# Cycle type and descent set in the hyperoctahedral groups

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March 6, 1995

**Résumé**

Nous exprimons le nombre d'éléments du groupe hyperoctaédral $B_n$ qui ont un ensemble de descentes donné, et dont l'inverse a un ensemble de descentes donné, comme le produit scalaire de deux représentations de $B_n$. On donne aussi le nombre d'éléments de $B_n$, qui sont dans une classe de conjugaison donnée et qui ont un ensemble de descentes donné, à l'aide d'un produit scalaire de deux représentations du groupe hyperoctaédral.

On a enfin, sous forme de séries génératrices de fonctions symétriques. des analogues des formules classiques qui donnent les s'eries génératrices exponentielles des éléments alternants des $B_n$.

**Abstract**

We express the number of elements of the hyperoctahedral group $B_n$, which have descent set $K$ and such that their inverses have descent set $J$, as a scalar product of two representations of $B_n$. We also give

*Supported by ISM (Montréal).

the number of elements of $B_n$, which have a prescribed descent set and which are in a given conjugacy class of $B_n$ by another scalar product of representations of $B_n$.

We finally give, by generating series of symmetric functions, some analogs of the classical formulas which express the exponential generating series of alternating elements in the $B_n$'s.

# 1 Introduction

Enumerating permutations according to certain statistics, as descent set, major index or cycle type is an old problem (see [1, 17] ). In [21], Solomon defines, for each subset $K$ of $\{1,\ldots,n-1\}$, a representation $\psi_K$ of $S_n$ such that the dimension of $\psi_K$ is the number of permutations in $S_n$ with descent set $K$. The characteristic symmetric function of this representation appears already in MacMahon's work [17] see also [13].

In [19] and [3] appear representations $X_\lambda$ of $S_n$, indexed by the partitions $\lambda$ of $n$, such that the number of permutations of cycle type $\lambda$ is the dimension of $X_\lambda$.

Moreover, Foulkes [7] and Gessel [8] have proved that the number of permutations $\sigma$ in $S_n$, with descent set $K \subseteq \{1,\ldots,n-1\}$ and such that the descent set of $\sigma^{-1}$ is $J \subseteq\{1,\ldots,n-1\}$, is the scalar product $< \psi_K, \psi_J >$. Gessel and Reutenauer have shown in [9] a related result giving the number of permutations with descent set $\{1,\ldots,n-1\}\backslash K$ and with cycle type $\lambda$ as the scalar product $< \psi_K, X_\lambda >$.

The literature also furnishes extensions of the enumeration of certain permutations to Coxeter groups (see [2, 21, 22]) or to wreath products (see [5, 18]).

In this communication we extend previous results to the case of the hyperoctahedral group. Our main results are Theorems 4, 7, and 10; Theorem 2 is a technical result. The *descent set* of $\sigma \in B_n$ is the set

$$des(\sigma) = \{i|0 \leq i \leq n-1, \sigma(i) > \sigma(i+1)\},$$

if we consider $B_n$ as the group of the permutations $\sigma$ of $\{-n,\ldots,-1,0,1,\ldots n\}$ such that for all $i \in\{0,\ldots,n,\}$ we have $\sigma(-i) = -\sigma(i)$. According to the terminology of Foata and Schützenberger [6], the descent set of $\sigma^{-1}$ will be called the *idown set* of $\sigma$. Theorem 4 expresses this number as the scalar product of two representations of $B_n$.

Theorem 7 gives, as a scalar product of two representations of $B_n$, the number of elements of the hyperoctahedral group, which have a prescribed descent set and which are in a given conjugacy class of $B_n$.

In the last section, we generalise to the hyperoctahedral group the notion of Eulerian symmetric functions, credited to Gessel by Désarménien [4, p. 283] and we state Theorem 10 which is an analog of a result of Springer [22, p. 35].

