Young's Orthogonal Form for Brauer's Centralizer Algebras

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SUMMARY

We consider the semi-simple algebra which arises as the centralizer of a tensor power of the fundamental representation of the orthogonal group. There is a canonical basis in every irreducible representation of this algebra; it is an analogue of the Young basis in an irreducible representation of the symmetric group. We evaluate the action of the generators of this algebra in the canonical basis. Then we introduce an analogue of the degenerate affine Hecke algebra for this centralizer algebra.

RÉSUMÉ

Nous considérons l'algèbre semi-simple qui apparaît comme commutant d'une puissance tensorielle de la répresentation fondamentale du groupe ortogonal. Il existe une base canonique dans toute représentation irréductible de cette algèbre; c'est un analogue de la base de Young d'une représentation irréductible du groupe symétrique. Nous calculons l'action des générateurs de cette algèbre sur la base canonique. Alors nous définissons un analogue de l'algèbre de Hecke affine dégénérée pour cette algèbre semi-simple.

Introduction

Let G be one of the classical groups $GL(N,\mathbb{C})$, $O(N,\mathbb{C})$, $Sp(N,\mathbb{C})$ acting on the vector space $U = \mathbb{C}^N$. The question how the n-th tensor power of the representation U decomposes into irreducible summands leads to studying the centralizer $C(n,N)$ in $\mathrm{End}(U)^{\otimes n}$ of the image of the group G. By the definition of the algebra $C(n,N)$ we have the ascending chain of subalgebras

$$C(1,N) \subset C(2,N) \subset \ldots \subset C(n,N).$$

Moreover, for the classical group G any irreducible representation of $C(n,N)$ appears at most once in the restriction of an irreducible representation of $C(n+1,N)$. Therefore a canonical basis exists in any irreducible representation V of $C(n,N)$. Its vectors are the eigenvectors for the subalgebra $X(n,N)$ in $C(n,N)$ generated by all the central elements in the members of the above chain.

For the group $G = GL(N,\mathbb{C})$ the centralizer $C(n,N)$ is generated by the permutational action of the symmetric group $S(n)$ in $U^{\otimes n}$. The action of $S(n)$ on the vectors of the canonical basis in V was described for the first time by A. Young [Y]. G. Murphy [Mp] rederived the formulas from [Y] by using the properties of the subalgebra $X(n,N)$.

Let us now suppose that G is the orthogonal group $O(N,\mathbb{C})$. To describe the corresponding centralizer algebra $C(n,N)$ explicitly, R. Brauer [Br] introduced certain complex associative algebra $B(n,N)$ along with a homomorphism onto $C(n,N)$. This homomorphism is injective if and only if $N \geqslant n$. There is also a chain of subalgebras

$$B(1,N) \subset B(2,N) \subset \ldots \subset B(n,N).$$

The group algebra $\mathbb{C}[S(n)]$ is contained in $B(n,N)$ as a subalgebra. The structure of the algebra $B(n,N)$ was investigated by P. Hanlon and D. Wales; see [HW] and references therein. In the present note we will also work with $B(n,N)$ and regard V as a representation of the latter algebra.

For $N \geqslant n$ an explicit description of the action of the algebra $B(n,N)$ on the vectors of the canonical basis in V was given by J. Murakami in [Mk]. His description was based on the results of [JMO]. In this note for any N we give a new description of this action based entirely on the properties of the subalgebra $X(n,N)$ in $C(n,N)$. We present our method as a sequence of propositions and theorems but omit their proofs. All the proofs shall be given in a more detailed publication. The case $G = Sp(N,\mathbb{C})$ is quite similar and shall be also considered elsewhere.

In Section 2 we introduce a remarkable family of pairwise commuting elements $x_1,\ldots,x_n$ of the algebra $B(n,N)$. For every n the element x_{n+1} belongs to the centralizer of the subalgebra $B(n,N)$ in $B(n+1,N)$. The elements $x_1,\ldots,x_n$ are the analogues of the pairwise commuting elements of $\mathbb{C}[S(n)]$ which were used in [Ju,Mu]. Their images in $C(n,N)$ belong to the subalgebra $X(n,N)$. The vectors of the canonical basis in V are eigenvectors of the elements $x_1,\ldots,x_n$ and we evaluate the respective eigenvalues; see Theorem 2.6.

There is a natural projection map $B(n+1,N) \to B(n,N)$ commuting with both left and right multiplication by the elements from $B(n,N)$; this map has been already used by H. Wenzl in [W]. The images of powers of the element x_{n+1} with respect to this map are certain central elements of the algebra $B(n,N)$. We evaluate the eigenvalues of these central elements in every irreducible representation V; see Theorem 3.8.

The algebra $B(n,N)$ comes with a family of generators $s_1,\dots,s_{n-1};\ \bar{s}_1,\dots,\bar{s}_{n-1}$. The elements $s_1,\dots,s_{n-1}$ are the standard generators of the symmetric group $S(n)$. Moreover, the quotient of the algebra $B(n,N)$ with respect to the ideal generated by $\bar{s}_1,\dots,\bar{s}_{n-1}$ is isomorphic to $\mathbb{C}[S(n)]$. We point out certain relations between the elements $x_1,\dots,x_n$ and the generators of $B(n,N)$; see Proposition 2.3. By using Proposition 2.3 and Theorems 2.6, 3.8 we describe the action of these generators on the vectors of the canonical basis in every representation V. For the representations which factorize through $\mathbb{C}[S(n)]$ our formulas coincide with those from [Y].

