} F^k$ in the representation $L(\lambda)$ had been found by Perelomov–Popov [PP]. Relation (4.2) is an immediate consequence of their formulae. On the other hand, this relation, as well as the corresponding relation for the Lie algebra $\mathfrak{gl}(N)$, can be proved independently by using the quantum Liouville formula [MNO], which provides another proof of the Perelomov–Popov formulae (see [M2]).

Quasi-determinants and noncommutative symmetric functions. For $m \leq N$ denote by $F^{(m)}$ the submatrix of F with the entries F_{ij}, where $i,j = 1,\ldots,m$. One has the following decomposition of the polynomial $\widetilde{C}(t) := t^{2n}\, C(t^{-1})$ in the algebra $\mathrm{U}(\mathfrak{o}(N))[[t]]$:

$$\widetilde{C}(t) = \prod_{m=2}^{N} |1 + (F^{(m)} + N/2 - m + 1)t|_{mm}, \tag{4.3}$$

where $|A|_{mm} := ((A^{-1})_{mm})^{-1}$ is the *mm-th quasi-determinant* of a matrix A [GR1], [GR2]. For any m the coefficients of the series $|1 + (F^{(m)} + N/2 - m + 1)t|_{mm}$ commute with each other and the set of these coefficients for all $m = 2,\ldots,N$ generates a commutative subalgebra in $\mathrm{U}(\mathfrak{o}(N))$. Let us introduce the elements $\Phi_k^{(m)}$ by the following formula:

$$\sum_{k=1}^{\infty} \Phi_k^{(m)} t^{k-1} = -\frac{d}{dt} \log |1 - (F^{(m)} + N/2 - m + 1)t|_{mm}.$$

They can be interpreted graphically in the following way. Let $\mathcal{F}^{(m)}$ denote the complete oriented graph with the vertices $\{1,\ldots,m\}$, the arrow from i to j is labelled by the ij-th matrix element of the matrix $F^{(m)} + N/2 - m + 1$. Every path in this graph defines a monomial in the matrix elements in a natural way. Then $\Phi_k^{(m)}$ is the sum of all monomials labelling paths in $\mathcal{F}^{(m)}$ of length k going from m to m, the coefficient of each monomial being the length of the first return to m; and also $\Phi_k^{(m)}$ is the sum of those monomials with the coefficients equal to the ratio of k to the number of returns to m.

The invariance of the coefficients of $\widetilde{C}(t)$ implies that the elements Φ_k defined by the formula

$$\sum_{k=1}^{\infty} \Phi_k t^{k-1} = -\frac{d}{dt} \log \widetilde{C}(t) \tag{4.4}$$

belong to the center of $\mathrm{U}(\mathfrak{o}(N))$. On the other hand, due to decomposition (4.3), they can be calculated by the formula

$$\Phi_k = \Phi_k^{(2)} + \cdots + \Phi_k^{(N)}.$$

It follows from (4.4) that $\Phi_{2k-1} = 0$ while the eigenvalue of $\Phi_{2k}/2$ in $L(\lambda)$ is $l_1^{2k} + \cdots + l_n^{2k}$.

Note that an analogous graphical interpretation can be obtained for the coefficients C_{2k} of the polynomial $C(u)$, as well as for the central elements whose eigenvalues in $L(\lambda)$ are the complete symmetric functions in $l_1^2,\ldots,l_n^2$ (see [M3]).

The elements $\Phi_k^{(m)}$ are a special case of the *noncommutative symmetric functions associated with a matrix.* A general theory of noncommutative symmetric functions has been developed in the paper [GKLLRT], where, in particular, the corresponding results for the Lie algebra $\mathfrak{gl}(N)$ are contained (see also [KL]). The arguments presented here have followed those of this paper.

Pfaffian-type elements. For a subset $I = \{i_1, \ldots, i_{2k}\}$ in $\{1, \ldots, N\}$, $(i_a < i_{a+1})$ denote by F^I the submatrix of F whose rows and columns enumerated by elements of the set I. Let $\mathrm{Pf}(F^I)$ denote the "Pfaffian" of the matrix F^I:

$$2^k\, k!\, \mathrm{Pf}(F^I) = \sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sgn}(\sigma)\, F_{i_{\sigma(1)}, i_{\sigma(2)}} \cdots F_{i_{\sigma(2k-1)}, i_{\sigma(2k)}}.$$

Then the elements

$$c_k := \sum_{I,\, |I| = 2k} (\mathrm{Pf}(F^I))^2$$

belong to the center of $\mathrm{U}(\mathfrak{o}(N))$ and one has the following decomposition of the Capelli-type determinant:

$$c(u) = 1 + \sum_{k=1}^{n} \frac{c_k}{(u^2 - \rho_1^2) \cdots (u^2 - \rho_k^2)}.$$

This implies that the eigenvalue of c_k in $L(\lambda)$ is given by the formula

$$(-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} (l_{i_1}^2 - \rho_{i_1+k-1}^2)(l_{i_2}^2 - \rho_{i_2+k-2}^2) \cdots (l_{i_k}^2 - \rho_{i_k}^2).$$

Connections of the element $c(u)$ with the Capelli identities will be discussed in [MN].

Remark. It was proved in [HU] that all the coefficients of the polynomial

$$\sum_{p \in \mathfrak{S}_N} \mathrm{sgn}(p)(u + F)_{p(1),1} \cdots (u - N + 1 + F)_{p(N),N}$$

belong to the center of $\mathrm{U}(\mathfrak{o}(N))$. However, the author does not know what the connection is between these coefficients and any of the elements discussed above.

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