The main tool for this purpose is a generalisation of the characteristic function of Frobenius, defined by Geissinger [10], see also [11]. Essentially, our characteristic function is an isomorphism between the $\mathbb{Z}$-module generated by the irreducible characters of the hyperoctahedral groups and the ring $\Lambda(X) \otimes \Lambda(Y)$, where $\Lambda(X)$ is the ring of symmetric functions on $X$. This is stated in Theorem 2, which is implicit in [16] and is an easy consequence of Proposition 5.1 in [23], see also [24].

# 2 Characteristic function

In the following, the elements of the hyperoctahedral groups will be called *signed permutations*. A signed permutation $\sigma$ in $B_n$ is determined by the sequence $\sigma(1)\sigma(2)\ldots\sigma(n)$. The group $B_n$ is the subgroup of $S_{\{-n,\ldots,-1,1,\ldots,n\}}$ of permutations which commute with $w_0 = (1,-1)\ldots(n,-n)$.

It is well known, and easy to verify, that a signed permutation $\sigma$, viewed as an element of $S_{\{-n,\ldots,-1,1,\ldots,n\}}$, has two kinds of cycles:

$$\begin{array}{rl} & (x_1,\ldots,x_k), \quad (-x_1,\ldots,-x_k) \\ \text{or} & (x_1,x_2\ldots,x_k,-x_1,-x_2,\ldots,-x_k). \end{array}$$

We will say that a couple of cycles of the first kind is an *even cycle of length* $k$, and that a cycle of the second kind is an *odd cycle of length* $k$. The *cycle type* $ct(\sigma)$ of a signed permutation $\sigma$ is a couple $(\lambda;\mu)$ of partitions where the parts of $\lambda$ (*resp.* $\mu$) are the lengths of the even cycles (*resp.* odd cycles) of $\sigma$. If $\sigma \in B_n$, one has $|\lambda| + |\mu| = n$, where $|\lambda| = \lambda_1 + \ldots + \lambda_k$. The following proposition gives a classical result (see [12]).

**Proposition 1** *Two signed permutations in $B_n$ are in the same conjugacy class if and only if they have the same cycle type.*

Let $X$ and $Y$ be infinite sets of variables, $\Lambda(X)$ denotes the ring of symmetric functions on $X$ with cœfficients in $\mathbf{Z}$. We define the scalar product $< -, - >$ on the ring $\Lambda(X) \otimes \Lambda(Y)$ by

$$< s_\lambda(X) \otimes s_\mu(Y), s_{\lambda'}(X) \otimes s_{\mu'}(Y) >= \delta_{\lambda\lambda'}\delta_{\mu\mu'},$$

where $s_\lambda$ denotes the Schur function associated to the partition $\lambda$.

For any group $G$, we denote by $R(G)$ the $\mathbf{Z}$-module generated by the irreducible characters of $G$. Let $R$ be the direct sum of the $R(B_n)$ for $n \geq 0$. Then, with the following multiplication, $R$ has a ring structure. If $f \in R(B_m)$ and $g \in R(B_n)$, then $f \times g$ is a character of $R(B_m) \times R(B_n)$. We embed $B_m \times B_n$ in $B_{m+n}$ and we define

$$f.g = ind_{B_m \times B_n}^{B_{m+n}}(f \times g).$$

One can verify that $R$ is a commutative, associative and graded ring. If we have $f = \sum f_n$ and $g = \sum g_n$, with $f_n, g_n$ in $R(B_n)$, we define the scalar product of $f$ and $g$ by

$$< f, g >= \sum_{n \geq 0} < f_n, g_n >_{B_n},$$

where $< f_n, g_n >_{B_n} = \frac{1}{n!2^n} \sum_{\sigma \in B_n} f_n(\sigma) g_n(\sigma^{-1})$.

For any character $f$ of $B_n$ we define $ch(f)$ by

$$ch(f) = \frac{1}{|B_n|} \sum_{\substack{\sigma \in B_n \\ ct(\sigma)=(\lambda;\mu)}} f(\sigma)(p_{\lambda_1}(X) + p_{\lambda_1}(Y)) \ldots (p_{\mu_1}(X) - p_{\mu_1}(Y)) \ldots .$$

Where $p_k(X)$ is the power-sum symmetric function on $X$.