In Section 4 we use the results of Sections 2 and 3 as a motivation to introduce a new algebra. This algebra is an analogue of the degenerate affine Hecke algebra $He(n)$ from [C1,C2] and [D]. We will denote the new algebra by $We(n,N)$ and call it the affine degenerate Wenzl algebra. The algebra $He(n)$ ia a quotient of $We(n,N)$; see Corollary 4.9. For each $m=0,1,2,\dots$ the algebra $We(n,N)$ admits a homomorphism to the centralizer of the subalgebra $B(m,N)$ in $B(m+n,N)$. The kernels of all these homomorphisms have the zero intersection; see Theorem 4.7. We use these homomorphisms to construct a linear basis in the algebra $We(n,N)$; see Theorem 4.6. The irreducible finite-dimensional representations of the algebra $We(n,N)$ will be considered elsewhere.

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1. Brauer Centralizer Algebra

Let n be a positive integer and N be an arbitrary complex parameter. Denote by $\mathcal{G}(n)$ be the set of all graphs with $2n$ vertices and n edges such that each vertex is incident with an edge. We will enumerate the vertices by $1,\dots,n,\bar{1},\dots,\bar{n}$. In other words, $\mathcal{G}(n)$ consists of all partitions of the set $\{1,\dots,n,\bar{1},\dots,\bar{n}\}$ into pairs. We will define the *Brauer algebra* $B(n,N)$ as an associative algebra over $\mathbb{C}$ with the basic elements $b(\gamma)$, $\gamma\in\mathcal{G}(n)$.

To describe the product $b(\gamma)\,b(\gamma')$ in $B(n,N)$ consider the graph obtained by identifying the vertices $\bar{1},\dots,\bar{n}$ of γ with the vertices $1,\dots,n$ of γ' respectively. Let q be the quantity of loops in this graph. Remove all the loops and replace the remaining connected components by single edges, retaining the numbers of the terminal vertices. Denote by $\gamma\circ\gamma'$ the resulting graph, then by definition

$$b(\gamma)\,b(\gamma')=N^{q}\cdot b(\gamma\circ\gamma'). \tag{1.1}$$

Evidently, the dimension of $B(n,N)$ is equal to $1\cdot3\cdot5\cdot\ldots\cdot(2n-1)$. The algebra $B(n,N)$ contains the group algebra of the symmetric group $S(n)$; one can identify an element s of $S(n)$ with $b(\gamma)$ where the edges of γ are $\{s(1),\bar{1}\},\dots,\{s(n),\bar{n}\}$.

An edge of the form $\{k,\bar{k}\}$ will be called *vertical.* We will regard $B(n-1,N)$ as a subalgebra of $B(n,N)$ with the basic elements $b(\gamma)$ where γ contains the vertical edge $\{n,\bar{n}\}$. Along with a transposition (k,l) in $S(n)$ we will consider the element $\overline{(k,l)}=b(\gamma)$ of $B(n,N)$ where the only non-vertical edges of γ are $\{k,l\}$ and $\{\bar{k},\bar{l}\}$.

We will sometimes write s_k and $\bar{s}_k$ instead of $(k,k+1)$ and $\overline{(k,k+1)}$ respectively. The elements $s_1,\ldots,s_{n-1};\bar{s}_1,\ldots,\bar{s}_{n-1}$ generate the algebra $B(n,N)$. One can directly verify the following relations for these elements:

$$s_k^2=1;\quad \bar{s}_k^2=N\,\bar{s}_k;\quad s_k\,\bar{s}_k=\bar{s}_k\,s_k=\bar{s}_k; \tag{1.2}$$

$$s_k\,s_{k+1}\,s_k=s_{k+1}\,s_k\,s_{k+1};\quad \bar{s}_k\,\bar{s}_{k+1}\,\bar{s}_k=\bar{s}_k;\quad \bar{s}_{k+1}\,\bar{s}_k\,\bar{s}_{k+1}=\bar{s}_{k+1}; \tag{1.3}$$

$$s_k\,\bar{s}_{k+1}\,\bar{s}_k=s_{k+1}\,\bar{s}_k;\quad \bar{s}_{k+1}\,\bar{s}_k\,s_{k+1}=\bar{s}_{k+1}\,s_k; \tag{1.4}$$

$$s_k\,s_l=s_l\,s_k,\quad \bar{s}_k\,s_l=s_l\,\bar{s}_k,\quad \bar{s}_k\,\bar{s}_l=\bar{s}_l\,\bar{s}_k,\qquad |k-l|>1. \tag{1.5}$$

Proposition 1.1. *The relations (1.2) to (1.5) are defining relations for $B(n,N)$.*

For the proof of this proposition see [BW, Section 5]. Now suppose that N is a positive integer. Consider the n-th tensor power of the representation $U=\mathbb{C}^N$ of the orthogonal group $G=O(N,\mathbb{C})$. Let $u(1),\ldots,u(N)$ be the standard orthogonal basis in U; denote by $u(i_1\ldots i_n)$ the vector $u(i_1)\otimes\ldots\otimes u(i_n)$ in $U^{\otimes n}$. Consider the centralizer algebra $C(n,N)=\operatorname{End}_G(U^{\otimes n})$.

