PARTITIONS AND SCHUBERT POLYNOMIALS

Extended Abstract

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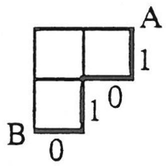
Abstract. In this paper we discuss a tabular method to compute the number of integer partitions lying inside a given one, λ, and prove that the Schubert polynomial having code 0λ counts exactly these partitions.

Résumé. Dans cet article, nous discutons une méthode tabulaire pour calculer le nombre de partitions d'entiers contenues dans une partition donnée λ, et nous montrons que le polynôme de Schubert de code 0λ compte exactement ces partitions.

1. Introduction. How many different positions can arise when playing *nim*, the well-known game which can be succinctly described as starting with an integer partition and removing as many elements as you want but from a *single* part? Try for instance with three parts (often called *piles*), one with 4 tokens and the other two with 2 tokens each.

To put it differently, suppose to play a rather silly game: to sort a (0,1)-array or string using *bubble-sort*. How many different strings can be obtained during the ordering? We are *rewriting* a (0,1)-word by using the unique *rule* $10 \rightarrow 01$, and we are asking for the cardinality of the *language* we can obtain from a given word. If we interpret the strings as *rim representations* [We] of integer partitions (1 for a vertical rim, 0 for a horizontal one, going from the upper right corner A to the lower left corner B), the former problem is equivalent to counting the number of integer partitions whose *Ferrers diagrams* lie inside that of a given partition. For example, in the case of Figure 1.1 this number is 5.



$$\begin{array}{ccccc} 1010 & \rightarrow & 0110 & & \\ \downarrow & & \downarrow & & \\ 1001 & \rightarrow & 0101 & \rightarrow & 0011 \end{array}$$

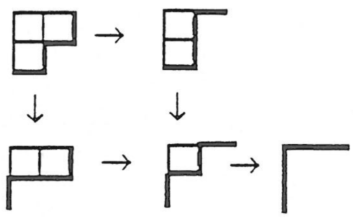


Figure 1.1.

This problem turns out to have less trifling applications, for instance in chemical graph theory (cf. e. g. [Ra]). It actually occurred to us when trying to count the number of matrices with given row and column sums [To]. We had to generate, store and retrieve partitions not greater than a given one, and we were looking for a simple way to determine their number and possibly a *ranking function*, so that it were easy to determine the place of each partition in an array, e. g. to record its presence or absence. We found we could do that by simply filling up the Ferrers diagram in much the same way as computing binomial coefficients in a slant version of Pascal's triangle. In fact, a rectangular diagram is filled up exactly with binomial coefficients.

Let us call such diagrams, filled with suitable numbers, *tableaux*. In the next Section we will precisely describe an algorithm to fill a tableau, and prove its correctness. As a preview, Figure 1.2 shows the tableau relative to partition 422 (the one we suggested for the game of nim: we shall use that partition as a standard example). The number of partitions whose diagram can be contained inside a given one is the sum of all such numbers, plus 1 if also the empty partition is to be included. In the case of partition 422 there are 21 smaller partitions.

1	1	1	1
4	3		
7	3		

$$1+1+1+1+4+3+7+3 = 21.$$

Figure 1.2.

If one considers the *Schubert polynomial* relative to code 0422, one discovers that it has exactly 22 terms. In §3 we prove that Schubert polynomials relative to codes of the form 0λ, where λ is a partition, count precisely the number of partitions not greater than λ, and a bijection is provided to associate each partition with a monomial of the Schubert polynomial.

2. Counting partitions. A *partition* λ is a nonincreasing sequence of nonnegative integers, having only a finite number p of positive elements, called *parts*. We shall write λ as $(\lambda_1, \lambda_2, ..., \lambda_p)$, or simply $\lambda_1\lambda_2... \lambda_p$, and say that $\lambda \leq \mu$ whenever $\lambda_i \leq \mu_i$ for each $i \geq 1$. A standard reference for partitions is Andrews' book [An].