The following result extends to $B_n$ the theory of the characteristic map (see [15]) and is an easy consequence of [23, proposition 5.1], this is also a restatement of [16, theorem (9.10)].

**Theorem 2** *One can index the irreducible characters $\chi^{(\lambda;\mu)}$ of the hyperoctahedral groups by the couples $(\lambda, \mu)$ of partitions such that $ch(\chi^{(\lambda;\mu)}) = s_\lambda(X) \otimes s_\mu(Y)$, so that ch is an isometric isomorphism.*

# 3 Signed permutations with given descent set and idown set

The group $B_n$, as a Coxeter group embedded in $S_{\{-n,\dots,-1,1,\dots n\}}$, is generated by $\{r_0, r_1, \dots r_{n-1}\}$ where $r_0 = (1,-1)$ and $r_i = (i, i+1)(-i, -i-1)$ for every $1 \le i \le n-1$. Let $I_n$ be the set $\{0, \dots, n-1\}$; if $K \subseteq I_n$ we denote by $W_K$ the subgroup of $B_n$ generated by the $r_k$, $k \in K$.

In the group algebra $\mathbf{Q}[B_n]$ of $B_n$ over $\mathbf{Q}$, we define, for all $K \subseteq I_n$, the two idempotents

$$\xi_K = \frac{1}{|W_K|} \sum_{w \in W_K} w, \tag{1}$$

$$\eta_K = \frac{1}{|W_K|} \sum_{w \in W_K} \varepsilon(w) w, \tag{2}$$

where $\varepsilon$ is the character of $B_n$ such that $\varepsilon(r_i) = -1$ for $i \in I_n$. If $K \subseteq I_n$, let $\psi_K$ be the character afforded by the left ideal $\mathbf{Q}[B_n]\xi_K \eta_{I_n \setminus K}$. The following proposition is due to Solomon [21].

**Proposition 3** *If $K$ is a subset of $I_n$ we have:*

*i) For all $g \in B_n$, $\psi_K(g) = \varepsilon(g)\psi_{I_n \setminus K}(g)$.*

*ii) The number of signed permutations having descent set $K$ is equal to the dimension of $\psi_{I_n \setminus K}$.*

We prove the next result, which extends, to the case of the hyperoctahedral group, results of Foulkes [7] and Gessel [8].

**Theorem 4** *Let $K$ and $J$ be subsets of $I_n$. Then the number of elements of $B_n$ having descent set $K$ and idown set $J$ is*

$$< \psi_{I_n \setminus K}, \psi_{I_n \setminus J} > = < \psi_K, \psi_J > .$$

To prove this Theorem, we define some $F_K(X, Y)$ ($K \subseteq I_n$) which generalise the quasisymmetric functions of Gessel (see [8]). If $\Pi \subseteq B_n$ we call *quasisymmetric generating function of* $\Pi$ the series

$$\sum_{\pi \in \Pi} F_{des(\pi)}(X, Y).$$

Let $\omega_0$, $\omega_1$ be the isometric automorphisms of $\Lambda(X) \otimes \Lambda(Y)$ defined by

$$\omega_0(s_\lambda(X) \otimes s_\mu(Y)) = s_{\lambda'}(X) \otimes s_\mu(Y)$$

$$\omega_1(s_\lambda(X) \otimes s_\mu(Y)) = s_\lambda(X) \otimes s_{\mu'}(Y)$$

where $\mu'$ is the partition conjugate to the partition $\mu$ (see [15, p.2]). A sequence of technical lemmas gives us the following result.

**Lemma 5** *i) If the quasisymmetric generating function $g$ of $\Pi$ is symmetric in $X$ and $Y$, then the number of elements of $\Pi$ which have descent set $K$ is $< g, \omega_1(ch(\psi_{I_n \setminus K})) >$.*

*ii) If $J \subseteq I_n$ and $\Pi = \{\sigma \in B_n | des(\sigma^{-1}) = J\}$ then the quasisymmetric function of $\Pi$ is $\omega_1(ch(\psi_{I_n \setminus J}))$.*

Theorem 4 is an easy consequence of lemma 5.

