# CATALAN PATH STATISTICS HAVING THE NARAYANA OR THE KREWERAS-POUPARD DISTRIBUTION

ROBERT A. SULANKE

ABSTRACT. Cet article étudiera le catalogage des statistiques sur les chemins de Dyck satisfaisantes la distribution de Narayana et celles satisfaisantes la distribution symétrique de Kreweras-Poupard. Ces deux distributions possedent plusieurs statistiques bien connues. Cet article étendra la liste considerablément et démontrera comment ces statistiques sont reliées par les bijections relativement simples.

This paper investigates cataloging the statistics on the Catalan lattice paths satisfying the Narayana distribution and those satisfying the symmetric Kreweras-Poupard distribution. For both these distributions, there are several known statistics. This paper extends the lists considerably and shows how these statistics relate to one another by moderately simple bijections.

## 1. INTRODUCTION

On $\mathbf{Z}^2$ consider lattice paths having positively directed vertical and horizontal unit steps with 0 denoting a vertical step and 1 denoting a horizontal step. For nonnegative integer $n$, the set of *Catalan paths*, $\mathcal{C}(n)$, is the set of all lattice paths from $(0,0)$ to $(n,n)$ that never run below the line $y = x$. Hence, $|\mathcal{C}(n)|$ is the $n^{th}$ Catalan number.

Here a lattice path *statistic* is either an integer-valued function or a vector-valued function with domain $\mathcal{C}(n)$. Our goal is to determine and catalog the statistics on $\mathcal{C}(n)$ that have the Narayana distribution or the symmetric Kreweras-Poupard distribution of Section 4. We will define the Narayana distribution as

$$N(n,k) = \binom{n-1}{k}\binom{n-1}{k} - \binom{n-1}{k-1}\binom{n-1}{k+1}.$$

Hence, $|\mathcal{C}(0)| = N(0,0) = 1$; $|\mathcal{C}(1)| = N(1,0) = 1$; $|\mathcal{C}(2)| = N(2,0) + N(2,1) = 1+1$; $|\mathcal{C}(3)| = N(3,0)+N(3,1)+N(3,2) = 1+3+1$; $|\mathcal{C}(4)| = N(4,0)+N(4,1)+N(4,2)+N(4,3) = 1+6+6+1$; etc.

For $P = P_1P_2\ldots P_h\ldots P_{2n} \in \mathcal{C}(n)$, **define** the following statistics:

(1) $\text{EVENA}(P) = |\{h : P_{2h} = 0\}|$ = the number of even ascents on $P$,
(2) $\text{VAL}(P) = |\{h : P_hP_{h+1} = 10\}|$ = the number of valleys on $P$,
(3) $\text{DOUBLEA}(P) = |\{h : P_hP_{h+1} = 00\}|$ = the number of double ascents on $P$,
(4) $\text{LONG}(P) = |\{h : P_hP_{h+1}P_{h+2} = 001\}| + |\{h : P_hP_{h+1}P_{h+2} = 110\}|$ = the number of nonfinal maximal constant sequences on $P$.

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Our starting point is the following result initiated by Narayana [1, 3, 6, 9]:

**Proposition 1.** *For fixed $n$, the statistics - EvenA, Val, DoubleA, and Long - are distributed by $N(n,k)$. E.g., $|\{P \in C(n) : \text{EVENA}(P) = k\}| = N(n,k)$.*

*Remark 1.* *Val* and *DoubleA* have been viewed as essentially the same statistic since, for any path in $C(n)$, if $k$ of the noninitial vertical steps are preceded by vertical steps then $n-k-1$ are preceded by horizontal steps. Notice that $N(n,k)$ is symmetric about $(n-1)/2$.

## 2. The even-odd-even-odd condition

We will consider path statistics, $\Theta(P)$, $P \in C(n)$, that count specified local behavior on all *even-odd-even-odd* positioned quadruples of steps of $P$. We will overline the *even-odd* pairs of steps to emphasize the parity of their positions. For example, the path, $P = 01001011 \in C(4)$, may be written as $0\ \overline{10}\ \overline{01}\ \overline{01}\ 1$.