With each partition one can associate a diagram, often called *Ferrers diagram*, the set of points $(i, j) \in \mathbf{N}^2$ such that $1 \leq j \leq \lambda_i$, $1 \leq i \leq p$. Points can be substituted by *squares*, and in the following we will refer to this latter kind of diagrams, as in Figures 1.1 and 1.2. Notice that $\lambda \leq \mu$ if and only if the diagram of λ can be *covered* by that of μ. This order relation defines a *partial* order on the set of partitions. With each partition λ one can associate the *Young's lattice* Y_λ of all partitions not greater than λ (cf. e. g. [SW], page 29). We want to count the number N_λ of elements in this lattice, and we will also devise a *ranking algorithm*, assigning to each partition a number in $\{0, ..., N_\lambda-1\}$, and vice versa.

With each partition we will associate a $(p + 1) \times (\lambda_1 + 1)$ matrix A of nonnegative integers. Rows are indexed 1 to $p + 1$, columns 0 to λ_1; the *algorithm* to fill the matrix is simply described by two nested **for** instructions:

for i := 1 **to** $p + 1$ **do**

  for j := 0 **to** λ_i **do**

$$A[i,j]:=1+\sum_{r=1}^{i-1}\sum_{c=j+1}^{\lambda_r}A[r,c] \tag{2.1}$$

with the usual convention that $\sum_{c=a}^{b} x = 0$ if $b < a$.

In other words: each element is one more than the sum of *all* elements *above and to the right* of itself. The following table will help to understand the algorithm, applied to the already familiar partition 422 ; here $p = 3$, $\lambda_1 = 4$.

col. row	col. 0	col. 1	col. 2	col. 3	col. 4
row 1	1	1	1	1	1
row 2	5	4	3		
row 3	12	7	3		
row 4	22				

Table 2.1.

Of course, one could use only the diagram of the partition (the shaded portion of Table 2.1), as we did in §1; however, it is convenient to add a column to the left, since the element in place $(p + 1, 0)$ will then contain exactly the number of partitions not greater than the given one.

A few shortcuts are also immediately evident, such as the fact that *each element is the sum of all elements in the row immediately above*, up to its column included $(22 = 12 + 7 + 3)$, or that *each element is the sum of the two adjacent elements* immediately above and to the right, if they are both present $(7 = 4 + 3)$. This last rule proves at once that a rectangular table (partition with equal parts) yields the binomial coefficients: Pascal's triangle is actually obtained by rotating the table 45 degrees counterclockwise.

Let us now prove our assertions. The simplest way to do this seems to require a different approach, which is useful, in any case, to extend the algorithm to skew partitions and to generalize the results. Consider the diagram of a partition as a *graph*: the edges are the edges of the squares, and the vertices are the vertices of the squares. We can orient the graph so that there are *oriented paths* between the upper right corner A and the bottom left corner B as in Figure 1.1. What we obtain is a *directed acyclic graph* (*DAG*). Now, the number of paths from A to B is precisely what we want to count.

Assign value 1 to vertex A. Assign the *sum* of the values of the incoming vertices to all other vertices (in *breadth-first* order). The value of each vertex is then exactly the number of paths from A to that vertex. Figure 2.1 shows the example of partition 422.

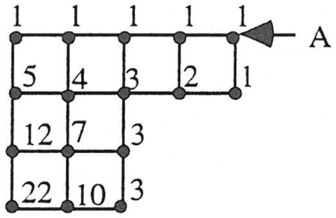
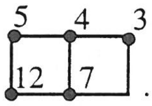


Figure 2.1.

If one inscribes each value into the square having the corresponding vertex as its *upper right* corner, then one obtains exactly Figure 1.2, or the matrix in Table 2.1 by adding column 0 and row 4. It is now quite evident that *each value is the sum of the two adjacent ones*, since paths can only reach adjacent points just before reaching the point under consideration. Alternatively, one can reach points on the *row above* the point of arrival, on condition that there be a connection (starting with a vertical step) with such a point, as in 

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It is immediate to prove *(2.1)* by induction on the row index i, using the formula for the row sum

$$A[i+1,j] = \sum_{c=j}^{\lambda_i} A[i,c]. \qquad (2.2)$$

However, it seems to be more useful to resort to a "geometrical" approach. Let $a, b, c, \ldots$ symbolically represent portions of the tableau and also the sum of values in that portion. Refer to Figure 2.2(a): by using *(2.1)* we get $c = d + 1$, $a = b + d + 1 = b + c$, and this is exactly what we had to prove to show the equivalence of *(2.1)* and *(2.2)*; precisely, that each element a is the sum of all elements in the row just above it, $b + c$.