# 4 Signed permutations with given cycle type and descent set

Let $A$ and $\bar{A}$ be two infinite alphabets, and $B$ be the disjoint union of $A$ and $\bar{A}$. Then $\mathbf{Q} < B >$ denotes the free associative (non commutative) $\mathbf{Q}$-algebra generated by $B$. The elements of $\mathbf{Q} < B >$ are called *polynomials*, and the set $B^*$ of the words on $B$ is a basis of $\mathbf{Q} < B >$. If $P$ and $Q$ are polynomials their *Lie bracket* is defined by

$$[P, Q] = PQ - QP.$$

The *free Lie algebra* $\mathcal{L}(B)$ is the smallest submodule of $\mathbf{Q} < B >$ containing $B$ and closed under Lie bracket; its elements are called *Lie polynomials.*

A Lie polynomial $P$ is said to be *even* (*resp. odd*) *and homogeneous of degree $i$*, if it is a linear combination of words of length $i$ having an even (*resp.* odd) number of letters in $\bar{A}$.

The *symmetric product* of $k$ polynomials $P_1, \ldots, P_k$ is defined by

$$(P_1, \ldots, P_k) = \frac{1}{k!} \sum_{\sigma \in S_k} P_{\sigma(1)} \ldots P_{\sigma(k)}.$$

For any couple of partitions $(\lambda, \mu)$, we denote by $U_{(\lambda,\mu)}$ the subspace of $\mathbf{Q} < B >$ linearly generated by the symmetric products $(P_1, \ldots, P_k, Q_1, \ldots, Q_l)$ with the two conditions

- $P_i$ is an even Lie polynomial of degree $\lambda_i$,
- $Q_i$ is an odd Lie polynomial of degree $\mu_i$.

The following proposition extends Lemma 8.22 in [20] and is a consequence of the theorem of Poincaré-Birkhoff-Witt.

**Proposition 6**

$$\mathbf{Q} < B >= \oplus_{\lambda,\mu} U_{(\lambda,\mu)}.$$

We now suppose that $\{1, \ldots, n\} \subset A$ and that $\{\bar{1}, \ldots, \bar{n}\} \subset \bar{A}$. From now on, we will write $\sigma(i) = \bar{j}$ instead of $\sigma(i) = -j$. Let $E_n$ be the subspace of $\mathbf{Q} < B >$ generated by the words $w_\sigma = \sigma(1) \ldots \sigma(n)$ for all $\sigma \in B_n$. We define the *absolute value* on $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ by

$$|i| = |\bar{i}| = i \text{ for all } i \in \{1, \ldots, n\}.$$

There is a natural action of $B_n$ onto $\mathbf{Q} < B >$. By the change of basis in $\mathbf{Q} < B >$ defined by $i \mapsto 1/2(i + \bar{i})$, $\bar{i} \mapsto 1/2(i - \bar{i})$, this gives another natural action which we use in the sequel.

For this action the spaces $E_n$ and $U_{(\lambda,\mu)}$ are invariant. Hence they define a representation of $B_n$ on the space $E_n \bigcap U_{(\lambda,\mu)}$; let $X_{(\lambda,\mu)}$ be the character of this representation.

If we write $\tilde{\omega}_i = ch^{-1} o \omega_i o ch$, for $i = 1, 2$, we have that $\tilde{\omega}_i$ maps irreducible characters onto irreducible characters and the following result holds

**Theorem 7** *The number of signed permutations having cycle type $(\lambda, \mu)$ and descent set $K \subseteq I_n$ is*

$$< X_{(\lambda,\mu)}, \tilde{\omega}_1(\psi_{I_n \backslash K}) >=< \varepsilon X_{(\lambda,\mu)}, \tilde{\omega}_0(\psi_K) > .$$

This Theorem extends Theorem 2.1 in [9]. The proof of this result is based on lemma 5 *i*) and on the following lemma.

