POLYNOMIAL SOLUTIONS TO THE REDUCED BKP HIERARCHIES

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ABSTRACT. An expression in terms of Schur's Q-functions of all weight vectors is given for the polynomial realization of the basic $A_{2l}^{(2)}$-module. Consequently it is shown that certain Q-functions solve the reduced BKP hierarchies.

RÉSUMÉ. Nous donnons une expression en termes de fonctions Q de Schur de tous les vecteurs de poids de la représentation basique de l'algèbre de Lie $A_{2l}^{(2)}$ dans sa réalisation polynomiale. Nous en déduisons que certaines fonctions Q de Schur résolvent les hiérarchies BKP réduites.

In 1981 Date et al. introduced a KP like hierarchy of nonlinear differential equations, which has a symmetry of the infinite orthogonal Lie algebra and is now called the KP hierarchy of B type or BKP hierarchy for short. In this presentation we will give an explicit expression of the weighted homogeneous polynomial solutions to the "reduced" BKP hierarchies.

Let $V = \mathbb{C}[t_1, t_3, t_5, \dots]$ be a polynomial ring of infinitely many variables $t = (t_1, t_3, t_5, \dots)$. The BKP hierarchy is described by the following generating function of Hirota bilinear equations [DJKM]:

$$\sum_{k=1}^{\infty} q_k(2z) q_k(-2\tilde{D}) \exp(z_1 D_1 + z_3 D_3 + \cdots)(\tau(t) \cdot \tau(t)) = 0.$$

Here $z = (z_1, z_3, \dots)$ are indeterminates and the polynomial $q_k(z)$ is defined by

$$\sum_{k=0}^{\infty} q_k(z) \lambda^k = \exp(z_1 \lambda + z_3 \lambda^3 + \cdots).$$

The symbol D_j stands for the Hirota derivative with respect to the variable t_j and we denote $\tilde{D} = (D_1, D_3/3, D_5/5, \dots)$.

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Fix a natural number l and put $r = 2l + 1$. The r-reduced BKP hierarchy means the Hirota equations where the variables t_{kr} $(k \geq 1)$ are thought to be 0. Hence the story of r-reduced BKP hierarchy is developed on the space

$$V_r = \mathbb{C}[\,t_j; j \geq 1 \text{ is odd and not a positive multiple of r}\,].$$

For a polynomial $P(t) \in V$, we denote by $P(t')$ the polynomial in V_r obtained by putting $t_{kr} = 0$ for $k \geq 1$. It is known that there is an acton of the affine Lie algebra of type $A_{2l}^{(2)}$ on the space V_r. This action is a highest weight irreducible representation and is called the basic representation of this affine Lie algebra [K]. The space of formal solutions to the r-reduced BKP hierarchy is obtained as the corresponding "group" orbit in a certain completion $\overline{V}_r$ through the highest weight vector $1 \in V_r$ under this representation [DJKM].

Next we give the weight vectors of the basic representation on V_r. To this end we recall Schur's Q-functions [M]. Let $X_1, \dots, X_n$ be indeterminates. The Hall-Littlewood symmetric function indexed by the Young diagram $Y = (m_1, \dots, m_n)$ $(m_1 \geq \cdots \geq m_n \geq 0)$ is defined by

$$Q_Y(X_1, \dots, X_n; q) = \prod_{j=1}^{\infty} (q;q)_{\mu_j} \sum_{w \in \mathfrak{S}_n/\mathfrak{S}_n^Y} w \left(X_1^{m_1} \cdots X_n^{m_n} \prod_{m_i > m_j} \frac{(X_i - qX_j)}{(X_i - X_j)} \right),$$