This geometrical approach is useful to prove some more properties of tableaux. For instance, if we do not want to add a further row and column to the diagram, we can however compute the number of included partitions more efficiently than summing all the elements and adding 1: it suffices to sum column 1 and row p (element $(p, 1)$ being counted twice). In Table 2.1, e. g., $(1 + 4 + 7) + (7 + 3) = 22$. In fact, referring to Figure 2.2(b), $e = a + b + c + d + 1$, but $c = b + 1$, therefore $e = (a + c) + (c + d)$.

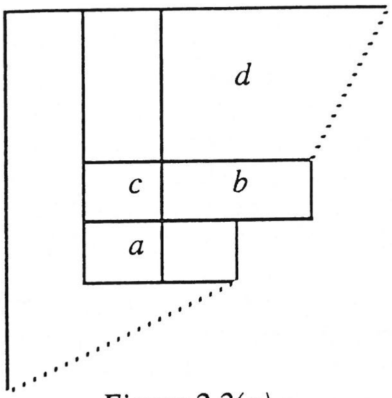


Figure 2.2(a).

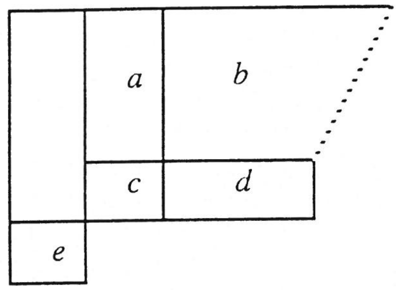


Figure 2.2(b).

Before we can introduce a *ranking* function mapping each partition $\mu \leq \lambda$ to a value between 0 and $rank(\lambda)$, we have to define a suitable *total* order on partitions. The "right" order is the so-called *colexicographic* [SW] or *inverse lexicographic* order: $\mu < \nu$ iff there exists an i such that $\mu_i < \nu_i$ while $\mu_j = \nu_j$ for each j *greater* than i. We shall call this order *colex order* and indicate it simply with <, since this symbol will distinguish the total colex order from the partial order $\leq$.

Example 2.1. The following is a list of the partitions $\mu \leq 422$ in increasing colex order, up to 211: $0 < 1 < 2 < 3 < 4 < 11 < 21 < 31 < 41 < 22 < 32 < 42 < 111 < 211$.

The ranking function $rank_\lambda(\mu)$ will simply yield the place of μ in the colex order list of the partitions not greater than λ. For instance, $rank_{422}(211) = 13$. Let us now remark that this rank coincides with the sum of the values covered, in the tableau of 422 (cf. Figure 1.2), by the tableau of 211, provided the two tableaux have coincident upper left corners: $1 + 1 + 4 + 7 = 13$. And this is by no means a... coincidence, as stated by the following theorem.

Theorem 2.1. $rank_\lambda(\mu) = \sum_{i>0} \sum_{j=1}^{\mu_i} A[i,j]$.

Proof. We will show that whenever ν immediately follows μ in the increasing colex order of partitions not greater than λ, by assigning $rank_\lambda(\mu)$ the value specified by the theorem we have $rank_\lambda(\nu) = rank_\lambda(\mu) + 1$, and of course $rank_\lambda(1) = 1$. To this end, suppose $\mu_j = \lambda_j$ for $0 \leq j \leq i$, while $\mu_{i+1} < \lambda_{i+1}$ ($j = 0$ accounts for the case $\mu_1 < \lambda_1$): then, to get the partition ν immediately following μ, we must have $\nu_j = \mu_{i+1} + 1$ for $1 \leq j \leq i + 1$, and $\nu_j = \mu_j$ for $j > i + 1$. This amounts to getting the diagram of ν by changing the diagram of μ with the addition of element $A[i + 1, \mu_{i+1} + 1]$ and removal of all the elements *above and to the right* of such an element. By (2.1), the element we add has a value which is exactly one more than the sum of the values of the elements we remove. ¶

Example 2.2. Again, let $\lambda = 422$, $\mu = 32$: then $i = 0$, since $\mu_1 < \lambda_1$, and the partition next to μ is $\nu = 42$. Now, let $\mu = 42$: then $i = 2$, and $\nu = 111$. In this latter case, passing form μ to ν we remove from the tableau elements with values $1 + 1 + 1 + 3 = 6$, while adding an element with value 7 (cf. Figure 1.2). Therefore $rank(111) = 1 + 4 + 7 = 12 = rank(42) + 1 = 1 + 1 + 1 + 1 + 4 + 3 + 1$.

