INCOMPLETE FLAG VARIETIES AND KAZHDAN–LUSZTIG POLYNOMIALS

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ABSTRACT. We give explicit formulas for the Kazhdan–Lusztig P- and R-polynomials for permutations related to incomplete flags $F_{1,n-1}$.

RÉSUMÉ. Nous donnons des formules explicites pour les P- et R-polynômes de Kazhdan–Lusztig pour les permutations rattaché à drapeaux incomplets.

1. INTRODUCTION

In [KL1] Kazhdan and Lusztig have associated with each Coxeter group W a family of so-called P-polynomials indexed by pairs of elements of W. These polynomials are determined by the geometry of the corresponding Schubert varieties and play an important role in representation theory. Explicit calculation of P-polynomials turned out to be a very hard problem, even for the case $W = S_n$. The most advanced result in this direction is a simple combinatorial algorithm for calculation of P-polynomials for Grassmann permutations (see [LS]). Several other particular cases are considered in [Br].

Another family of polynomials defined in [KL1], so called R-polynomials, often helps to calculate P-polynomials (see [KL1, De, Br]). These polynomials also have a transparent geometrical interpretation (see [SSV1, Cu]). Their explicit calculation is, in general, a simpler problem than that for P-polynomials; nevertheless, one encounters here rather complicated combinatorial problems ([De, SSV2]).

In this note we give explicit formulas for P- and R-polynomials for two classes of permutations related to incomplete flags consisting of a line and a hyperplane. This case is not covered by results of [Ze], since the corresponding Schubert varieties do not admit small resolutions of singularities.

Let us denote by $F_{i_1,\dots,i_k}$ the variety of all flags of type $V^{i_1} \subset V^{i_2} \subset \cdots \subset V^{i_k} \subseteq \mathbb{C}^n$. For brevity, the variety $F_{1,2,\dots,n}$ of complete flags is denoted by F_n. There exists a natural

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bundle $F_n \to F_{i_1,\dots,i_k}$ that just drops redundant subspaces. Evidently, the fiber of this bundle is diffeomorphic to $F_{i_1} \times F_{i_2-i_1} \times \cdots \times F_{n+1-i_k}$. Each complete flag defines a decomposition of $F_{i_1,\dots,i_k}$ into Schubert cells. This decomposition is consistent with our bundle, i.e. the inverse image of a Schubert cell in $F_{i_1,\dots,i_k}$ is the union of some Schubert cells in F_n. It is easy to see that the index set of this union is an interval in the Bruhat order on S_n. Thus, with each $F_{i_1,\dots,i_k}$ we associate two sets of permutations, namely, the maximal and the minimal elements of the corresponding intervals. These sets are denoted $\overline{\mathcal{M}}_{i_1,\dots,i_k}$ and $\underline{\mathcal{M}}_{i_1,\dots,i_k}$, respectively.

We are interested in the variety $F_{1,n-1}$; each point of this variety is a flag consisting of a line and a hyperplane. Below we provide explicit expressions for the polynomials $P_{x,y}(q)$ in the cases $y \in \overline{\mathcal{M}}_{1,n-1}$, x arbitrary and $y \in \underline{\mathcal{M}}_{1,n-1}$, x arbitrary. Besides, we present explicit expressions for the polynomials $R_{x,y}(q)$ in the cases $x, y \in \overline{\mathcal{M}}_{1,n-1}$ and $x, y \in \underline{\mathcal{M}}_{1,n-1}$.

2. Results

It is easy to see that permutations in $\overline{\mathcal{M}}_{1,n-1}$ are of the form $(n-1, n-2, \dots, 1, \dots, n, \dots, 3, 2)$ while those in $\underline{\mathcal{M}}_{1,n-1}$ of the form $(2, 3, \dots, 1, \dots, n, \dots, n-2, n-1)$. Recall that $P_{x,y}(q) \equiv P_{x^{-1},y^{-1}}(q)$ (see [Dy]) and $R_{x,y}(q) \equiv R_{x^{-1},y^{-1}}(q)$. Therefore, it is possible to state all the results in terms of inverse permutations, which seems to us more convenient.