where q is a parameter, $\mu_j = \#\{i; m_i = j\}$ is the multiplicity of j in Y, $(q;q) = \prod_{i=1}^{k}(1-q^i)$ and $\mathfrak{S}_n^Y$ is the subgroup of the symmetric group $\mathfrak{S}_n$ consisting of permutations w such that $m_{w(i)} = m_i$ for all i. As a specialization $q = -1$ we get Schur's Q-function $Q_Y(X_1, \dots, X_n)$. It is obvious that $Q(X_1, \dots, X_n) = 0$ unless Y is a strict Young diagram, i.e., $\mu_j \leq 1$ for $j \geq 1$. If we put $t_j = 2(X_1^j + \ldots + X_n^j)/j$ for positive odd integers j, the Q-functions turn out to be weighted homogeneous polynomials of $t = (t_1, t_3, \dots)$:

$$Q_Y(t) = \sum_{Y'} X_{Y'}^{Y}(-1) \frac{t_1^{\mu_1} t_3^{\mu_3} \cdots}{\mu_1! \mu_3! \cdots} \in V.$$

Here the summation runs over all Young diagrams $Y' = (m'_1, \dots, m'_n)$ consisting of odd numbers m'_j, μ'_j denotes the multiplicity of j in Y', and $X_{Y'}^{Y}(-1)$ is an integer which, although we do not make explicit, can be computed directly from the character table of projective representations of the symmetric group. Our first claim is the following.

Proposition 1. *The basic representation of* $A_{2l}^{(2)}$ *on* V_r $(r = 2l+1)$ *has a base consisting of the weight vectors* $Q_(t')$, *where* $Y = (m_1, \dots, m_{2d})$ $(m_1 > \cdots > m_{2d} \geq 0)$ *is a strict Young diagram such that every* m_j *is not a positive multiple of* r.

Every affine Lie algebra has the so-called fundamental imaginary root δ. Let $\alpha_0, \dots \alpha_l$ be the simple roots of $A_{2l}^{(2)}$ in the standard notation. Then we have $\delta = 2(\alpha_0 + \cdots +$

$\alpha_{l-1}) + \alpha_l$ [K]. Next we describe a recipe to choose weight vectors of weight $\delta \pm \lambda$ for a given weight vector of weight λ. Suppose we are given a weight vector $Q_Y(t')$ of weight λ, where $Y = (m_1, \dots, m_{2d})$ is a Young diagram as in Proposition 1. Then one of the following possible "moves" $Y \longmapsto Y'$ is permitted to have weight $\lambda - \delta$:

(1) Replace m_j by $m_j + r$ for some j and arrange the sequence into decreasing order to have a new strict Young diagram Y'.

(2) Add 2 rows $(k, r-k)$ $(1 \leq k \leq l)$ and arrange to have a new strict Young diagram Y'.

One can say that it is nothing but to remove an "r-bar" to obtain strict Young diagram of weight $\lambda + \delta$ [O].

The weight λ is said to be maximal if $\lambda + \delta$ is not a weight. It is known that the totality of maximal weights of the basic representation of $A_{2l}^{(2)}$ is given by the Weyl group orbit through the highest weight, and hence each maximal weight is of multiplicity 1. Since the Weyl group can be seen as a subgroup of the group corresponding to the affine Lie algebra, the maximal weight vectors are polynomial solutions to the r-reduced BKP hierarchy. In view of the above rule, we have the following description of maximal weight vectors.

Theorem 2. *Schur's Q-function $Q_Y(t')$ is a maximal weight vector of the basic representation of $A_{2l}^{(2)}$ if and only if the strict Young diagram Y has no r-bars; equivalently, the shift symmetric Young diagram $\tilde{Y}$ of Y has no r-hooks.*

Taking the limit $l \to \infty$ $(r \to \infty)$ we see that the Q-function $Q_Y(t)$ of any strict Young diagram Y solves the non-reduced BKP hierarchy [Y].

A similar argument is available for "$2l$-reduction" of the BKP hierarchy, in which case the basic representation of the affine Lie algebra $D_{l+1}^{(2)}$ plays a role of symmetry [NY]. And also one can choose a suitable weight base of the basic representation of $A_1^{(1)}$ realized on the space V, which consists of reduced Schur functions. This representation corresponds to the KdV hierarchy and in this case the Littlewood-Richardson rule appears in the linear relations among weight vectors [ANY].

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