Notice that, given a rank r and a partition λ, we can easily construct the partition μ such that $rank_\lambda(\mu) = r$. It suffices to take squares in the diagram of λ, from top down and left to right, in such a way that the sum of their values is exactly r. Theorem 2.1 ensures that if it is possible to do so, then it can be done in only one way.

The tabular method just described works as well for *skew diagrams*, i. e. diagrams which are difference of two others. More precisely, suppose $\mu \leq \lambda$: then, we shall denote by λ/μ the diagram associated with the set of squares which are associated with λ and *not* with μ.

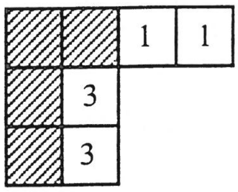


Figure 2.3(a)

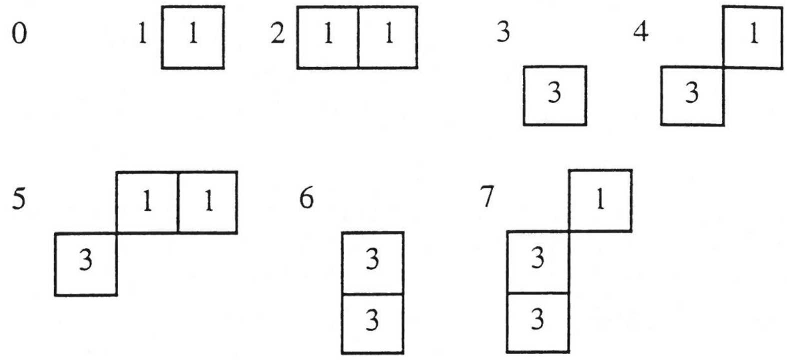


Figure 2.3(b).

In Figure 2.3(a) we show the tableau relative to 422/211. It is obtained by applying (2.1), which is absolutely valid, while now it is no more true that each element is the sum of elements in the row immediately above! However, the meaning of values inscribed in the tableau is the same as before: there are 1 + 1 + 3 + 3 = 8 skew partitions or diagrams smaller than or inside the diagram of λ/μ, and they are orderly reproduced in Figure 2.3(b). But this also means that there are 8 partitions *between* 211 and 422 (including one of the two and excluding the other): 211, 311, 411, 221, 321, 421, 222, 322 (just add the skew partitions to the smaller one!). Therefore, we are able to count the number of partitions which are comprised between two others, and to attribute them a rank: the case of partitions smaller than a given one, examined before, is simply associated with $\lambda/0$.

The approach of counting partitions as paths inside a DAG still works: one simply has to imagine a *unique* path connecting possible disconnected components, and this justifies the use of algorithm (2.1). One can even imagine partitions with *increasing* parts, or stranger diagrams filled according to (2.1). In any case, by specifying a *construction rule* for the tableau and a particular *sequence of squares* inside the tableau itself, one can obtain a numerical sequence which, in some cases, can have a well-known, or a new and interesting, combinatorial meaning. For instance, one can easily obtain more than 40 sequences reported in Sloane's book [Sl]. In [Co] this point is illustrated and formulae are given to directly compute the number of partitions, without having to fill a table.

3. Certain Schubert polynomials enumerate partitions. Schubert polynomials were introduced in [LS1] by means of *divided difference operators* (cf. [LS2], [Ma]). A few combinatorial characterizations have also been given (cf. [BJ], [BB]). However, well before Schubert polynomials were called so, they were studied in a paper by D. Monk [Mo], who devised an "intersection formula" which can be used very conveniently to compute the polynomials [KK]. A particular case of that formula is the following rule. The polynomials, with variables $x_1, x_2 \ldots, x_n$, are indexed by permutations and will be denoted with X_π.

Monk's rule: $$X_\mu = x_j \cdot X_\nu + \sum_\pi X_\pi. \qquad (3.1)$$

The rule is valid under more general conditions, but we shall *assume* that $\mu = \ldots \mu_i \ldots \mu_j \ldots \mu_k \ldots$, where j is the *maximum* index such that $\mu_j > \mu_k$ while $j < k$, and k is the *maximum* index such that $\mu_j > \mu_k$. Then, ν is obtained from μ by exchanging μ_j and μ_k, so that $\nu = \ldots \mu_i \ldots \mu_k \ldots \mu_j \ldots$, and the *sum* is over all permutations π which can be obtained from ν by exchanging μ_i and μ_k whenever $i < k$, $\mu_i < \mu_k$ and for every index h between i and k μ_h is *external to the interval* $[\mu_i, \mu_k]$.