Equivalent to the consideration of catpaths with even-odd pairs emphasized is the study of pairs of lattice paths that only intersect initially and terminally. There, diagonally opposing steps are emphasized. Bijections between pairs of nonintersecting lattice paths and catpaths are recorded in [8].

**Defining path statistics by a "matrix code":** Let $M$ be a 4 by 4 matrix with integer entries. Here we index the rows and the columns of $M$ by $00, 01, 10, 11$ in lieu of the usual $1, 2, 3, 4$. For $P = P_1 \dots P_h \dots P_{2n} \in C(n)$, define $M(P)$ to be a sum over selected entries of $M$ as follows:

$$M(P) = \sum_{1 \le h \le n-2} (M)_{P_{2h}P_{2h+1},\ P_{2h+2}P_{2h+3}}.$$

Moreover, for $t \in \{0,1\}$, define $M_t(P) = M(P) + \chi(P_2 = t)$, where $\chi(A) = 1$ if $A$ is true and 0, otherwise.

That a statistic has a matrix code is equivalent to the *even-odd-even-odd condition* that, given $P \in C(n)$, $M(P)$ depends additively only on the following multiset of subpaths (i.e., a multiset of 4-words on $\{0,1\}$),

$$\{P_{2h}P_{2h+1}P_{2h+2}P_{2h+3} : 1 \le h \le n-3\},$$

and does not depend on the relative positions of these subpaths on $P$ or directly on the length of $P$. Accounting for each second step, $P_2$, is essentially needed for $C(2)$. Once $t \in \{0,1\}$ is fixed, the same summand $\chi(P_2 = t)$ is retained for all $n$.

**Example 1.** Let

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \hat{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the entry $(M')_{10,\ 01} = \hat{1}$ is in the 10-row and the 01-column of $M'$ and thus $M'(P)$ can count *all* noninital odd-even positioned ascents *occurring as* $\overline{10}\ \overline{01}$ on a given path. Moreover, $M'(P)$ counts the noninitial odd-even ascents on a path $P$, while $M(P)$ counts all even-odd ascents. The summand, $\chi(P_2 = 0)$, will account for an initial odd-even ascent on $P$. Hence, with $D = M + M'$,

$$\text{DOUBLEA}(P) = D_0(P) \equiv \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_0 (P).$$

In particular, if $P = 0\ 00\ 01\ 00\ 01\ 11\ 10\ 01\ 11\ 1$, then $\text{DOUBLEA}(P) = D_0(P)$ $= (D)_{00\ 01} + (D)_{01\ 00} + (D)_{00\ 01} + (D)_{01\ 11} + (D)_{11\ 10} + (D)_{10\ 01} + (D)_{01\ 11} + \chi(P_2 = 0)$ $= 2 + 0 + 2 + 0 + 0 + 1 + 0 + 1$.

## 3. Statistics with the Narayana Distribution

**Proposition 2.** *There are exactly* 56 *path statistics,* $\Theta$*, satisfying the* even-odd-even-odd condition *with the summand* $\chi(P_2 = 0)$ *for which* $|\{P \in \mathcal{C}(n) : \Theta(P) = k\}| = N(n, k)$.

**Proposition 3.** *There are exactly* 56 *path statistics,* $\Theta$*, satisfying the* even-odd-even-odd condition *with the summand* $\chi(P_2 = 1)$ *for which* $|\{P \in \mathcal{C}(n) : \Theta(P) = k\}| = N(n, k)$

A matrix code, $M_t$, is called *Narayana code*, if $M_t$ represents a statistic having the Narayana distribution. For $t = 0$, there are 66 Narayana codes; however, each of ten of these represents a statistic that is represented by another matrix. For $t = 1$, there are 91 Narayana codes; however, each of 35 of these represents a statistic that is represented by another matrix.