**Lemma 8** *If $(\lambda, \mu)$ is a couple of partitions and if $\Pi$ is the set of signed permutations having cycle type $(\lambda, \mu)$, then the quasisymmetric generating function of $\Pi$ is $ch(X_{(\lambda,\mu)})$.*

To prove this lemma, we use a bijection between a basis of $U_{(\lambda,\mu)}$ and a subset of multisets of Lyndon words (see [14, p. 67 and 77] and [20, p. 166]). We also use the equality of the generating series of $U_{(\lambda,\mu)}$ and $ch(X_{(\lambda,\mu)})$.

# 5 Alternating signed permutations and trigonometric symmetric functions.

A *rising alternating* (*resp. falling alternating*) signed permutation $\sigma$ is a signed permutation having descent set $K_n^r = \{1,3,\ldots\} \subset I_n$ (*resp.* $K_n^f = \{0,2,\ldots\} \subset I_n$).

**Example** The signed permutation $\sigma = 3\,1\,4\,\bar{6}\,7\,\bar{2}\,5$ is a rising alternating element of $B_7$ and $\sigma = \bar{1}\,4\,3\,7\,\bar{6}\,\bar{2}\,\bar{5}$ is a falling alternating element of $B_7$.

It is well known that the exponential generating series of rising alternanting permutations is

$$\frac{1+\sin x}{\cos x}. \tag{3}$$

Let $b_n$ denotes the number of rising alternating signed permutations in $B_n$. If one takes $b_0 = 1$, one then has, see [22, p. 35]

$$\sum_{n\geq 0} \frac{b_n}{n!} x^n = \frac{\sin x + \cos x}{\cos 2x}. \tag{4}$$

For any set of variables $X$, if $h_n(X)$ is the complete symmetric function on $X$, and $H_X(t) = \sum_{n\geq 0} h_n(X)t^n$; we define the symmetric cosinus and sinus, as in [4] by

$$COS_X(t) = \frac{H_X(it) + H_X(-it)}{2} \tag{5}$$

$$SIN_X(t) = \frac{H_X(it) - H_X(-it)}{2i}. \tag{6}$$

We then have the next lemma

**Lemma 9** *One has the three following relations*

*i)* $COS_X(t)^2 + SIN_X(t)^2 = H_X(it)H_X(-it)$

*ii)* $COS_{X\cup Y}(t) = COS_X(t)COS_Y(t) - SIN_X(t)SIN_Y(t)$

*iii)* $SIN_{X\cup Y}(t) = COS_X(t)SIN_Y(t) + SIN_X(t)COS_Y(t)$.

Note that the classical trigonometric formulas follow by the specializations $p_1(X)t \mapsto a$, $p_1(Y)t \mapsto b$ and the other power-sums are mapped to zero. In

[4], Désarménien gives a symmetric analog of the generating series in equation (3) of the form

$$\frac{1 + SIN_X(t)}{COS_X(t)}. \tag{7}$$

The following Theorem extends (7) to the case of the hyperoctahedral groups and gives symmetric analogs of (4).

**Theorem 10**

$$\begin{aligned}
\sum_{n\geq 0} ch(\psi_{K^r_{2n+1}})t^{2n+1} &= H_X(it)H_X(-it)\frac{SIN_Y(t)}{COS_{X\cup Y}(t)} \\
\sum_{n\geq 0} ch(\psi_{K^f_{2n+1}})t^{2n+1} &= \frac{SIN_X(t)}{COS_{X\cup Y}(t)} \\
1 + \sum_{n\geq 1} ch(\psi_{K^r_{2n}})t^{2n} &= \frac{COS_X(t)}{COS_{X\cup Y}(t)} \\
1 + \sum_{n\geq 1} ch(\psi_{K^f_{2n}})t^{2n} &= H_X(it)H_X(-it)\frac{COS_Y(t)}{COS_{X\cup Y}(t)}.
\end{aligned}$$

To prove this result we use the following formula, due to Solomon [21]

$$\psi_K = \sum_{K\subseteq J\subseteq I_n} (-1)^{|J\setminus K|}\phi_J$$

where the $\phi_J$ are certain representations of the group $B_n$. We also use the fact that $ch(\phi_J)$ can be expressed as a product of complete symmetric functions on $X$ and $X \cup Y$.