By recursively applying the rule, one can get to the identical permutation *1*, and by substituting $X_1 = 1$ one can use Monk's rule to obtain any Schubert polynomial.

Example 3.1. Let us take X_{1423}. For $\mu = 1423$ one has $j = 2$, $k = 4$, so that $\nu = 1324$ and $\pi = 3124$. Therefore $X_{1423} = x_2 X_{1324} + X_{3124}$. Applying the rule once more $X_{1324} = x_2 X_{1234} + X_{2134}$. Now substitute $X_{1234} = 1$. The other terms give $X_{3124} = x_1 X_{2134} = x_1^2 X_{1234} = x_1^2$ and $X_{2134} = x_1$, so that one obtains $X_{1423} = x_1^2 + x_1x_2 + x_2^2$.

An alternative way to identify a permutation is to specify its *Lehmer code* (or one of many possible similar codes [Se]). Lehmer code is "the right one" since it gives the lexicographically *smallest* sequence of exponents for the monomials of the associated Schubert polynomial (if the variables are ordered $x_1 > x_2 > ... > x_n$). Given a permutation π, its Lehmer code $L(\pi)$ is the list $(c_1, c_2, ..., c_n)$, where $c_i := |\{j > i : \pi_j < \pi_i\}|$, $1 \le i \le n$. For example, $L(1423) = 0200$, and in fact the lexicographically smallest monomial in X_{1423} is x_2^2. We shall occasionally *use the code as an index* to specify a Schubert polynomial in place of the corresponding permutation.

We will now consider only those permutations (and associated polynomials) whose code consists of a *zero* followed by a partition. Let us examine in detail a permutation μ of at least 4 elements, whose code is of the form $(0, \lambda_1, \lambda_2, ..., \lambda_p)$, with $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_p > 0$. First of all, $\mu_1 = 1$, since the first element must not be followed by any bigger element. Suppose $1 < i < j < k$ and $\mu = 1... \mu_i ... \mu_j ... \mu_k ...$: then $\mu_j > \mu_k > \mu_i$ is *impossible*, since otherwise one can choose μ_i to be the *smallest* element between 1 and μ_j, and then all greater elements must occur *after* μ_j, implying $\lambda_i < \lambda_j$, contrary to the hypothesis. But this means that when using Monk's formula the *only* term contributing to the sum is the one exchanging the initial element 1. We can therefore reformulate

Monk's rule for codes of the form 0λ: $X_A = x_{p+1} X_B + X_C$, (3.2)

where A, B and C are codes, and precisely $A = 0\lambda_1...\lambda_{p-1}\lambda_p$, $B = 0\lambda_1...\lambda_{p-1}\lambda_p{-}1$, $C = \lambda_p\lambda_1...\lambda_{p-1}0$.

Example 3.2. Let the code be $A = 042200$. Then $\mu = 164523$. Let us apply Monk's rule: in this case $j = p + 1 = 4$, $k = 6$, $\nu = 164325$, $\pi = 364125$. Notice that $B = L(\nu) = 042100$ and $C = L(\pi) = 242000$.

Consider now $\mu = 53412$, $L(\mu) = 422$ (henceforth we shall *omit final zeros in the code*), $j = 3$, $k = 5$, $\nu = 53214$, $L(\nu) = 421$, and there is no permutation π. In fact, an interesting consequence of Monk's rule is the fact that when a permutation has a code which is a *partition* λ (without the initial zero), then there is no contribution to the sum, and therefore the Schubert polynomial is a single *monomial*, whose exponent vector coincides with the partition itself. This happens whenever the permutation has no subpermutation isomorphic to 132. As shown e. g. in [SS], the number of n-permutations having this characteristic is the n-th *Catalan number*, and therefore the number of n-permutations having code 0λ is simply the (n−1)-th Catalan number.

To go on, we need to know how to operate on X_C, whose code is *not* of the form 0λ.