*The scheme used to prove Propostion 2: The first step:* With computer aid, we can routinely consider all 4 by 4 matrices that are plausible as Narayana codes (with the $\chi(P_2 = 0)$ summand) for $\mathcal{C}(n)$ for small values of $n$.

For $n = 3$, we observe that there are at most eight categories of matrices, $M_0$, that are plausible under the Narayana distribution, $(1, 3, 1)$. Here we used the fact that the $\chi(P_2 = 0)$ summand implies $M_0(0\ \overline{00}\ \overline{11}\ 1) > 0$, $M_0(0\ \overline{01}\ \overline{01}\ 1) > 0$, and $M_0(0\ \overline{01}\ \overline{10}\ 1) > 0$.

Only six of the plausible categories are eventually realized as Narayana codes. They are represented as follows:

$$\begin{bmatrix} - & - & - & 0 \\ - & 1 & 0 & - \\ - & 0 & 1 & - \\ - & - & - & - \end{bmatrix}_0, \begin{bmatrix} - & - & - & 0 \\ - & 0 & 1 & - \\ - & 0 & 1 & - \\ - & - & - & - \end{bmatrix}_0, \begin{bmatrix} - & - & - & 1 \\ - & 0 & 0 & - \\ - & 1 & 0 & - \\ - & - & - & - \end{bmatrix}_0, \begin{bmatrix} - & - & - & 0 \\ - & 1 & 0 & - \\ - & 1 & 0 & - \\ - & - & - & - \end{bmatrix}_0, \begin{bmatrix} - & - & - & 0 \\ - & 0 & 1 & - \\ - & 1 & 0 & - \\ - & - & - & - \end{bmatrix}_0, \begin{bmatrix} - & - & - & 0 \\ - & 0 & 0 & - \\ - & 2 & 0 & - \\ - & - & - & - \end{bmatrix}_0$$

We continue checking in a similar manner using $\mathcal{C}(n)$, for $n = 4 \ldots$, until a set of plausible matrices is completely determined. During our checking stability was apparently reached at $n = 7$, with 66 "candidate matrices" satisfying the Narayana distribution on $\bigcup_{n=0}^{7} \mathcal{C}(n)$.

*The second step:* As expected, we find that three of the candidates are the known Narayana codes, namely those corresponding to the statistics, EVENA, DOUBLEA, and LONGS, of Proposition 1, which are represented as

$$E_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}_0, D_0 = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_0, L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}_0.$$

We then show that any other candidate, say $M_0$, is indeed a Narayana code by establishing a bijection, $\text{BIJ} : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$, such that

$$M_0(\text{BIJ}(P)) = M_0'(P),$$

where $M_0'$ has been shown previously to be a Narayana code. In general, when there is a bijection, BIJ, for which $M_0(\text{BIJ}(P)) = M_0'(P)$, for all $P \in \cup_{n \geq 0}\mathcal{C}(n)$, we will define, using common graph theoretic notions, an *edge* between the *vertices*, $M$ and $M'$. We will write:

$$M \overset{\text{BIJ}}{\longleftrightarrow} M'.$$

Hence, we complete the proof by forming a connected graph with *all* the candidates as vertices and with the bijections (defined in Section 6) producing the edges. A portion of the graph appears in Figure 1.

*Bijection 1.* Interestingly, during the process of constructing the graph proving Proposition 2, two connected components developed. The only known way to "bridge" the two components was to use the neat bijection recently found by Benchekroun and Moszkowski [1]. It is denoted here as:

$$\mu : \{P \in \mathcal{C}(n) : \text{LONG}(P) = k\} \rightarrow \{P \in \mathcal{C}(n) : \text{VAL}(P) = k\}.$$

$\mu$ is defined so that, for $P \in \mathcal{C}(n)$, $\text{LONG}(P) = k$ and $0 < h \leq k$, if $((x_h, y_h), (u_h, v_h))$ is the coordinate designation for the last step of the $h^{th}$ nonfinal long sequence (i.e., long ascents or nonfinal long descents) of $P$, then $(x_h + 1, y_h)$ are the coordinates of the $h^{th}$ valley of $\mu(P)$. See the location of $\mu$ in Figure 1.