Proposition 1.2. *a) There is a homomorphism $B(n,N)\to C(n,N)$ where the action of (k,l) and $\overline{(k,l)}$ in $U^{\otimes n}$ for $k<l$ is defined by*

$$\begin{aligned}(k,l)\cdot u(i_1\ldots i_k\ldots i_l\ldots i_n)&=u(i_1\ldots i_l\ldots i_k\ldots i_n),\\ \overline{(k,l)}\cdot u(i_1\ldots i_k\ldots i_l\ldots i_n)&=\delta(i_k\,i_l)\cdot\sum_{i=1}^{N}u(i_1\ldots i\ldots i\ldots i_n).\end{aligned} \tag{1.6}$$

b) This homomorphism is surjective for any positive integer N.
c) This homomorphism is injective if and only if $N\geqslant n$.

The algebra $C(n,N)$ is semisimple by its definition; the irreducible representations of $C(n,N)$ are parametrized [Wy, Theorem 5.7.F] by the Young diagrams with at most N boxes in the first two columns and with $n-2r$ boxes altogether where $r=0,1,\ldots,[n/2]$. Denote the set of all such diagrams by $\mathcal{O}(n,N)$. Let $V(\lambda,n)$ be the representation of $C(n,N)$ corresponding to a diagram $\lambda\in\mathcal{O}(n,N)$. The next proposition is contained in [L, Theorem I]; see also [Ki, Section 3].

Proposition 1.3. *The restriction of $V(\lambda,n)$ onto $C(n-1,N)$ decomposes into the direct sum $\bigoplus_{\mu} V(\mu,n-1)$ where μ ranges over all the diagrams $\mu\in\mathcal{O}(n-1,N)$ obtained from λ by removing or adding a box.*

Corollary 1.4. *Each irreducible representation of $C(n-1,N)$ appears at most once in the restriction onto $C(n-1,N)$ of an irreducible representation of $C(n,N)$.*

2. Jucys-Murphy Elements for $B(n,N)$

By definition for any complex parameter N we have the chain of subalgebras

$$B(1,N)\subset B(2,N)\subset\ldots\subset B(n,N). \tag{2.1}$$

In this section we will introduce a remarkable family of pairwise commuting elements in $B(n,N)$ corresponding to this chain; cf. [Ju, Mu]. For every $k=1,\dots,n$ consider the element of $B(k,N)$

$$x_k = \frac{N-1}{2} + \sum_{l=1}^{k-1} (k,l) - \overline{(k,l)}. \tag{2.2}$$

Lemma 2.1. *The element x_k commutes with all the elements of $B(k-1,N)$.*

Corollary 2.2. *The elements $x_1,\dots,x_n$ of $B(n,N)$ pairwise commute.*

Proposition 2.3. *The following relations hold in the algebra $B(n,N)$:*

$$s_k\,x_l = x_l\,s_k,\quad \bar{s}_k\,x_l = x_l\,\bar{s}_k;\qquad l\neq k,k+1; \tag{2.3}$$

$$s_k\,x_k - x_{k+1}\,s_k = \bar{s}_k - 1,\quad s_k\,x_{k+1} - x_k\,s_k = 1-\bar{s}_k; \tag{2.4}$$

$$\bar{s}_k\,(x_k + x_{k+1}) = 0,\quad (x_k+x_{k+1})\,\bar{s}_k = 0. \tag{2.5}$$

Corollary 2.4. *The elements $x_1^i+\dots+x_n^i$ with $i=1,3,\dots$ are central in $B(n,N)$.*

It follows from the definition (1.1) that for any $b\in B(k,N)$ there is a unique element $b'\in B(k-1,N)$ such that

$$\bar{s}_k\,b\,\bar{s}_k = b'\,\bar{s}_k\,; \tag{2.6}$$

cf. [W, Proposition 2.2]. Moreover, the map $b\mapsto b'$ evidently commutes with the left and right multiplication by elements from the subalgebra $B(k-1,N)\subset B(k,N)$. In particular, due to Lemma 2.1 we have

$$\bar{s}_k\,x_k^i\,\bar{s}_k = z_k^{(i)}\,\bar{s}_k\,;\qquad i=0,1,2,\dots \tag{2.7}$$

where $z_k^{(0)}=N$ and $z_k^{(1)},z_k^{(2)},\dots$ are central elements of the algebra $B(k-1,N)$. In Section 4 we will provide explicit formulas for these elements; see Corollary 4.3 and the subsequent remark. Here we will point out only some relations that the definition (2.7) implies.

Lemma 2.5. *We have the relations*

$$-2\,z_k^{(i)} = z_k^{(i-1)} + \sum_{j=1}^{i} (-1)^j\,z_k^{(i-j)}\,z_k^{(j-1)};\qquad i=1,3,\dots. \tag{2.8}$$

Consider the generating series

$$Z_k(u) = \sum_{i\geqslant 0} z_k^{(i)}\,u^{-i} \in B(n,N)[[u^{-1}]].$$

From the relations (2.3) to (2.7) we obtain that

$$Z_k(u) = Z_k(-u) + Z_k(u)\,Z_k(-u)/u - \big(Z_k(u)+Z_k(-u)\big)/2u\,. \tag{2.9}$$

Therefore for the series $Q_k(u)$ determined by the equality

$$Q_k(u)\cdot(u+1/2)=Z_k(u)+u-1/2 \tag{2.10}$$

we obtain the relation $Q_k(u)\,Q_k(-u)=1$.