Lemma 3.1. Let $C = (c_1, c_2, ..., c_p)$ be a Lehmer code with $c_i > 0$, $1 \le i \le p$, and $C-1 = (c_1-1, c_2-1, ..., c_p-1)$. Then $X_C = x_1 x_2 ... x_p X_{C-1}$.

Remark. This lemma is similar to the *Block Decomposition Formula* of § 1.5 in [BJ], but the conditions of the two formulae are in fact different: permutation 546132, having code 43301, has a block decomposition, while our lemma is not applicable; on the other hand, 24513 is not decomposable, but its code is 122 and the lemma can be applied. The proof of the lemma is not difficult by using difference operators, and it is immediate by resorting to a suitable combinatorial interpretation.

Iterating the application of Monk's rule (3.2) and Lemma 3.1, one gets to the further lemma, which is crucial to prove the final result.

Lemma 3.2. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{p-1}, \lambda_p)$, $\lambda' = (\lambda_1, \lambda_2, ..., \lambda_{p-1})$, $\lambda - 1 = (\lambda_1-1, \lambda_2-1, ..., \lambda_{p-1}-1, \lambda_p-1)$. Then $X_{0\lambda} = x_{p+1}^{\lambda_p} X_{0\lambda'} + x_1 x_2 ... x_p X_{0\lambda-1}$.

Now, let Y_C denote the value of the Schubert polynomial of code C when assigning *value 1 to each variable*: then, from Lemma 3.2, $Y_{0\lambda} = Y_{0\lambda'} + Y_{0\lambda-1}$. Both λ' and $\lambda-1$ are partitions smaller than λ (in the partial order), therefore if one assumes that $Y_{0\mu}$ gives the number of partitions not greater than μ when $\mu \le \lambda$, $\mu \ne \lambda$, one has a valid induction argument, provided the recurrence above is valid for partitions and $Y_0 = 1$. Since X_0 is the Schubert polynomial associated with the permutation with no inversions, $X_0 = 1$, and $Y_0 = 1$ is trivial. For the recurrence, let us resort to our "geometric" approach, cf. Figure 3.1(a).

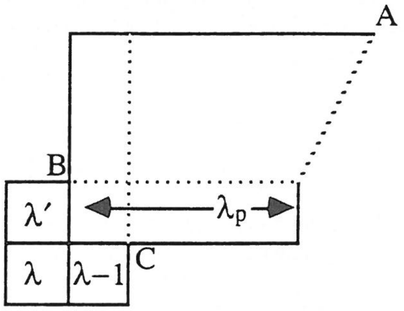


Figure 3.1(a).

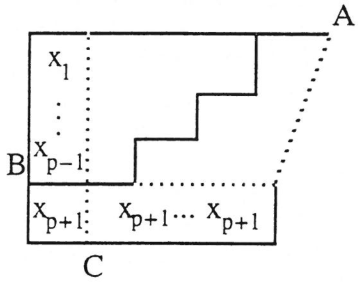
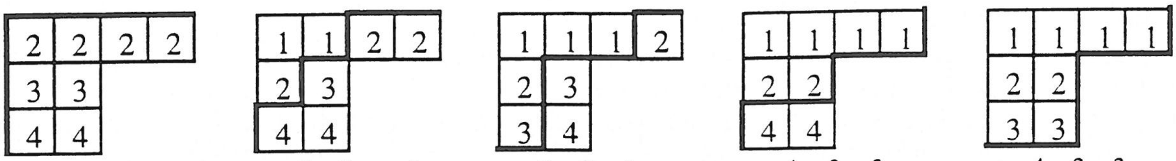


Figure 3.1(b).

There is almost no need for words: $Y_{0\lambda'}$ counts partitions inside the "trapezoid" at the upper right of point B, constituted by the diagram of λ without the last part λ_p; in the tableau, $Y_{0\lambda'}$ should be inscribed into the square marked λ'. $Y_{0\lambda-1}$ counts partitions inside the region at the upper right of point C, and should be inscribed into the square marked $\lambda - 1$. $Y_{0\lambda}$ should appear in the square marked λ: as we know, $Y_{0\lambda}$ is exactly the sum of its adjacent values.

