BI-INFINITARY CODES AND FORMAL POWER SERIES

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Abstract - In this paper, we obtain two characterizations of bi-infinitary codes using formal power series as well as unambiguous finite automata with the associated characteristic series.

Résumé - Dans cet article, nous obtenons deux caractérisations des codes bi-infinitaires, au moyen de séries formelles et d'automates finis non-ambigus ainsi que des séries caractéristiques associées.

1. INTRODUCTION

Combinatorics on words is an area which grew from group theory and probability theory, but which in recent times appears frequently in problems of computer science dealing with automata and formal languages. The theory of automata and formal languages and subsequently, the theory of codes were developed, taking motivation from computer science and information theory.

The theory of codes was initiated by Schützenberger in the mid fifties [10] and extensive investigations were made in depth by him and many others [1,8]. Do Long Van [3,4,5] introduced the notion of infinitary codes which is a natural generalization of the notion of codes for finitary languages (sets of finite words) to infinitary languages (sets consisting of finite and infinite words). Motivated by this and the study of bi-infinite words [9], the notion of bi-infinitary codes which is the most general notion of codes to bi-infinitary languages (sets consisting of finite, left-infinite, right-infinite or infinite, bi-infinite words) has been introduced in [6] and generalized results were established. A different approach to the study of codes in the context of infinite or bi-infinite words is to consider infinite factorization using only finite words. This approach is examined in [2].

In this paper, the study of bi-infinitary codes is continued. Formal power series are used as a tool to establish results in the study. We extend the notion of formal power series on the monoid ${}^{\infty}A^{\infty}$. We give a characterization for a bi-infinitary language to be a bi-infinitary code in terms of formal power series. Again, we consider finite automata recognizing bi-infinitary languages and the formal power series associated with the automata. We define star operation on finite automata, thereby, on unambiguous finite automata recognizing bi-infinitary languages and establish a characterization for such bi-infinitary languages to be bi-infinitary codes, using the characteristic series associated with the automata.

2 PRELIMINARIES

Let A be a finite alphabet. A^* denotes the set of all finite words on A, e is the empty word and $A^+ = A^* - \{e\}$. The length of a word x in A^* is denoted by $|x|$.

A^N is the set of all right–infinite words $\alpha = \alpha_1\alpha_2 ... \alpha_n ...$ ($\alpha_n \in A$). In the same way, A^{-N} is the set of all left–infinite words $\alpha = ... \alpha_{-n} ... \alpha_{-2}\alpha_{-1}\alpha_0$ ($\alpha_{-n} \in A$). A^Z denotes the set of all bi-infinite words $\alpha = ... \alpha_{-(n+1)}\ \alpha_{-n} ... \alpha_{-2}\ \alpha_{-1}\ \alpha_0\ \alpha_1\ \alpha_2 ... \alpha_n\ \alpha_{n+1} ...$ ($\alpha_i \in A$). Every (bi–)infinite word has a countable length ω.

For any $X \subseteq A^+$, X^ω (${}^\omega X$, ${}^\omega X^\omega$), denotes the set of all right–infinite (left–infinite, bi–infinite) words of the form $x_1 x_2 ...$ $(...x_{-2}x_{-1}x_0, ...x_{-2}x_{-1}x_0x_1x_2...)$ for $x_i \in X$. In particular, if $x \in A^+$, then, $x^\omega = xxx...$, ${}^\omega x = ...xxx$ and ${}^\omega x^\omega = ...xxx...$. We write $A^\infty = A^* \cup A^N$, ${}^\infty A = A^* \cup A^{-N}$ and ${}^\infty A^\infty = A^* \cup A^N \cup A^{-N} \cup A^Z$.

A product operation . on elements of ${}^\infty A^\infty$ is defined as follows:

$$\alpha.\beta = \begin{cases} \alpha, \text{ if } \alpha \in A^N \cup A^Z \\ \alpha\beta, \text{ if } \alpha \in A^* \cup A^{-N}, \beta \in A^* \cup A^N \\ \beta, \text{ if } \alpha \in A^* \cup A^{-N}, \beta \in A^{-N} \cup A^Z. \end{cases}$$

The product is associative and therefore ${}^\infty A^\infty$ is a monoid. This monoid has A^*, A^∞ and ${}^\infty A$ as its submonoids. For simplicity, instead of $\alpha.\beta$, we write $\alpha\beta$. For any $X \subseteq {}^\infty A^\infty$, we denote by X^*, the submonoid of ${}^\infty A^\infty$ generated by X and write $X^+ = X^* - \{e\}$. If α is a word, instead of $\{\alpha\}^*$ ($\{\alpha\}^+$), we write $\alpha^*(\alpha^+)$.

By a bi–infinitary language on an alphabet A, we mean, a subset of ${}^\infty A^\infty$.