From now on until the end of Section 3 we will assume that the parameter N is a positive integer. We will then have the chain of semisimple algebras

$$C(1,N)\subset C(2,N)\subset\ldots\subset C(n,N). \tag{2.11}$$

Consider the subalgebra $X(n,N)$ in $C(n,N)$ generated by all the central elements of $C(1,N),C(2,N),\ldots,C(n,N)$. It follows from Corollary 1.4 that the subalgebra $X(n,N)$ is maximal commutative.

There is a canonical basis in every representation space $V(\lambda,n)$ of $C(n,N)$ corresponding to the chain (2.11); it consists of the eigenvectors of the subalgebra $X(n,N)$. The basic vectors are parametrized by the sequences

$$\Lambda=\bigl(\Lambda(1),\ldots,\Lambda(n)\bigr)\in\mathcal{O}(1,N)\times\ldots\times\mathcal{O}(n,N)$$

where $\Lambda(n)=\lambda$ and each two neighbouring terms of the sequence differ by exactly one box. Denote by $\mathcal{L}(\lambda,n)$ the set of all such sequences. Let $v(\Lambda)$ be the basic vector in $V(\lambda,n)$ corresponding to a sequence $\Lambda\in\mathcal{L}(\lambda,n)$. Up to a scalar multiplier, it is uniquely determined by the following condition: $v(\Lambda)\in V\bigl(\Lambda(k),k\bigr)$ in the restriction of $V(\lambda,n)$ onto $C(k,N)$ for any $k=1,\ldots,n-1$.

We will regard $V(\lambda,n)$ as a representation of the algebra $B(n,N)$ also. In the next section we will use the elements $x_1,\ldots,x_n\in B(n,N)$ to describe the action of the generators $s_1,\ldots,s_{n-1};\bar{s}_1,\ldots,\bar{s}_{n-1}$ of $B(n,N)$ on the vector $v(\Lambda)\in V(\lambda,n)$. It follows from Corollary 1.4 and Lemma 2.1 that the images in $C(n,N)$ of the elements $x_1,\ldots,x_n$ belong to the subalgebra $X(n,N)$. Denote by $x_k(\Lambda)$ the eigenvalue of x_k corresponding to the vector $v(\Lambda)$. For any $\Lambda\in\mathcal{L}(\lambda,n)$ we will define $\Lambda(0)$ as the empty partition. If a box of the diagram λ occurs in the row i and the column j then the difference $j-i$ is called the *content* of this box.

Theorem 2.6. *Suppose that the diagrams $\Lambda(k-1)$ and $\Lambda(k)$ differ by the box occuring in the row i and the column j. Then*

$$x_k(\Lambda)=\pm\left(\frac{N-1}{2}+j-i\right) \tag{2.12}$$

where the upper sign in $\pm$ corresponds to the case $\Lambda(k)\supset\Lambda(k-1)$ while the lower sign corresponds to $\Lambda(k)\subset\Lambda(k-1)$.

Corollary 2.7. *Suppose that N is odd or $N\geqslant 2n-1$. Then:*

a) *the images in $C(n,N)$ of the elements $x_1,\ldots,x_n$ generate the algebra $X(n,N)$;*

b) *the images in $C(n,N)$ of the elements $x_1^i+\ldots+x_n^i$ with $i=1,3,\ldots$ generate the centre of the algebra $C(n,N)$.*

For $N=2,4,\ldots,2n-2$ the statements a) and b) of Corollary 2.7 are no longer valid. However, the elements $x_1,\ldots,x_n$ will suffice to describe the action in $V(\lambda,n)$ of the generators $s_1,\ldots,s_{n-1};\bar{s}_1,\ldots,\bar{s}_{n-1}$ of $B(n,N)$ for any positive integer N.

3. Young Ortogonal Form for $C(n,N)$

It this section we will make explicit the matrix elements $s_k(\Lambda,\Lambda'), \bar{s}_k(\Lambda,\Lambda')$ of the generators $s_k, \bar{s}_k \in B(n,N)$ in the canonical basis of the representation $V(\lambda,n)$:

$$s_k \cdot v(\Lambda) = \sum_{\Lambda' \in \mathcal{L}(\lambda,n)} s_k(\Lambda,\Lambda')\, v(\Lambda'), \qquad \bar{s}_k \cdot v(\Lambda) = \sum_{\Lambda' \in \mathcal{L}(\lambda,n)} \bar{s}_k(\Lambda,\Lambda')\, v(\Lambda').$$

Note that each of the vectors $v(\Lambda) \in V(\lambda,n)$ here is defined up to a scalar multiplier. Before specifying these multipliers we will determine the diagonal matrix elements $s_k(\Lambda,\Lambda)$, $\bar{s}_k(\Lambda,\Lambda)$ along with all the products $s_k(\Lambda,\Lambda')\, s_k(\Lambda',\Lambda)$, $\bar{s}_k(\Lambda,\Lambda')\, \bar{s}_k(\Lambda',\Lambda)$.