# References


[1] André D. (1881). Sur les permutations alternées, *Journal de mathématiques pures et appliquées*, **7**, 161-184.

[2] Arnold V.I. (1992). Springer numbers and morsification spaces, *Journal of algebraic geometry*, **1**, **2**, 197-214.

[3] Bergeron, F., Bergeron, N. and Garsia A. M. (1988). Idempotents for the free Lie algebra and $q$-enumeration. In *invariant theory and tableaux* (ed D. Stanton), p.166-190. IMA Volumes in Mathematics and its Applications, V. 19. Springer, Berlin.

[4] Désarménien J. (1983). Fonctions symétriques associées à des suites classiques de nombres, *Annales scientifiques de l'École Normale supérieure,* **16**,271-304.

[5] Ehrenborg R., Readdy M. (1994). Sheffer posets and $r$-signed permutations, *preprint.*

[6] Foata D. and Schützenberger M.-P. (1978). Major index and inversion number of permutations, *Mathematische Nachrichten*, **83**, 143-159.

[7] Foulkes H.O. (1976). Enumeration of permutations with prescribed up-down and inversion sequences, *Discrete Math.*,**15**, 235-252.

[8] Gessel I. (1984). Multipartite P-partitions and inner product of skew Schur functions. *Contemporary Mathematics,* **34**, 289-301.

[9] Gessel I. and Reutenauer C. (1993). Counting permutations with given cycle structure and descent set. *Journal of combinatorial theory,* **A**,**64**, 2.

[10] Geissinger L. (1977). Hopf algebras of symmetric functions and class functions, *in Combinatoire et représentation du groupe symétrique,* Lecture notes in Math., Springer-Verlag, 579, 168-181.

[11] Geissinger L. and Kinch D. (1978). Representations of the hyperoctahedral groups, *Journal of algebra*, **53**, 1-20.

[12] Kerber A. (1971). *Representations of Permutation Groups I.* Springer Lecture Notes in Mathematics, Vol. 240. Springer-Verlag, Berlin.

[13] Littlewood D. E. (1950). *The theory of group characters and matrix representations of groups.* 2nd ed, Clarendon Press, Oxford.

[14] Lothaire M. (1983). *Combinatorics on words.* Encyclopedia of Mathematics, Vol. 17. Addison-Wesley, Reading, MA.

[15] Macdonald I.G. (1979). *Symmetric functions and Hall ploynomial.* Oxford University Press.

[16] Macdonald I.G. (1980). Polynomial functors and wreath products. *Journal of pure and applied algebra*, **18**, 173-204.

[17] MacMahon P.A. (1915-1916). *Combinatory Analysis*, Cambridge: *reprinted* Chelsea, New-York, 1960.

[18] Reiner V. (1993). Signed permutations statistics and cycle type, *European Journal of Combinatorics*, **14**, 569-579.

[19] Reutenauer C. (1986). Theorem of Poincaré-Birkhoff-Witt, logarithm and representations of the symmetric group whose order are the Stirling numbers. In *Combinatoire Énumérative*, Proceedings, Montréal, (1985), (ed G. Labelle et P.Leroux). Lecture notes in mathematics, Springer, Berlin, Vol. 1234, 216-284.

[20] Reutenauer C. (1993). *Free Lie algebras*. Oxford science publications.

[21] Solomon L. (1968). A decomposition of the group algebra of a finite Coxeter group. *Journal of algebra* **9**, 220-239.

[22] Springer T. A. (1971). Remarks on a combinatorial problem, *Nieuw Archiev voor Wiskunde (1)*, **19**, 30-36.

[23] Stembridge J. (1992). The projective representations of the hyperoctahedral group, *Journal of Algebra*,**145**, 396-453.

[24] Zelevinski A. V. (1981). *Representations of finite classical groups*. Springer Lecture Notes in Mathematics, Vol. 869. Springer-Verlag, Berlin.