For any $X \subseteq {}^\infty A^\infty$, we write $X_{fin} = X \cap A^*$,

$X_{inf} = X \cap A^N$, $X_{-inf} = X \cap A^{-N}$, $X_{biinf} = X \cap A^Z$,

$X^{(0)} = \{e\}$, $X^{(1)} = X$,

$X^{(\overrightarrow{n})} = \{(x_1,x_2,...,x_n) \mid x_1,x_2,...,x_{n-1} \in X_{fin}, x_n \in X_{fin} \cup X_{inf}\}$ for $n \geq 2$,

$X^{(\overleftarrow{n})} = \{(x_1,x_2,...,x_n) \mid x_1 \in X_{fin} \cup X_{-inf}, x_2,x_3,...,x_n \in X_{fin}\}$ for $n \geq 2$,

$X^{(\overleftrightarrow{n})} = \{(x_1,x_2,...,x_n) \mid x_1 \in X_{-inf}, x_n \in X_{inf}, x_2,x_3,...,x_{n-1} \in X_{fin}\}$ for $n \geq 2$

$X^{(n)} = X^{(\overrightarrow{n})} \cup X^{(\overleftarrow{n})} \cup X^{(\overleftrightarrow{n})}$ for $n \geq 2$, $\quad X^{(*)} = \bigcup_{n \geq 0} X^{(n)}$

We say that a word $\alpha \in {}^\infty A^\infty$ has a factorization on elements of X if $\alpha = x_1x_2...x_n$ for some $(x_1, x_2,..., x_n) \in X^{(*)}$. Here we note that α is expressed as a finite product of elements of X.

Definition 2.1: A subset X of ${}^{\omega}A^{\omega}$ is called a bi–infinitary code if every word $\alpha \in {}^{\omega}A^{\omega}$ has atmost one factorization on elements of X. More precisely, X is a bi–infinitary code if for any n, m $\geq$ 1 and for any $(x_1,x_2,...,x_n) \in X^{(n)}$, $(x'_1, x'_2, ..., x'_m) \in X^{(m)}$, the equality $x_1x_2...x_n = x'_1x'_2...x'_m$ implies $n = m$ and $x_i = x'_i$ $(i=1,2,...,n)$.

Example 2.2: If A = {a,b}, the subset X = $\{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ba\}$ is a bi–infinitary code whereas the subset Y = $\{{}^{\omega}(ab)^{\omega}, {}^{\omega}a, b^{\omega}, ab\}$ is not a bi–infinitary code, since we have, ${}^{\omega}ab^{\omega} = {}^{\omega}a.ab.b^{\omega} = {}^{\omega}a.b^{\omega}$.

Finite automata recognizing separately languages of finite words, languages of right–infinite words and languages of bi–infinite words are known [7,9]. Based on these notions, we now define a finite automaton recognizing a bi–infinitary language.

Definition 2.3: A finite automaton M over an alphabet A is a seven–tuple
$M = (Q, A, E_M, I_{fin}, I_{linf}, T_{fin}, T_{rinf})$ where

- Q is a finite set of states;
- A is a finite alphabet;
- E_M is a subset of $Q \times A \times Q$, called the set of arrows;
- $I_{fin} \subseteq Q$ is a set of initial states;
- $I_{linf} \subseteq Q$ is a set of left–infinite repetitive states;
- $T_{fin} \subseteq Q$ is a set of final states;
- $T_{rinf} \subseteq Q$ is a set of right–infinite repetitive states.

An arrow (p,a,q) is also denoted by $p \xrightarrow{a} q$. A path in M is either a finite or right–infinite or left–infinite or bi–infinite sequence, namely, $c = c_1c_2...c_m$ or $c = c_1c_2...$ or $c = ...\ c_{-2}c_{-1}c_0$ or $c = ...\ c_{-2}c_{-1}c_0c_1c_2\ ...$ respectively. The arrows in the path are consecutive in the sense $c_{i+1} : q_i \xrightarrow{a_{i+1}} q_{i+1}$ where

(i) for finite c, $0 \leq i \leq m-1$,
(ii) for right–infinite c, $i \in N \cup \{0\}$ (N is the set of all positive integers),
(iii) for left–infinite c, $i \in Z^-$ (the set of all negative integers),
(iv) for bi–infinite c, $i \in Z$ (the set of all integers).

The word α corresponding to a path c is either $\alpha = a_1a_2\ ...\ a_m$ or $\alpha = a_1a_2\ ...$ or $\alpha = ...\ a_{-2}a_{-1}a_0$ or $\alpha = ...\ a_{-2}a_{-1}a_0a_1a_2...$ and is called the label of the path c.

We denote by $c : p \xrightarrow{f} q$, a finite path c of label f which begins at $p \in Q$ and terminates at $q \in Q$.

$c : p \xrightarrow{u}> T_{rinf}$ denotes a right–infinite path c of label u which begins at $p \in Q$ and passes through the set T_{rinf} infinitely often on the right.

$c : I_{linf} >\xrightarrow{v} q$ denotes a left–infinite path c of label v which passes through the set I_{linf} infinitely often on the left and terminates at $q \in Q$.

$c : I_{linf} >\xrightarrow{w}> T_{rinf}$ denotes a bi–infinite path c of label w which passes through I_{linf} infinitely often on the left and through T_{rinf} infinitely often on the right.