However, the proof through the two lemmas is a little involved. After all, one would like to associate a partition with each monomial and vice versa! Paths are easily associated with monomials, after Gessel & Viennot [GV], but here the number of steps does not coincide with the number of squares, which is the homogeneous degree of the monomials. Actually, we were not able to find the correspondence before discovering how to exploit Lemma 3.2. The lemma seems to suggest that paths inside the diagram, starting from A and reaching B, should be associated with monomials having a factor $x_{p+1}^{\lambda_p}$, while paths from A to C should be associated with monomials containing $x_1 x_2 ... x_p$.

Figure 3.1(b) should make things clear: squares on the last row *under* the path from A to B should be labeled x_{p+1}, while, going instead to point C, the first column would be left *above* the path, and squares should be labeled $x_1 x_2 ... x_p$. Labeling should *increment the index when crossing the path.* Figure 3.2 reports some examples for partition 422, code 0422: only indices are reported in the diagram; the corresponding monomial and partition are written below each diagram.



$x_2^4x_3^2x_4^2$	$x_1^2x_2^3x_3x_4^2$	$x_1^3x_2^2x_3^2x_4$	$x_1^4x_2^2x_4^2$	$x_1^4x_2^2x_3^2$
0	21	311	42	422

Figure 3.2.

While it is quite evident that with each path is associated *one and only one monomial*, the vice versa is not entirely obvious. But observe that the exponent of x_1 in the monomial gives the first part μ_1 of the partition μ we are constructing inside the diagram of λ, then μ_2 equals the exponent of x_2 minus $\lambda_1 - \mu_1$, and so on: therefore the partition is completely determined by the monomial. In this way we have proved not only that this class of Schubert polynomials counts partitions inside a given one, but also that *each polynomial associated with a code of the form 0λ has all its positive coefficients equal to 1*. This fact is interesting by itself, but may also be relevant when trying to compute such polynomials, since one can then generate the monomials without having to check, in order to increment the coefficient, if a monomial has already been generated.

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References

[An] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Ma, 1976

[BB] N. Bergeron, S.C. Billey, "RC-Graphs and Schubert Polynomials", *Experiment. Math.* **2** (1993) 257-269

[BJ] S.C. Billey, W. Jockusch, R.P. Stanley, "Some Combinatorial Properties of Schubert Polynomials", *J. Algebraic Combin.* **2** (1993) 345-374

[Co] S. Colombo, *Partizioni e algoritmi per l'enumerazione di strutture combinatorie,* Tesi di Laurea in Scienze dell'Informazione, Università di Milano, 1992

[GV] I. Gessel, G. Viennot, "Binomial Determinants, Paths, and Hook Length Formulae", *Advances in Math.* **58** (1985) 300-321

[Ke] A. Kerber, *Algebraic Combinatorics via Finite Group Actions*, BI-Wissenschaftsverlag, Mannheim, 1991

[KK] A. Kerber, A. Kohnert, A. Lascoux, "SYMMETRICA, an Object-Oriented Computer-Algebra System for the Symmetric Group", *J. Symb. Computation* **14** (1992) 195-203

[LS1] A. Lascoux, M.P. Schützenberger, "Polynômes de Schubert", *C.R. Acad. Sc. Paris* Série I **294** (1982) 447-450

[LS2] A. Lascoux, M.P. Schützenberger, "Décompositions dans l'algèbre des différences divisées", *Discrete Math.* **99** (1992) 165-179

[Ma] I.G. Macdonald, *Notes on Schubert Polynomials*, LACIM, Montreal, 1991

[Mo] D. Monk, "The Geometry of Flag Manifolds", *Proc. London Math. Soc.* **9** (1959) 253-286

[Ra] M. Randic´, "On Enumeration of Complete Matchings in Hexagonal Lattices", in *Graph Theory, Combinatorics and Applications*, Y. Alavi et al. Eds., Wiley, New York, 1991

[Se] R. Sedgewick, "Permutation Generation Methods", *ACM Comp. Surveys* **9** (1977) 137-164

[Sl] N.J.A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York, 1973

[SS] R. Simion, F.W. Schmidt, "Restricted Permutations", *Europ. J. Combin.* **6** (1985) 383-406

[SW] D. Stanton, D. White, *Constructive Combinatorics*, Springer, New York, 1986

[To] M. Torelli, "Computing the Number of Nonnegative Integer Matrices with Prescribed Row and Column Sums through Symmetric Functions", to appear in *J. Comb. Inf. Sys. Sciences* (1995)

[We] M.B. Wells, *Elements of Combinatorial Computing*, Pergamon, Oxford, 1971