## 4. Statistics for the symmetric Kreweras-Poupard distribution

Next consider bivariate statistics on $\mathcal{E}(n,k) = \{P \in \mathcal{C}(n) : |\{h : P_{2h} = 0\}| = k\}$. Given a pair of 4 by 4 matrices, $M$ and $M'$, define $M|M'$ so that $M|M'(P) = (M(P), M'(P))$.

Let

$$symKP(a,b,i,j) = \binom{a}{i}\binom{b}{i}\binom{a}{j}\binom{b}{j} - \binom{a+1}{i+1}\binom{b-1}{i-1}\binom{a-1}{j-1}\binom{b+1}{j+1}.$$

Our reference point here is the symmetric variation of the Kreweras-Poupard theorem [5, 4, 8], which states that the next proposition holds for each of the three pairs:

$$\bar{M}|\bar{M}' = \left[\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0 \\ 1&1&0&0&0&0&0&0 \\ 1&1&0&0&1&0&1&0 \end{array}\right], \quad \bar{\bar{M}}|\bar{\bar{M}}' = \left[\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&0 \\ 1&0&0&0&1&1&1&0 \end{array}\right], \quad \widetilde{M}|\widetilde{M'} = \left[\begin{array}{cccc|cccc} 1&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0 \end{array}\right].$$

**Proposition 4.** *There are exactly* 35 *bivariate statistics defined by matrix pairs,* $M|M'$, *that satisfy*

$$|\{P \in \mathcal{E}(n,k) : (i,j) = M|M'(P)\ \}| = symKP(n-k-1, k, i, j).$$

Here $M|M'$ and $M'|M$ are considered the same. While there are 61 $M|M'$ pairs satisfying the symmetric Kreweras-Poupard distribution on $\mathcal{E}(n,k)$, each of 26 of these represents a bivariate statistic that is represented by another pair.

$$
\begin{tikzcd}
\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{EX3}"] \arrow[d, leftrightarrow, "\text{EX2}"'] & \begin{bmatrix}1&1&0&0\\1&0&1&0\\1&1&0&0\\1&1&0&0\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{EX1}"] \arrow[d, leftrightarrow, "\text{EX2}"'] & \begin{bmatrix}1&0&1&0\\1&0&1&0\\1&1&0&0\\1&1&0&0\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{SL1}"] & \begin{bmatrix}1&1&1&0\\0&0&1&0\\1&1&0&0\\1&1&0&0\end{bmatrix}_0 \arrow[d, leftrightarrow, "\text{ZAP}"'] \\
\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&1&0&0\\1&0&1&0\end{bmatrix}_0 \arrow[d, leftrightarrow, "\text{REV}"'] & \begin{bmatrix}1&1&0&0\\1&0&1&0\\1&1&0&0\\1&0&1&0\end{bmatrix}_0 \arrow[d, leftrightarrow, "\text{REV}"'] & \begin{bmatrix}0&0&0&1\\0&0&0&1\\1&1&0&1\\1&1&0&1\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{IDY}"] \arrow[d, leftrightarrow, "\beta"'] & \begin{bmatrix}1&1&1&1\\0&0&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}_0 \\
\begin{bmatrix}1&1&0&0\\0&1&0&0\\2&1&0&0\\1&1&0&0\end{bmatrix}_0 & \begin{bmatrix}1&1&0&0\\0&0&0&0\\2&2&0&0\\1&1&0&0\end{bmatrix}_0 & \begin{bmatrix}1&1&0&1\\0&0&0&1\\1&1&0&1\\0&0&0&1\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{IDY}"] \arrow[d, Leftrightarrow, "\text{COM}"'] & \begin{bmatrix}2&2&1&1\\0&0&0&0\\1&1&0&0\\0&0&0&0\end{bmatrix}_0 \\
 & \begin{bmatrix}0&0&0&0\\1&0&1&0\\1&1&0&0\\2&1&1&0\end{bmatrix}_0 \arrow[r, leftrightarrow, "\mu"] \arrow[d, leftrightarrow, "\beta"'] & \begin{bmatrix}0&0&1&0\\1&1&1&0\\0&0&1&0\\1&1&1&0\end{bmatrix}_1 & \\
\begin{bmatrix}1&1&0&0\\0&0&0&0\\1&2&0&1\\0&1&0&1\end{bmatrix} \arrow[r, leftrightarrow, "\text{REV}"] & \begin{bmatrix}2&1&1&0\\1&0&1&0\\1&1&0&0\\0&0&0&0\end{bmatrix}_0 \arrow[r, leftrightarrow, "\text{IDY}"] & \begin{bmatrix}1&0&0&0\\1&0&1&1\\1&0&1&1\\0&0&0&1\end{bmatrix}_0 &
\end{tikzcd}
$$