We omit *the label* in the arrow of a path when it is not needed. We note that the finite automaton moves only from left to right. We let

$L^*(M) = \{f \in A^* \mid \exists\, (c : i \xrightarrow{f} t), \text{ with } i \in I_{fin},\ t \in T_{fin}\}$

$L^N(M) = \{u \in A^N \mid \exists\, (c : i \xrightarrow{u}> T_{rinf}) \text{ with } i \in I_{fin}\}$

$L^{-N}(M) = \{v \in A^{-N} \mid \exists\, (c : I_{linf} >\xrightarrow{v} q) \text{ with } q \in T_{fin}\}$

$L^Z(M) = \{w \in A^Z \mid \exists\, (c : I_{linf} >\xrightarrow{w}> T_{rinf})\}$

and $L(M) = L^*(M) \cup L^N(M) \cup L^{-N}(M) \cup L^Z(M)$.

We define a bi–infinitary language $L \subseteq {}^\infty A^\infty$ to be recognizable if there is a finite automaton M over A such that $L = L(M)$.

Example 2.4: Consider the finite automaton $M=(Q,A,E_M,I_{fin},I_{linf},T_{fin},T_{rinf})$ over A where $Q = \{2,3,4\}$; $A = \{a,b,c\}$; $I_{fin} = I_{linf} = \{2\}$; $T_{fin} = T_{rinf} = \{4\}$. E_M is given by Fig.1.

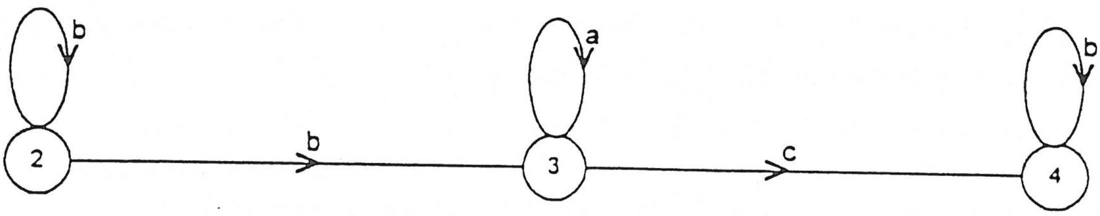

Fig.1

Now, $L(M) = b^+a^*cb^* \cup {}^\omega ba^*cb^\omega \cup {}^\omega ba^*cb^* \cup b^+a^*cb^\omega$.

We now define a reduced finite automaton.

Definition 2.5: A finite automaton $M = (Q,A,E_M,I_{fin},I_{linf},T_{fin},T_{rinf})$ over A is called reduced if the following four conditions hold:

(i) Every $q \in Q$ is accessible from $I_{fin} \cup I_{linf}$, i.e., there exists a finite path beginning at a state of $I_{fin} \cup I_{linf}$ and terminating at q.

(ii) Every $q \in Q$ is coaccessible from $T_{fin} \cup T_{rinf}$, i.e., there exists a finite path beginning at q and terminating at a state in $T_{fin} \cup T_{rinf}$.

(iii) The set T_{rinf} is contained in $\{q \in Q \mid \exists\, (c : q \longrightarrow\!\!> T_{rinf})\}$.

(iv) The set I_{linf} is contained in $\{q \in Q \mid \exists\, (c : I_{linf} >\!\!\longrightarrow q)\}$.

We note that the finite automaton given in example 2.4 is reduced.

3 FORMAL POWER SERIES ON $^{\omega}A^{\omega}$

In this section, we obtain a characterization of bi–infinitary codes in terms of formal power series on $^{\omega}A^{\omega}$.

Definition 3.1: A formal power series (or simply 'series') on $^{\omega}A^{\omega}$ is a function $S: {^{\omega}A^{\omega}} \to N \cup \{0\} \cup \{\omega\}$. The series S can be written in the form $S = \sum_{\alpha \in {^{\omega}A^{\omega}}} (S, \alpha)\alpha$, where (S, α) denotes the image of α by S.

The set $\operatorname{Supp} S = \{ \alpha \in {^{\omega}A^{\omega}} \mid (S, \alpha) \neq 0 \}$ is called support of the series S. A series S is called characteristic if $(S, \alpha) \leq 1$ for all α.

To every bi–infinitary language X, we associate a characteristic series $\underline{X}$ defined by

$$(\underline{X}, \alpha) = \begin{cases} 1 \text{ if } \alpha \in X \\ 0 \text{ if } \alpha \notin X \end{cases}$$

Given two formal power series S and T on $^{\omega}A^{\omega}$, we define S+T and ST as follows: for any $\alpha \in {^{\omega}A^{\omega}}$,

$$(S+T, \alpha) = (S, \alpha) + (T, \alpha)$$

$$(ST, \alpha) = \sum_{\substack{\beta\tau=\alpha \\ \beta \in A^N \cup A^Z \Rightarrow \tau = e \\ \tau \in A^{-N} \cup A^Z \Rightarrow \beta = e}} (S, \beta)\, (T, \tau)$$