Theorem 1. *Let $y = (i, n, n-1, \dots, 1, j)$, $x = (x_1, \dots, x_n)$. Then*

(i) *$P_{x,y}(q) = 1$ for any x if $i < j$;*

(ii)

$$P_{x,y}(q) = \begin{cases} 1 & \text{if } j \leqslant x_1 \text{ or } i \geqslant x_n, \\ 1 + q^{i-j} & \text{otherwise.} \end{cases}$$

Theorem 2. *Let $y = (i, 1, 2, \dots, n, j)$, $x = (x_1, \dots, x_n)$. Then $P_{x,y}(q) = (1+q)^r$, where r is the number of solutions of the following equation and two inequalities in z:*

$$\sum_{p=1}^{z} x_p = \frac{z(z+1)}{2}, \qquad j+1 \leqslant z \leqslant i-2.$$

Theorem 3. *Let $x = (i, n, n-1, \dots, 1, j)$, $y = (k, n, n-1, \dots, 1, l)$. Then:*

1. *If $k + j < n+1$ or $l + i > n+1$, then $R_{x,y}(q) \equiv 0$.*

2. *Let $l + i \leqslant n+1 \leqslant k+j$. Denote by $\Sigma_{j,k}$ the segment $[n+1-j, k]$, and by $\Sigma_{l,i}$ the segment $[l, n+1-i]$.*

(i) *If $\Sigma_{j,k} \cap \Sigma_{l,i} = \varnothing$, then*

$$R_{x,y}(q) = (q-1)^{(k+j)-(l+i)}.$$

(ii) *If $\Sigma_{j,k} \cap \Sigma_{l,i} \neq \varnothing$ and at least one of $\Sigma_{j,k}$ and $\Sigma_{l,i}$ degenerates to a point, then*

$$R_{x,y}(q) = (q-1)^{|k+i-n-1|+|l+j-n-1|-1}.$$

(iii) *If* $\Sigma_{j,k} \cap \Sigma_{l,i} \neq \varnothing$ *and both* $\Sigma_{j,k}$ *and* $\Sigma_{l,i}$ *are nondegenerate, then* $R_{x,y}(q) = (q-1)^a(q^2-q+1)^b$, *where*

$$a = |k+i-n-1| + |l+j-n-1| - 1,$$
$$b = \frac{1}{2}((k+j)-(l+i)-|k+i-n-1|-|l+j-n-1|) - 1.$$

Observe that Theorem 3 allows to calculate R-polynomials for $x = (i,1,2,\ldots,n,j)$, $y = (k,1,2,\ldots,n,l)$, since by Lemma 2.1(iv) of [KL1] one has

$$R_{x,y}(q) \equiv R_{(n+1-k,n,n-1,\ldots,1,n+1-l),(n+1-i,n,n-1,\ldots,1,n+1-j)}(q).$$

3. Sketches of proofs

Theorem 1. It is easy to see that the natural projection $\pi\colon F_n \to F_{1,n-1}$ has a smooth fiber diffeomorphic to F_{n-2}. Let σ be a Schubert cycle in $F_{1,n-1}$, and $\delta = \pi^{-1}(\sigma)$. Then the stalk of the IH sheaf on δ at an arbitrary point x is isomorphic to the stalk of the IH sheaf on σ at the point $\pi(x)$. Therefore, by [KL2], in order to find P-polynomials we have to calculate the local intersection homology for Schubert cycles in $F_{1,n-1}$. Each Schubert cycle in this variety is either a smooth manifold or an even-dimensional cone. Such a cone may be considered as a suspension of the spherization of the tangent bundle to an odd dimensional sphere. The intersection homology of this object can be computed easily using formula (3.3) from [KL2].

Theorem 2. Below we describe geometrical cycles that yield nontrivial IH classes. Let $y = (i,1,\ldots,n,j)^{-1} \in \underline{\mathcal{M}}_{1,n-1}$; the corresponding Schubert cycle $V_y = \{\varphi \in F_n\}$ is defined by the following relations:

$$\varphi^1 \subseteq f^i,\ \varphi^2 \supset f^1,\ \varphi^3 \supset f^2,\ \ldots,$$
$$\dim(\varphi^{j+1} \cap f^{j+1}) \geqslant j, \dim(\varphi^{j+2} \cap f^{j+2}) \geqslant j+1, \ldots, \dim(\varphi^i \cap f^i) = i, \ldots,.$$

We consider an arbitrary point $\theta \in V_y$. Let $j+1 \leqslant m_1 < m_2 < \cdots < m_s \leqslant i-2$ be all the indices such that $\theta^{m_\alpha} = f^{m_\alpha}$, $\alpha = 1,\ldots,s$. We fix a set of 1-dimensional spaces $\tau_1,\ldots,\tau_s$, $\tau_\alpha \in f^{m_\alpha}$, $\tau_\alpha \notin f^{m_\alpha - 1}$. Let $1 \leqslant l_1 < \cdots < l_k \leqslant s$; denote by $X_{l_1,\ldots,l_k}$ the subvariety in V_y defined by the conditions

$$X_{l_1\ldots l_k} = \{\varphi \in V_y\colon \tau_{l_\alpha} \subset f^{m_{l_\alpha}}, \alpha = 1,\ldots,k\}, \qquad \operatorname{codim}_{\mathbb{C}} X_{l_1\ldots l_k} = k.$$

These subvarieties form a basis for the stalk of the IH sheaf at θ.

Theorem 3. The proof follows from the general combinatorial procedure of finding R-polynomials described in [SSV2].

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