**Figure 1: A portion of the graph of the Narayana codes with $\chi(P_2 = 0)$.**

As before, we can prove Proposition 4 using the scheme of finding candidates and bijective verification. If $\text{BIJ} : \mathcal{E}(n,k) \to \mathcal{E}(n,k)$ is an arbitrary bijection satisfying $M|M'(\text{BIJ}(P)) = \widehat{M}|\widehat{M'}(P)$, we will write $M|M' \overset{\text{BIJ}}{\longleftrightarrow} \widehat{M}|\widehat{M'}$. A portion of the resulting graph in the proof of Proposition 4 appears in Figure 2.

$$
\begin{tikzcd}
\bar{M}|\bar{M}' \arrow[r, leftrightarrow, "\beta"] & \left[\begin{array}{cccc|cccc} 1&1&0&0&1&0&1&0\\ 0&0&0&0&1&0&1&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right] \arrow[r, leftrightarrow, "\mathrm{Sl1}"] \arrow[d, leftrightarrow, "\mathrm{REV}"'] & \left[\begin{array}{cccc|cccc} 1&0&0&0&1&1&1&0\\ 1&0&0&0&0&0&1&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right] \arrow[r, leftrightarrow, "\beta"] \arrow[d, leftrightarrow, "\mathrm{IDY}"'] & \bar{\bar{M}}|\bar{\bar{M}}' \\
& \left[\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&1&0&1&0&0&0&0\\ 0&1&0&1&0&0&1&1 \end{array}\right] & \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&1\\ 1&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{array}\right] &
\end{tikzcd}
$$

**Figure 2: Part of the graph of Kreweras-Poupard codes.**

*Remark 2.* In this section we have limited our attention to bivariate statistics satisfying the even-odd-even-odd condition and the $symKP$ distribution on $\mathcal{E}(n,k)$. There are other bivariate statistics satisfying this condition and this distribution on some other subsets of $\mathcal{C}(n)$ whose cardinality satisfies the Narayana distribution. However, it appears that, for such other subsets, the bivariate statistics are much less abundant.

## 5. Complementary codes

Now we forge a link between some of the Narayana codes with summand $\chi(P_2 = 0)$ and some with $\chi(P_2 = 1)$. Define the *complement* of a matrix $M$, denoted by $M^c$, so that $(M^c)_{ij} = 1 - (M)_{ij}$ for all entries.

For any $P \in \mathcal{C}(n), n > 0$, and for any 4 by 4 matrix $M$, whose components belong to $\{0,1\}$, observe that

$$[M^c(P) + \chi(P_2 = 1)] + [M(P) + \chi(P_2 = 0)] = n - 1.$$

Since $N(n,k)$ is symmetric about $(n-1)/2$ in $k$, it follows that, for any 0-1 matrix, $M$, $M_0$ is a Narayana statistical code if, and only if, $M_1^c$ is a Narayana statistical code. We will write "$M_1^c \overset{\mathrm{COM}}{\Longleftrightarrow} M_0$" for such a pair of equivalent statistical codes.