Proposition 3.2

For any formal power series T, S, R on $^*A^*$, we have,

(i) $T+S = S+T$; $(T+S) + R = T + (S+R)$

(ii) $(TS)R = T(SR)$

(iii) $\mathrm{Supp}\,(S+T) = \mathrm{Supp}\, S \cup \mathrm{Supp}\, T$

(iv)
$$\mathrm{Supp}(ST) = \begin{cases} \mathrm{Supp}\, S.[(\mathrm{Supp}\, T)_{fin} \cup (\mathrm{Supp}\, T)_{inf}] & \text{if } (T,e)\neq 0, (S,e)=0. \\ [(\mathrm{Supp}\, S)_{fin} \cup (\mathrm{Supp}\, S)_{-inf}].\mathrm{Supp}\, T & \text{if } (T,e)=0, (S,e)\neq 0. \\ \mathrm{Supp}\, S.\mathrm{Supp}\, T & \text{if } (T,e)\neq 0, (S,e)\neq 0. \\ [(\mathrm{Supp}\, S)_{fin} \cup (\mathrm{Supp}\, S)_{-inf}].[(\mathrm{Supp}T)_{fin} \cup (\mathrm{Supp}\, T)_{inf}] & \text{if } (T,e)=0, (S,e)=0. \end{cases}$$

The statements can be proved easily.

By virtue of proposition 3.2, we can consider nth power S^n of a series S and therefore, we introduce the star operation on formal power series, $S^* = \sum_{n\geq 0} S^n$ where $S^0 = \underline{\{e\}}$.

Proposition 3.3: For any formal power series S on $^*A^*$, $\mathrm{Supp}(S^*) = (\mathrm{Supp}\, S)^*$.

Proof: We have two cases.

Case (i): Suppose $(S,e) \neq 0$. Then, using proposition 3.2 (iv),

$$\mathrm{Supp}(S^*) = \bigcup_{n\geq 0} \mathrm{Supp}(S^n) = \bigcup_{n\geq 0} (\mathrm{Supp}\, S)^n = (\mathrm{Supp}\, S)^*.$$

Case (ii): Suppose $(S,e) = 0$.

Now, $$\mathrm{Supp}(S^*) = \bigcup_{n\geq 0} \mathrm{Supp}(S^n) = \{e\} \cup (\mathrm{Supp}\, S) \cup \Big(\bigcup_{n\geq 2} \mathrm{Supp}(S^n)\Big)$$

$$= \{e\} \cup (\mathrm{Supp}\, S) \cup (\mathrm{Supp}\, S)_{inf} \cup (\mathrm{Supp}\, S)_{-inf} \cup (\mathrm{Supp}\, S)_{biinf} \cup \Big(\bigcup_{n\geq 2} \mathrm{Supp}(S^n)\Big)$$

On using proposition 3.2 (iv), we have

$$
\begin{aligned}
\mathrm{Supp}(S^*) = \{e\} &\cup (\mathrm{Supp}\ S) \cup (\mathrm{Supp}\ S)_{inf} \cup (\mathrm{Supp}\ S)_{-inf} \cup (\mathrm{Supp}\ S)_{biinf} \\
&\cup \Big[\bigcup_{n\geq 2} \Big[((\mathrm{Supp} S)_{fin})^n \cup \Big(\bigcup_{i=1}^{n-1} ((\mathrm{Supp}\ S)_{fin})^i\, (\mathrm{Supp}\ S)_{inf}\Big) \\
&\cup \Big(\bigcup_{i=1}^{n-1} (\mathrm{Supp}\ S)_{-inf}\, ((\mathrm{Supp}\ S)_{fin})^i\Big) \\
&\cup \Big(\bigcup_{i=0}^{n-2} (\mathrm{Supp}\ S)_{-inf}\, ((\mathrm{Supp}\ S)_{fin})^i\, (\mathrm{Supp}\ S)_{inf}\Big)\Big]\Big]
\end{aligned}
$$

But we note that

$$
\begin{aligned}
(\mathrm{Supp}\ S)^n = &\Big[\bigcup_{i=1}^{n-1} ((\mathrm{Supp}\ S)_{fin})^i\, (\mathrm{Supp}\ S)_{inf}\Big] \cup \Big[\bigcup_{i=1}^{n-1} (\mathrm{Supp}\ S)_{-inf}((\mathrm{Supp}\ S)_{fin})^i\Big] \\
&\cup \Big[\bigcup_{i=0}^{n-2} (\mathrm{Supp}\ S)_{-inf}((\mathrm{Supp}\ S)_{fin})^i(\mathrm{Supp}\ S)_{inf}\Big] \cup [(\mathrm{Supp}\ S)_{fin}]^n \\
&\cup (\mathrm{Supp}\ S)_{inf} \cup (\mathrm{Supp}\ S)_{-inf} \cup (\mathrm{Supp}\ S)_{biinf} \text{ for all } n \geq 2.
\end{aligned}
$$