Let an index $k \in \{1,\ldots,n-1\}$ and a sequence $\Lambda \in \mathcal{L}(\lambda,n)$ be fixed. Denote by $V(\Lambda,k)$ the subspace in $V(\lambda,n)$ spanned by the vectors $v(\Lambda')$ such that $\Lambda'(l) = \Lambda(l)$ for any $l \neq k$. The action of s_k and $\bar{s}_k$ in $V(\lambda,n)$ preserves this subspace.

Proposition 3.1. *Suppose that* $\Lambda(k-1) \neq \Lambda(k+1)$. *Then* $\bar{s}_k \cdot v(\Lambda) = 0$.

Proposition 3.2. *Suppose that* $\Lambda(k-1) \neq \Lambda(k+1)$. *Then* $x_k(\Lambda) \neq x_{k+1}(\Lambda)$ *and* $s_k(\Lambda,\Lambda) = \big(x_{k+1}(\Lambda) - x_k(\Lambda)\big)^{-1}$.

Observe that if $\Lambda(k-1) \neq \Lambda(k+1)$ then the space $V(\Lambda,k)$ has the dimension at most two. Therefore due to the relation $s_k^2 = 1$ we get

Corollary 3.3. *Suppose that* $\Lambda(k-1) \neq \Lambda(k+1)$ *and* $v(\Lambda') \in V(\Lambda,k)$ *with* $\Lambda \neq \Lambda'$. *Then* $s_k(\Lambda,\Lambda')\, s_k(\Lambda',\Lambda) = 1 - \big(x_{k+1}(\Lambda) - x_k(\Lambda)\big)^{-2}$.

Two Young diagrams are *associated* if the sum of the lengths of their first columns equals N while the lengths of their other columns respecively coinside. In paticular, for an even N a diagram is *self-associated* if its first column consists of $N/2$ boxes.

Lemma 3.4. *For any* $v(\Lambda') \in V(\Lambda,k)$ *we have* $x_k(\Lambda) + x_k(\Lambda') \neq 0$ *unless* N *is odd and* $\Lambda' = \Lambda$ *where the diagrams* $\Lambda(k-1), \Lambda(k)$ *are associated.*

Let us now consider the case $\Lambda(k-1) = \Lambda(k+1)$. Due to Theorem 2.6 we then have $x_k(\Lambda') + x_{k+1}(\Lambda') = 0$ for any $v(\Lambda') \in V(\Lambda,k)$. The next two lemmas are contained in [RW, Theorem 2.4(b)].

Lemma 3.5. *Suppose that* $\Lambda(k-1) = \Lambda(k+1)$. *Then*

$$\bar{s}_k(\Lambda,\Lambda) = \frac{\dim U\big(\Lambda(k),N\big)}{\dim U\big(\Lambda(k+1),N\big)}\,.$$

Lemma 3.6. *Suppose that* $\Lambda(k-1) = \Lambda(k+1)$. *Then the image of the action of* $\bar{s}_k$ *in the subspace* $V(\Lambda,k)$ *is one-dimensional.*

Corollary 3.7. *Suppose that* $\Lambda(k-1) = \Lambda(k+1)$ *and* $v(\Lambda') \in V(\Lambda,k)$. *Then* $\bar{s}_k(\Lambda,\Lambda')\, \bar{s}_k(\Lambda',\Lambda) = \bar{s}_k(\Lambda,\Lambda)\, \bar{s}_k(\Lambda',\Lambda')$.

There are well known explicit formulas for the dimension of the irreducible representation $U(\lambda,N)$ of the orthogonal group G; see for instance [EK, Section 3]. Due to Lemma 3.5 these formulas already provide certain expressions for the matrix element $\bar{s}_k(\Lambda, \Lambda)$. In this section we will employ the relations (2.4) and (2.7) to determine $\bar{s}_k(\Lambda, \Lambda)$ independently of any explicit formulas for $\dim U(\lambda, N)$.

Suppose that $\Lambda(k-1) = \Lambda(k+1) = \mu$. Let l be the quantity of pairwise distinct rows (or columns) in the diagram μ. Then one can obtain $l+1$ diagrams by adding a box to μ and l diagrams by removing a box from μ. Let $c_1, \ldots, c_{l+1}$ and $d_1, \ldots, d_l$ be the contents of these boxes respectively. Denote by $b_1, \ldots, b_{2l+1}$ the numbers

$$(N-1)/2 + c_1\,, \ldots, (N-1)/2 + c_{l+1}\,, \; -(N-1)/2 - d_1\,, \ldots, -(N-1)/2 - d_l$$

taken in an arbitrary order; then

$$b_1 + \ldots + b_{2l+1} = (N-1)/2 + c_1 + \ldots + c_{l+1} - d_1 - \ldots - d_l = (N-1)/2\,. \tag{3.1}$$

Denote by $z_k^{(i)}(\mu)$ the eigenvalue of the central element $z_k^{(i)} \in B(k-1, N)$ defined by (2.7) in the representation $V(\mu, k-1)$. Consider the formal power series in u^{-1}

$$Q(\mu, u) = \sum_{i \geqslant 0} q_i(\mu)\, u^{-i} = \prod_{j=1}^{2l+1} \frac{u + b_j}{u - b_j}\,;$$

the coefficients $q_1(\mu), q_2(\mu), \ldots$ are the symmetric *Schur q-functions* in $b_1, \ldots, b_{2l+1}$.