Since, by Proposition 1, the Narayana number counts catpaths with respect to valleys and since

$$|\{h : P_{2h+2}P_{2h+3} = \overline{10}\} \cup \{h : P_{2h+1}P_{2h+2} = 10\}| + \chi(P_1P_2P_3 = 010) = \begin{bmatrix} 0&0&1&0\\ 1&1&1&0\\ 0&0&1&0\\ 1&1&1&0 \end{bmatrix}_1 (P),$$

the matrix on the right, denoted by $V_1$, is a Narayana code. Locate $V_1$ and $D_0$ in Figure 1.

*Bijection 2.* The following explicit bijection is both interesting and used for the Bijection 9: Let $\alpha : \{P \in \mathcal{C}(n) : \mathrm{Val}(P) = k\} \to \{P \in \mathcal{C}(n) : \mathrm{DoubleA}(P) = k\}$ be **defined** as follows: Each $R = R_1 \dots R_h \dots R_{2n} \in \{P \in \mathcal{C}(n) : \mathrm{Val}(P) = k\}$ is determined uniquely by the sequence of its valleys, $(x_1, y_1) \dots (x_h, y_h) \dots (x_k, y_k)$, say. If we consider the set complements, $\{x'_1, \dots x'_{n-k-1}\} = \{1, \dots, n-1\} - \{x_1, \dots x_k\}$ and $\{y'_1, \dots y'_{n-k-1}\} = \{1, \dots, n-1\} - \{y_1 \dots y_k\}$, then $(y'_1, x'_1) \dots (y'_h, x'_h) \dots (y'_{n-k-1}, x'_{n-k-1})$ will be the valleys of $\alpha(R) = s_1 s_2 \dots s_{2n} \in \{P \in \mathcal{C}(n) : \mathrm{DoubleA}(P) = k\}$.

## 6. Some catpath bijections.

We will now introduce some of the bijections from $\mathcal{C}(n)$ to $\mathcal{C}(n)$ that establish the edges to prove the Propositions 2, 3, and 4. Figures 1 and 2 witness their effects.

*Bijection 3.* The *reverse* of a path is defined as $\text{REV} : \mathcal{C}(n) \to \mathcal{C}(n) : \text{REV}(P) = P'$ so that $P'_h = 1 - P_{2n-h}$. E.g. $\text{REV}(010011) = 001101$.

**Lemma 1.** *For any* $P \in \mathcal{C}(n)$,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_0 (\text{REV}(P)) + \begin{bmatrix} p & h & l & d \\ n & f & j & b \\ o & g & k & e \\ m & e & i & a \end{bmatrix} (\text{REV}(P)) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_0 (P) + \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} (P).$$

*Bijection 4.* $\text{IDY} : \mathcal{C}(n) \to \mathcal{C}(n)$ denotes the identity map, i.e., $\text{IDY}(P) = P$.

The following shows that there are path statistics that have more than one representation as a matrix code.

**Lemma 2.** *For any* $P \in \mathcal{C}(n)$,

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (P) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} (P)$$

*Proof.* Since a given path will have the same number of $\overline{00}$'s as $\overline{11}$'s, it will have the same number of $\overline{00}$'s as $\overline{11}$'s that do not belong to a $\overline{00}\ \overline{11}$ quadruple.

*Bijection 5.* The *exchange of a boomerang, No. 1,* $\text{EX1} : \mathcal{C}(n) \to \mathcal{C}(n)$, is a path isomorphism defined so each $\overline{01}$ or $\overline{10}$ that occurs immediately after any $\overline{00}$ is replaced by the other step pair. E.g. $\text{EX1}(0\ 00\ 10\ 10\ 01\ 11\ 1) = 0\ 00\ 01\ 10\ 01\ 11\ 1$.

EX2 is defined is same way with "$\overline{11}$" replacing "$\overline{00}$."