As a consequence, we have,

$$
\begin{aligned}
\mathrm{Supp}(S^*) &= \{e\} \cup (\mathrm{Supp}\ S) \cup \Big(\bigcup_{n\geq 2} (\mathrm{Supp}\ S)^n\Big) \\
&= \{e\} \cup \Big(\bigcup_{n\geq 1} (\mathrm{Supp}\ S)^n\Big) = (\mathrm{Supp}\ S)^*.
\end{aligned}
$$

Definition 3.4: For any $X \subseteq {}^{\infty}A^{\infty}$ and any $\alpha \in {}^{\infty}A^{\infty}$, we define

$F_X^n(\alpha) = \{(x_1, x_2, \ldots, x_n) \in X^{(n)} \mid \alpha = x_1x_2\ldots x_n\}$ $(n \geq 1)$ and $F_X(\alpha) = \bigcup_{n\geq 1} F_X^n(\alpha)$.

Obviously, X is a bi–infinitary code if and only if Card $(F_X(\alpha)) \leq 1$ for any $\alpha \in {}^{\infty}A^{\infty}$, where Card(Y) stands for cardinality of a set Y.

We now obtain a characterization of bi–infinitary codes in terms of formal power series on ${}^{\infty}A^{\infty}$.

Theorem 3.5: For any subset X of ${}^{\infty}A^{\infty} - \{e\}$, X is a bi–infinitary code if and only if the series $(\underline{X})^*$ is characteristic.

Proof: We first claim that for any $\alpha \in {}^{\infty}A^{\infty}$, $((\underline{X})^n,\alpha) = \mathrm{Card}\,(F_X^n(\alpha))$ for all $n\geq 1$. For $n=1$, the statement is true. We assume that the statement is true for n. We prove it for $n+1$. We have, $((\underline{X})^{n+1}, \alpha) = ((\underline{X})^n\, \underline{X}, \alpha)$

$$
= \sum_{\substack{\beta\tau = \alpha \\ \beta \in A^{N} \cup A^{Z} \Rightarrow \tau = e \\ \tau \in A^{-N} \cup A^{Z} \Rightarrow \beta = e}} ((\underline{X})^n, \beta)\, (\underline{X}, \tau) \qquad = \sum_{\substack{\beta\tau = \alpha \\ \beta \in {}^{\infty}A - \{e\},\ \tau \in A^{\infty} - \{e\}}} ((\underline{X})^n, \beta)\, (\underline{X}, \tau)
$$

since $(\underline{X},e) = 0$ and $((\underline{X})^n,e)=0$

$$= \sum_{\substack{\beta\tau = \alpha \\ \beta \in (X_{fin})^n \cup X_{-inf}(X_{fin})^{n-1} \\ \tau \in X}} ((\underline{X})^n, \beta)\,(\underline{X}, \tau) \qquad = \sum_{\substack{\beta\tau = \alpha \\ \beta \in (X_{fin})^n \cup X_{-inf}(X_{fin})^{n-1} \\ \tau \in X}} \text{Card}\,(F^n_X(\beta))\,.\,\text{Card}\,(F^1_X(\tau)),$$

since $(\underline{X}, \tau) = 0$ for $\tau \notin X$ and $((\underline{X})^n, \beta) = 0$ for $\beta \notin (X_{fin})^n \cup X_{-inf}(X_{fin})^{n-1}$

$$= \text{Card}\,(F^{n+1}_X(\alpha)).$$

We now prove that if X is a bi–infinitary code, then the series $(\underline{X})^*$ is characteristic. The converse of the statement follows by retracing the steps.

If X is a bi–infinitary code, then for any $\alpha \in {}^{\omega}A^{\omega}$, we have,

$$1 \geq \text{Card}\,(F_X(\alpha)) = \text{Card}\Big(\bigcup_{n\geq 1} F^n_X(\alpha)\Big) = \sum_{n\geq 1} \text{Card}\,(F^n_X(\alpha))$$

$$= \sum_{n\geq 1} ((\underline{X})^n,\alpha) = \sum_{n\geq 0} ((\underline{X})^n,\alpha) \text{ since } ((\underline{X})^n,e) = 0 \text{ for all } n \geq 1$$

$$= ((\underline{X})^*,\alpha).$$ Hence, $(\underline{X})^*$ is a characteristic series.

4 FINITE AUTOMATON M^* OVER A

In this section, we exhibit a characterization of bi–infinitary codes in terms of unambiguous finite automata with star operation, through characteristic series associated with the automata.