Theorem 3.8. *For every $i = 1, 2, \ldots$ we have the equality*

$$z_k^{(i)}(\mu) = q_{i+1}(\mu) + q_i(\mu)/2\,.$$

Corollary 3.9. *Suppose that $\Lambda(k-1) = \Lambda(k+1) = \mu$ and let $x_k(\Lambda) = b$. Then*

$$\bar{s}_k(\Lambda, \Lambda) = \begin{cases} (\,2b+1\,) \displaystyle\prod_{b_j \neq b} \frac{b + b_j}{b - b_j} & \textit{if } b \neq -1/2\,; \\ - \displaystyle\prod_{b_j \neq b} \frac{b + b_j}{b - b_j} & \textit{if } b = -1/2\,. \end{cases}$$

Proposition 3.10. *Suppose that $\Lambda(k-1) = \Lambda(k+1)$ and $v(\Lambda') \in V(\Lambda, k)$. Then*

$$s_k(\Lambda, \Lambda') = \left(\bar{s}_k(\Lambda, \Lambda') - \delta(\Lambda, \Lambda')\right) \left(x_k(\Lambda) + x_k(\Lambda')\right)^{-1} \tag{3.2}$$

unless N is odd and $\Lambda' = \Lambda$ where the diagrams $\Lambda(k), \Lambda(k-1)$ are associated. In the latter case $s_k(\Lambda, \Lambda) = 1$.

Now let the index k run through the set $\{1, \ldots, n-1\}$ while the sequences Λ, Λ' run through the set $\mathcal{L}(\lambda, n)$. If $v(\Lambda') \notin V(\Lambda, k)$ then $s_k(\Lambda, \Lambda') = \bar{s}_k(\Lambda, \Lambda') = 0$.

Suppose that $v(\Lambda') \in V(\Lambda, k)$. As we have already mentioned, the vectors $v(\Lambda), v(\Lambda') \in V(\lambda, n)$ are defined up to scalar multipliers. Up to the choice of these multipliers Proposition 3.1 and Corollaries 3.7, 3.9 describe the matrix element $\bar{s}_k(\Lambda, \Lambda')$ while Propositions 3.2, 3.10 and Corollary 3.3 describe the matrix element $s_k(\Lambda, \Lambda')$. The following theorem completes the description of these matrix elements.

Theorem 3.11. *Suppose that $v(\Lambda') \in V(\Lambda, k)$ and $\Lambda \neq \Lambda'$. Then one can assume:*

$$s_k(\Lambda, \Lambda') = s_k(\Lambda', \Lambda) > 0 \quad \textit{if } \Lambda(k-1) \neq \Lambda(k+1), \tag{3.3}$$

$$\bar{s}_k(\Lambda, \Lambda') = \bar{s}_k(\Lambda', \Lambda) > 0 \quad \textit{if } \Lambda(k-1) = \Lambda(k+1). \tag{3.4}$$

4. Degenerate Affine Wenzl Algebra

In this section we will be again assuming that N is an arbitrary complex number. We will now use the results of Section 2 as a motivation to introduce a new object. This is the complex associative algebra generated by the algebra $B(n,N)$ along with the pairwise commuting elements $y_1, \dots, y_n$ and central elements $w_1, w_2, \dots$ subjected to the following relations. We impose the relations

$$s_k\, y_l = y_l\, s_k\,, \quad \bar{s}_k\, y_l = y_l\, \bar{s}_k\,; \qquad l \neq k, k+1; \tag{4.1}$$

$$s_k\, y_k - y_{k+1}\, s_k = \bar{s}_k - 1\,, \quad s_k\, y_{k+1} - y_k\, s_k = 1 - \bar{s}_k\,; \tag{4.2}$$

$$\bar{s}_k\,(y_k + y_{k+1}) = 0\,, \quad (y_k + y_{k+1})\,\bar{s}_k = 0\,. \tag{4.3}$$

Moreover, we impose the relations

$$\bar{s}_1\, y_1^i\, \bar{s}_1 = w_i\, \bar{s}_1\,; \qquad i = 1, 2, \dots\,. \tag{4.4}$$

We will view this algebra as an analogue of the degenerate affine Hecke algebra $He(n)$ considered in [C1,C2] and [D]; see Corollary 4.9 below. We will denote the above introduced algebra by $We(n,N)$ and call it the *degenerate affine Wenzl algebra* in honour of H. Wenzl who has used the maps

$$B(k,N) \to B(k-1,N) : \; b \mapsto b'\,; \qquad k = 1, 2, \dots, n-1$$

defined by (2.6) to prove that the algebra $B(n,N)$ is semisimple when N is not an integer.

It is convenient to put $w_0 = N$. The equality (4.4) is then valid for $i = 0$ also. The assignements

$$y_k \mapsto x_k, \quad w_i \mapsto z_1^{(i)}$$

define a homomorphism

$$\pi : We(n,N) \to B(n,N) \tag{4.5}$$

identical on $B(n,N)$ by the relations (2.3) to (2.5) and (2.7). The relations (4.1) to (4.4) imply that

$$-2\, w_i = w_{i-1} + \sum_{j=1}^{i} (-1)^j\, w_{i-j}\, w_{j-1}; \qquad i = 1, 3, \dots\,. \tag{4.6}$$

In particular, we have $w_1 = N(N-1)/2$. Morover, the following proposition holds.