*Bijection 6.* $\text{EX3} : \mathcal{C}(n) \to \mathcal{C}(n)$, is a path isomorphism defined as follows: Let $\overline{x_1} \ldots \overline{x_m}$ be a maximal subpath of $P$ such that $\overline{x_h} \in \{\overline{01}, \overline{10}\}$. Let $\overline{y_1} \ldots \overline{y_m}$ denote its image under EX3, also with $\overline{y_h} \in \{\overline{01}, \overline{10}\}$. Put $\overline{y_1} = \overline{x_1}$. For $1 \leq h < m$, put $\overline{y_{h+1}} = \overline{y_h}$ when $\overline{x_{h+1}} = \overline{01}$ and $\overline{y_{h+1}} \neq \overline{y_h}$, otherwise.

*Bijection 7.* The *slides of* $\overline{01}$*'s with respect to ascent* on a path results, essentially, when the maximal (possibly empty) subpaths of 01's on both sides of each $\overline{00}$ are interchanged. More precisely, define $\text{SL1} : \mathcal{C}(n) \to \mathcal{C}(n)$ so that each maximal subpath of $P$ of the form $W_1W_2W_3$, where (1) $W_1$ and $W_3$ are maximal and consists only of $\overline{01}$ step pairs and (2) $W_2$ is nonempty and consists only of step pairs from $\{\overline{00}, \overline{01}\}$, is replaced by $W_3W_2W_1$ on $\text{SL1}(P)$.

*Bijection 8.* $\text{ZAP} : \mathcal{C}(n) \to \mathcal{C}(n)$, is a path isomorphism defined so each quadruple $\overline{10}\ \overline{01}$ is exchanged for $\overline{00}\ \overline{11}$, and vice versa. E.g. $\text{ZAP}(00100111) = 00101101$.

*Bijection 9.* For brevity, we will only informally define the useful bijection: $\beta : \mathcal{E}(n,k) \to \mathcal{E}(n,k)$. We will view the "skeleton" of a path, $P$, as the reduced path consisting only of the $\overline{00}$'s and $\overline{11}$'s after all other even-odd step pairs have been removed from $P$. $\beta$ will map the skeleton of $P$ into a new skeleton under $\alpha$ (from Definition 2) where $\alpha$ treats each $\overline{00}$ as just a single ascent $\hat{0}$ and $\overline{11}$ as just a single descent $\hat{1}$. Further, $\beta$ will map each maximal subpath in $\{\overline{01}, \overline{10}\}$ located between two $\overline{00}$'s on $P$ to a valley on the new skeleton, and vice versa. Likewise, $\beta$ will map each maximal subpath in $\{\overline{01}, \overline{10}\}$ located between two $\overline{11}$'s on $P$ to a peak on the new skeleton, and vice versa.

As an example, $\alpha(\hat{0}\hat{1}\hat{0}\hat{0}\hat{1}\hat{0}\hat{1}\hat{1}) = \hat{0}\hat{0}\hat{0}\hat{1}\hat{1}\hat{0}\hat{1}\hat{1}$, by Definition 2. If each $W_h$'s is a maximal subpath on $\{\overline{01}, \overline{10}\}$, then

$$\beta(\ 0\ W_1\ \overline{00}\ W_2\ \overline{11}\ W_3\ \overline{00}\ W_4\ \overline{00}\ W_5\ \overline{11}\ W_6\ \overline{00}\ W_7\ \overline{11}\ W_8\ \overline{11}\ W_9\ 1\ )$$
$$= 0\ W_1\ \overline{00}\ W_3\ \overline{00}\ W_6\ \overline{00}\ W_8\ \overline{11}\ W_2\ \overline{11}\ W_4\ \overline{00}\ W_9\ \overline{11}\ W_5\ \overline{11}\ W_7 1$$

**Proposition 5.**

$$\begin{bmatrix} a & b & c & d \\ e & f & g & d \\ i & j & k & d \\ m & n & o & d \end{bmatrix}_0 \overset{\beta}{\longleftrightarrow} \begin{bmatrix} m & n & o & d \\ e & f & g & d \\ i & j & k & d \\ a & b & c & d \end{bmatrix}_0$$

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Boise State University, Boise, ID, USA 83725
*E-mail address*: sulanke@math.idbsu.edu