Definition 4.1: To every given finite automaton $M=(Q,A,E_M,I_{fin},I_{linf},T_{fin},T_{rinf})$, we associate a finite automaton M^* defined by the following steps:

(i) Let $B = (Q \cup \{\mu\}, A, E_B, \{\mu\}, \phi, \{\mu\}, \phi)$ be a finite automaton with μ a new state and $E_B = E_M \cup J \cup K \cup G$ where

$$\begin{array}{lll} J & = & \{ (\mu,a,q) \mid \exists\, i \in I_{fin} : (i, a, q) \in E_M \} \\ K & = & \{ (p,a,\mu) \mid \exists\, t \in T_{fin} : (p, a, t) \in E_M \} \\ G & = & \{ (\mu,a,\mu) \mid \exists\, i \in I_{fin}, \exists\, t \in T_{fin} : (i,a,t) \in E_M \} \end{array}$$

(ϕ stands for empty set)

(ii) Let $C = (Q \cup \{\mu'\}, A, E_C, \{\mu'\}, \phi, \phi, T_{rinf})$ be a finite automaton with μ', a new state and $E_C = E_M \cup R$ where $R = \{ (\mu', a, q) \mid \exists\, i \in I_{fin} : (i, a, q) \in E_M \}$

(iii) Let $D = (Q \cup \{\mu''\}, A, E_D, \phi, I_{linf}, \{\mu''\}, \phi)$ be a finite automaton with μ'', a new state and $E_D = E_M \cup S$ where $S = \{(p, a, \mu'') \mid \exists\, t \in T_{fin} : (p, a, t) \in E_M \}$.

(iv) Let $H = (Q, A, E_H, \phi, I_{linf}, \phi, T_{rinf})$ be a finite automaton with $E_H = E_M$.

Let F be the disjoint union of the finite automata B, C, D and H, keeping

(i) the states μ, μ' and μ'' equal, i.e., $\mu = \mu' = \mu''$.

(ii) the sets of left–infinite repetitive states of the finite automata H and D, the same,

(iii) the sets of right–infinite repetitive states of the finite automata H and C, the same.

The finite automaton M^* over A is defined as a reduced part of F, taking $\mu = 1$.

Example 4.2: Let M be the finite automaton considered in example 2.4. The finite automaton F as in the definition 4.1 is given by