Proposition 4.1. *The elements $y_1^i + \ldots + y_n^i$ with $i = 1, 3, \ldots$ are central in the algebra $We(n,N)$.*

We have the ascending chain of algebras $We(1,N) \subset We(2,N) \subset \ldots$ by definition.

Proposition 4.2. *For each $k = 1, 2, \ldots$ we have the equalities*

$$\bar{s}_k\, y_k^i\, \bar{s}_k = w_k^{(i)}\, \bar{s}_k\,; \qquad i = 0, 1, 2, \ldots \tag{4.7}$$

where $w_k^{(i)}$ is a central element of the algebra $We(k-1, N)$. The generating series

$$W_k(u) = \sum_{i \geqslant 0} w_k^{(i)}\, u^{-i}$$

satisfy

$$\frac{W_{k+1}(u) + u - 1/2}{W_k(u) + u - 1/2} = \frac{(u + y_k)^2 - 1}{(u - y_k)^2 - 1} \cdot \frac{(u - y_k)^2}{(u + y_k)^2}\,. \tag{4.8}$$

Consider the series $Z_k(u)$ and $Q_k(u)$ defined by (2.9) and (2.10) respectively. Since $x_1 = (N-1)/2$ we have

$$Q_1(u) = \frac{u + (N-1)/2}{u - (N-1)/2}\,.$$

Furthermore,

$$\pi : W_k(u) \mapsto Z_k(u)$$

for every $k = 1, 2, \ldots$ by (4.7). Thus we obtain the following corollary to Proposition 4.2.

Corollary 4.3. *For every $k = 1, 2, \ldots$ we have*

$$Q_k(u) = \frac{u + (N-1)/2}{u - (N-1)/2} \cdot \prod_{l=1}^{k-1} \frac{(u + x_l)^2 - 1}{(u - x_l)^2 - 1}\, \frac{(u - x_l)^2}{(u + x_l)^2}\,.$$

When N is a positive integer this statement can be also derived from Corollary 2.4 and Theorems 2.6 , 3.8 due to the next observation. Consider a Young diagram μ with m boxes and l pairwise distinct rows. Let $c_1, \ldots, c_{l+1}$ and $d_1, \ldots, d_l$ be respectively the contents of the boxes that can be added to or removed from μ. Let $e_1, \ldots, e_m$ be the contents of the boxes of μ. Then for any $h \in \mathbb{C}$ and $i \geqslant 0$

$$\sum_{j=1}^{l+1} (h + c_j)^i - \sum_{j=1}^{l} (h + d_j)^i = h^k + \sum_{j=1}^{m} (h + e_j + 1)^i - 2(h + e_j)^i + (h + e_j - 1)^i.$$

In the remaing part of this section will construct a linear basis in the algebra $We(n, N)$. Let us equip the algebra $We(n, N)$ with an ascending filtration by defining the degrees of its generators in the following way:

$$\deg\, s_k = \deg\, \bar{s}_k = 0, \quad \deg\, y_k = 1, \quad \deg\, w^{(i)} = 0.$$

Denote by u_k the image of the element $y_k \in We(n, N)$ in the corresponding graded algebra $\operatorname{gr} We(n, N)$. In the latter algebra by the relations (4.1) to (4.3) we have

$$s\, u_k\, s^{-1} = u_{s(k)}\,, \quad s \in S(n). \tag{4.9}$$

These relations along with (4.1) and (4.3) imply that

$$(4.10) \qquad \overline{(k,l)}\, u_m = u_m\, \overline{(k,l)}\,, \quad m \neq k, l\,;$$

$$(4.11) \qquad \overline{(k,l)} \cdot (u_k + u_l) = 0\,, \quad (u_k + u_l) \cdot \overline{(k,l)} = 0\,; \qquad k \neq l.$$

Furthermore, due to the relations (4.4) and (4.9) we have

$$(4.12) \qquad \overline{(k,l)}\, u_k^i\, \overline{(k,l)} = 0\,; \quad i = 1, 2, \ldots\,; \qquad k \neq l.$$

By definition, the elements $b(\gamma)$ where γ runs through the set of graphs $\mathcal{G}(n)$, constitute a linear basis in the algebra $B(n, N)$. Any edge of a graph $\gamma \in \mathcal{G}(n)$ of the form $\{k, l\}$ or $\{\bar{k}, \bar{l}\}$ will be called *horizontal*. If $k < l$ then the vertex k or $\bar{k}$ will be called the *left end* of the horizontal edge $\{k, l\}$ or $\{\bar{k}, \bar{l}\}$ respectively. The vertex l or $\bar{l}$ will be then called the *right end*.

The number of horizontal edges in a graph $\gamma \in \mathcal{G}(n)$ is even. If this number is $2r$, the element $b(\gamma) \in B(n, N)$ has the form of the product $\overline{(k_1, l_1)} \ldots \overline{(k_r, l_r)} \cdot s$ where $s \in S(n)$ and all $k_1, l_1, \ldots, k_r, l_r$ are pairwise distinct. The elements $b(\gamma)$ where the graph γ has $2r$ horizontal edges or more, span a two-sided ideal in $B(n, N)$.