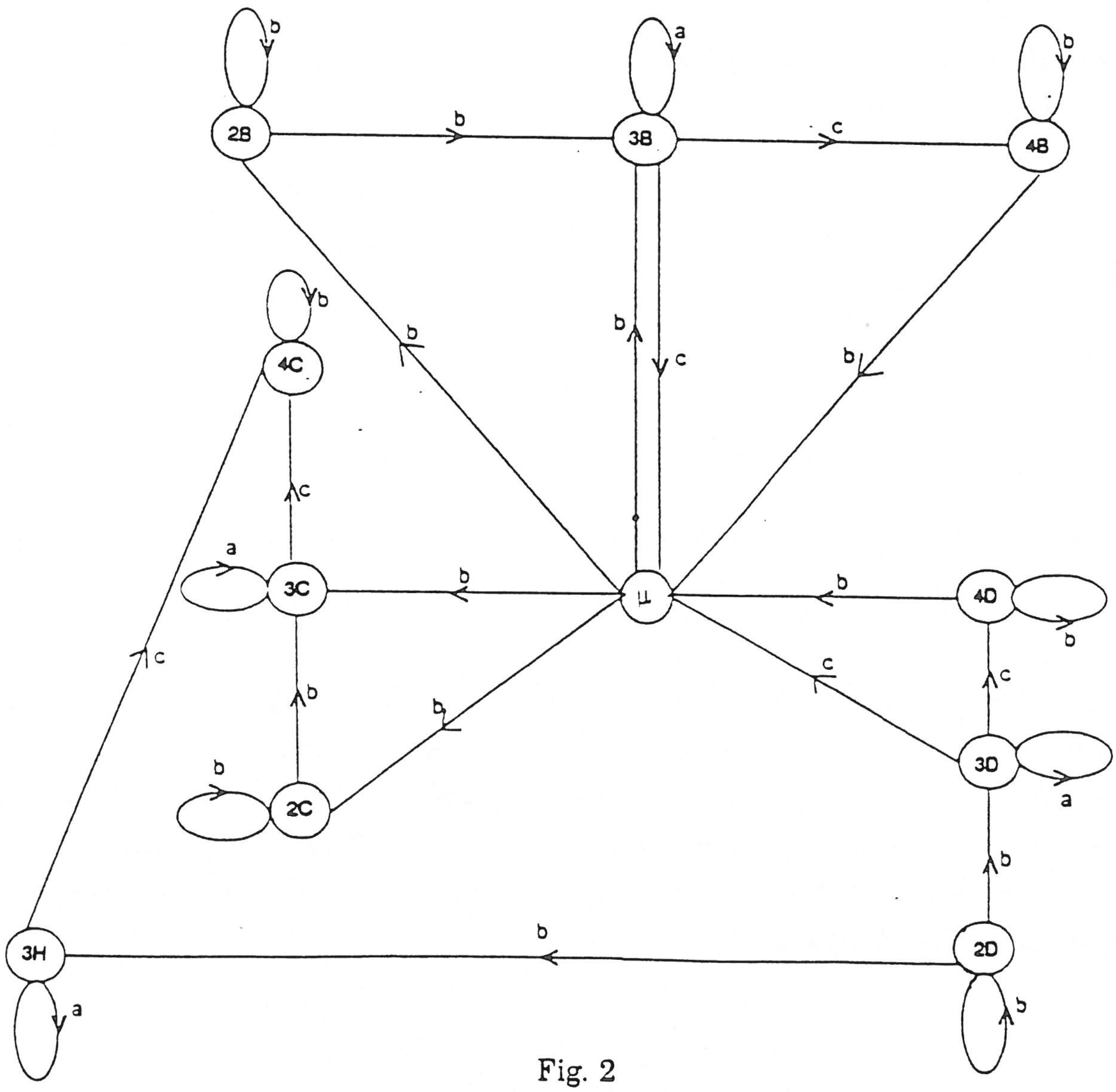

Fig. 2

$$F = (Q', A, E_F, \{\mu\}, I'_{linf}, \{\mu\}, T'_{rinf})$$

where

$$\begin{aligned} Q' &= \{2B, 3B, 4B, 2C, 3C, 4C, 2D, 3D, 4D, 3H, \mu\}; \\ A &= \{a,b,c\}; \\ I'_{linf} &= \{2D\}; \\ T'_{rinf} &= \{4C\}; \end{aligned}$$

E_F is given by Fig.2.

Since the reduced part of F is itself, the finite automaton M^* is obtained from F by simply taking $\mu = 1$. Clearly,

$$\begin{aligned} L(M^*) = \; & (b^+a^*cb^*)^* \cup (b^+a^*cb^*)^* (b^+a^*cb^\omega) \\ & \cup ({}^\omega ba^*cb^*)\,(b^+a^*cb^*)^* \\ & \cup ({}^\omega ba^*cb^*)\,(b^+a^*cb^*)^*\,(b^+a^*cb^\omega) \\ & \cup {}^\omega ba^*cb^\omega. \end{aligned}$$

Definition 4.3: Given a finite automaton $M = (Q, A, E_M, I_{fin}, I_{linf}, T_{fin}, T_{rinf})$ over A, we denote by $|M|$, the formal power series defined as follows:

$$(|M|,\alpha) = \begin{cases} \text{Card}\,\{c : i \xrightarrow{\alpha} t \mid i \in I_{fin},\ t \in T_{fin}\} & \text{if } \alpha \in A^* \\ \text{Card}\,\{c : i \xrightarrow{\alpha}\!> T_{rinf} \mid i \in I_{fin}\} & \text{if } \alpha \in A^N \\ \text{Card}\,\{c : I_{linf} >\!\!\xrightarrow{\alpha} t \mid t \in T_{fin}\} & \text{if } \alpha \in A^{-N} \\ \text{Card}\,\{c : I_{linf} >\!\!\xrightarrow{\alpha}\!> T_{rinf}\} & \text{if } \alpha \in A^Z \end{cases}$$

Obviously, we have $L(M) = \text{Supp}\,|M|$. (4.1)

Proposition 4.4: Let M be a finite automaton over A and $X \subseteq {}^\omega A^\omega - \{e\}$.

(i) If $L(M) = X$, then $L(M^*) = X^*$

(ii) If $|M| = \underline{X}$, then $|M^*| = (\underline{X})^*$.

Proof: We consider the finite automaton $F = (Q', A, E_F, I'_{fin}, I'_{linf}, T'_{fin}, T'_{rinf})$ used in the definition of the finite automaton M^* over A. We say that a path $c : \mu \longrightarrow \mu$ ($c : \mu \longrightarrow\!> T'_{rinf}$ or $c : I'_{linf} >\!\!\longrightarrow \mu$ or $c : I'_{linf} >\!\!\longrightarrow\!> T'_{rinf}$) in F is simple if it is non–empty and none of its interior vertices is μ. Evidently, every path $c : \mu \longrightarrow \mu$ ($c : \mu \longrightarrow\!> T'_{rinf}$ or $c : I'_{linf} >\!\!\longrightarrow \mu$ or $c : I'_{linf} >\!\!\longrightarrow\!> T'_{rinf}$) in F has a unique factorization into simple paths.

Let now S be the formal power series defined by

$$(S,\alpha) = \begin{cases} \text{Card}\ \{c : \mu \xrightarrow{\alpha} \mu \ \mid c \text{ is simple}\} \text{ if } \alpha \in A^* \\ \text{Card}\ \{c : \mu \xrightarrow{\alpha}> T'_{rinf} \mid c \text{ is simple}\} \text{ if } \alpha \in A^N \\ \text{Card}\ \{c : I'_{linf} >\!> \xrightarrow{\alpha} \mu \mid c \text{ is simple}\} \text{ if } \alpha \in A^{-N} \\ \text{Card}\ \{c : I'_{linf} >\!> \xrightarrow{\alpha}> T'_{rinf} \mid c \text{ is simple}\} \text{ if } \alpha \in A^Z. \end{cases}$$

Then, we have $|F| = S^*$. (4.2)

We now prove that Supp S = X. (4.3)

Since $X \subseteq {}^*A^* - \{e\}$ and a simple path is non–empty, the empty word e does not belong to both X and Supp S. For $a \in A$, we have $(S,a) = 1$ if and only if $a \in X$ by the construction of F in the definition 4.1. For α with $|\alpha| \geq 2$, we distinguish two cases.

Case (i): Suppose $|\alpha| < \omega$.

Let $\alpha = afb$ for $a,b \in A$, $f \in A^*$. Every simple path $\mu \xrightarrow{\alpha} \mu$ in F has a unique decomposition of the form $c : \mu \xrightarrow{a} p \xrightarrow{f} q \xrightarrow{b} \mu$ (4.4)

with $p,q \in Q$. So, there exists atleast a successful path in M with the same label α, i.e.,

$i \xrightarrow{a} p \xrightarrow{f} q \xrightarrow{b} t$, $i \in I_{fin}$, $t \in T_{fin}$ (4.5)

Conversely, every successful path of label α in M corresponds with a simple path $\mu \xrightarrow{\alpha} \mu$ in F. We have thus shown that $(S,\alpha) \neq 0$ if and only if $\alpha \in X$ (4.6)

Case (ii): Suppose $|\alpha| = \omega$.

Let $\alpha = au$ for $a \in A$, $u \in A^N$. Then the argument is analogous to case (i), in which instead of (4.4) and (4.5), we use the following (4.4a) and (4.5a),

$c: \quad \mu \xrightarrow{a} p \xrightarrow{u}> T'_{rinf}$ (4.4a)

$i \xrightarrow{a} p \xrightarrow{u}> T_{rinf}$, $i \in I_{fin}$ (4.5a)

with $p \in Q$ and obtain (4.6).

Let $\alpha = va$ for $a \in A$, $v \in A^{-N}$. Then, similar to (4.4a) and (4.5a), we have the following (4.4b) and (4.5b),

$c: \quad I'_{linf} >\!> \xrightarrow{v} p \xrightarrow{a} \mu$ (4.4b)

$I_{linf} >\!> \xrightarrow{v} p \xrightarrow{a} t$, with $t \in T_{fin}$, $p \in Q$ (4.5b)

and the argument is similar to the case when $\alpha = au$, where $a \in A$, $u \in A^N$ and obtain (4.6).

Let α = vau, where a $\in$ A, v $\in A^{-N}$ and u $\in A^N$. Then we have

$$c: \quad I'_{linf} \gg\!\xrightarrow{v}> p \xrightarrow{a} q \xrightarrow{u}\gg T'_{rinf} \tag{4.4c}$$

$$I_{linf} \gg\!\xrightarrow{v}\gg p \xrightarrow{a} q \xrightarrow{u}\gg T_{rinf}, \text{ with } p, q \in Q \tag{4.5c}$$

The argument can be done similar to previous cases and we can obtain (4.6).

Now, combining proposition 3.3, and the equations (4.1), (4.2) and (4.3), we obtain, $L(M^*)=\mathrm{Supp}\ |M^*| = \mathrm{Supp}\ |F| = \mathrm{Supp}\ S^* = (\mathrm{Supp}\ S)^* = X^*$. Thus, (i) holds.

We now prove (ii). From $|M| = \underline{X}$, uniqueness of the paths 4.5, 4.5a, 4.5b, 4.5c will follow corresponding to the paths 4.4, 4.4a, 4.4b, 4.4c respectively. Hence $(S,\alpha) \le 1$ and $(S,\alpha) = 1$ if and only if $\alpha \in X$, i.e., if and only if $(\underline{X},\alpha) = 1$. Thus, $S = \underline{X}$ and so, we have, $|M^*| = |F| = S^* = (\underline{X})^*$.

Definition 4.5: A finite automaton $M = (Q, A, E_M, I_{fin}, I_{linf}, T_{fin}, T_{rinf})$ is called unambiguous if the following conditions hold:

(i) $\forall\, p, q \in Q, \forall\, f \in A^*$: Card $\{ c : p \xrightarrow{f} q \} \le 1$

(ii) $\forall\, p \in Q, \forall\, u \in A^N$: Card $\{ c : p \xrightarrow{u}\gg T_{rinf}\} \le 1$

(iii) $\forall\, p \in Q, \forall\, v \in A^{-N}$: Card $\{ c : I_{linf} \gg\!\xrightarrow{v}> p \} \le 1$

(iv) $\forall\, w \in A^Z$: Card $\{ c : I_{linf} \gg\!\xrightarrow{w}\gg T_{rinf}\} \le 1$.

Proposition 4.6: Let $M = (Q, A, E_M, \{i\}, I_{linf}, \{t\}, T_{rinf})$ be a reduced finite automaton having a unique initial state i and a unique final state t. Then, M is unambiguous if and only if $|M|$ is a characteristic series.

Theorem 4.7: Let $X \subseteq {}^\omega A^\omega - \{e\}$ and M be a finite automaton such that $|M| = \underline{X}$. Then, X is a bi–infinitary code if and only if M^* is unambiguous.

Proof: By virtue of proposition 4.4 (ii), we have, $|M^*| = (\underline{X})^*$. Then, by theorem 3.5 and proposition 4.6, we have, X is a bi–infinitary code if and only if $(\underline{X})^*$ is characteristic and so if and only if M^* is unambiguous.

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