Lemma 4.4. *Let u be a monomial in $u_1, \ldots, u_n$. For any two graphs $\gamma, \gamma' \in \mathcal{G}(n)$ we have the equality in the algebra* gr $We(n, N)$

$$(4.13) \qquad b(\gamma)\, u\, b(\gamma') = \varepsilon \cdot u'\, b(\gamma)\, b(\gamma')\, u''$$

where $\varepsilon \in \{1, 0, -1\}$ and u', u'' are certain monomials in $u_1, \ldots, u_n$.

Consider any graph $\gamma \in \mathcal{G}(n)$ with exactly $2r$ horizontal edges. Let

$$k_1, \ldots, k_r\,,\ \bar{k}'_1, \ldots \bar{k}'_r \quad \text{and} \quad l_1, \ldots, l_r\,,\ \bar{l}'_1, \ldots \bar{l}'_r$$

be all the left ends and the right ends of the horizontal edges respectively.

Lemma 4.5. *For any two monomials u and u' in $u_1, \ldots, u_n$ we have the equality in the algebra* gr $We(n, N)$

$$u\, b(\gamma)\, u' = \varepsilon \cdot u_1^{i_1} \ldots u_n^{i_n}\, b(\gamma)\, u_1^{j_1} \ldots u_n^{j_n}$$

where $\varepsilon \in \{1, 0, -1\}$ and

$$(4.14) \qquad k \in \{l_1, \ldots, l_r\} \Rightarrow i_k = 0; \quad j_k \neq 0 \Rightarrow k \in \{l'_1, \ldots, l'_r\}.$$

Any product in the algebra $We(n, N)$ of the form

$$(4.15) \qquad y_1^{i_1} \ldots y_n^{i_n}\, b(\gamma)\, y_1^{j_1} \ldots y_n^{j_n} \cdot w_2^{h_2} w_4^{h_4} \ldots$$

will be called a *regular monomial* if the exponents $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ satisfy the conditions (4.14). The two theorems below are the main results of this section.

Theorem 4.6. *All the regular monomials (4.15) constitute a basis in $We(n,N)$.*

By the relations (2.3) to (2.5) and (2.7) for every $m = 0, 1, 2, \ldots$ the assignements

$$s_k \mapsto s_{m+k}, \quad \bar{s}_k \mapsto \bar{s}_{m+k}, \quad y_k \mapsto x_{m+k}, \quad w_i \mapsto z^{(i)}_{m+1}$$

define a homomorphism

$$\pi_m : We(n,N) \to B(m+n,N).$$

In particular, the homomorphism π_0 coincides with (4.5). Furthermore, by Lemma 2.1 the image of the homomorphism π_m commutes with the subalgebra $B(m,N)$ in $B(m+n,N)$.

Theorem 4.7. *The kernels of $\pi_0, \pi_1, \pi_2, \ldots$ have the zero intersection.*

Due to Lemmas 4.4, 4.5 and to the equalities (4.6) every element of the algebra $We(n,N)$ can be expressed as a linear combination of regular monomials. Thus both Theorems 4.6 and 4.7 follow from the next proposition; cf. [O, Lemma 2.1.11].

Proposition 4.8. *Given a finite set $\mathcal{F}$ of regular monomials in $We(n,N)$ there exists $m \in \{0,1,2,\ldots\}$ such that the images in $B(m+n,N)$ of all the monomials from $\mathcal{F}$ with respect to the homomorphism π_m are linearly independent.*

We will now compare the algebra $We(n,N)$ with the *degenerate affine Hecke algebra* $He(n)$ from [C1,C2] and [D]. The latter algebra is generated by the group algebra $\mathbb{C}[S(n)]$ and the pairwise commuting elements $v_1, \ldots, v_n$ subjected to the relations

$$s_k\, v_l = v_l\, s_k\,, \quad l \neq k, k+1;$$
$$s_k\, v_k - v_{k+1}\, s_k = -1\,, \quad s_k\, v_{k+1} - v_k\, s_k = 1\,.$$

By the relations (4.1) to (4.4) we have the following corollary to Theorem 4.6.

Corollary 4.9. *For any $f_2, f_4, \ldots \in \mathbb{C}$ the assignements $s_k \mapsto s_k$, $\bar{s}_k \mapsto 0$, $y_k \mapsto v_k$ and*

$$w_i \mapsto f_i, \quad i = 2, 4, \ldots$$

determine a homomorphism of the algebra $We(n,N)$ onto $He(n)$.

The subalgebra in $He(n)$ generated by the elements $v_1, \ldots, v_n$ is maximal commutative. The centre of the algebra $He(n)$ consists of all symmetric polynomials in $v_1, \ldots, v_n$. For the proofs of these two statements see [C2, Section 1]. The next corollary provides analogues of these statements for the algebra $We(n,N)$.

Corollary 4.10. *The subalgebra in $We(n,N)$ generated by the elements $y_1, \ldots, y_n$ and $w_1, w_2, \ldots$ is maximal commutative. The elements $y_1^i + \ldots + y_n^i$ with $i = 1, 3, \ldots$ and w_i with $i = 2, 4, \ldots$ generate the centre of the algebra $We(n,N)$.*

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