Homotopy Tools with Applications to some Combinatorial Problems

- EXTENDED ABSTRACT -

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Abstract

We provide a "toolkit" of basic lemmas for the comparism of homotopy types of (homotopy) limits of diagrams of spaces over finite partially ordered sets, among them several new ones. In the setting of this paper, we obtain simple inductive proofs that provide explicit homotopy equivalences. We show how this toolkit of old and new diagram lemmas can be used on quite different fields of applications. In this paper we demonstrate this with respect to the "generalized homotopy-complementation formula" by Björner [4], the geometry and combinatorics of toric varieties (which turn out to be homeomorphic to homotopy limits, and for which the homotopy limit construction provides a suitable spectral sequence), and in the study of the combinatorics and homotopy types of arrangements of subspaces.

Abstract

Nous présentons un outillage de lemmes fondamentaux servant à la comparaison de types d'homotopie de limites (d'homotopie) de diagrammes d'espaces sur un ensemble fini ordonné partiellement. La formulation choisie dans cet article nous permet de déduire des preuves inductives simples fournissant explicitement des équivalences d'homotopie. Nous montrons comment appliquer cet outillage de lemmes anciens et nouveaux à de différents resultats — notamment à la "formule de complémentation de homotopie" de Björner [4], à la géometrie et la combinatoire de variétés toriques (qui se révèlent être homéomorphes à des limites d'homotopie et pour lesquelles la construction de limites d'homotopie fournit une suite spectrale propre) et finalement à l'étude de la combinatoire et des types d'homotopie d'arrangements de sous-espaces.

1 Introduction

A diagram of spaces is a functor from a small category to a category of topological spaces. In the following, we will consider the special case where the small category is a finite partially ordered

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set P, and a P-diagram is an assignment of topological spaces D_p to the elements $p \in P$, and compatible maps $d_{pp'} : D_p \to D_{p'}$ to the order relations $p \geq p'$ in P.

In various geometric, algebraic and combinatorial situations one has to deal with structures that can profitably be interpreted as limits or homotopy limits of diagrams over finite posets. In fact, if a space is written as a finite union of (simpler) pieces, then it is the limit of a corresponding diagram of spaces. While limits do not have good functorial properties in homotopy theory, they can usually be replaced by homotopy limits (Puppe [18] may have been the first to exploit this). Homotopy limits have much better functorial properties. Thus there is a wide variety of possible techniques to manipulate diagrams of spaces in such a way that the homotopy limit is preserved (up to homotopy type).

Basic work on homotopy limits has been done by Segal [21], Bousfield & Kan [6], tom Dieck [25], Vogt [27], and Dwyer & Kan [9, 10]. See Hollender & Vogt [15] for a recent survey.

Two key results in this setting are the "Projection Lemma" [21] [6, XII.3.1(iv)], which allows one to replace limits by homotopy limits, and the "Homotopy Lemma" [25] [6, XII.4.2] [27] which compares the homotopy types of diagrams *over the same partially ordered set*. These tools have found striking applications, for example, in the study of subspace arrangements [30] [24].

There are two objectives in our mind, first we want to provide a "tool kit" for the manipulation of limits and homotopy limits: basic lemmas that allow one to compare the homotopy types of diagrams defined *over different partial orders*. Not all of these results are new: in particular, Dwyer & Kan [10, Sect. 9] provide a list of some (apparently) "well-known" results, for which a proof can be found in Hollender & Vogt [15, Prop. 4.4]. However, in our attempt to isolate the "central results" we also describe new tools, in particular the UPPER FIBER LEMMA, Theorem 3.8 below.

Our second objective is to reveal the very combinatorial nature of the homotopy limit itsself and its power in describing the combinatorial structure of mathematical objects, like toric varieties, subspace arrangements and subgroup complexes.

We provide several applications of our methods to various areas within mathematics. As a first application, in the field of topological combinatorics, we present a new proof of a result by Björner on the homotopy type of complexes [4], which generalizes the HOMOTOPY COMPLEMENTATION FORMULA of Björner and Walker [5]; a tool which has proved to be very powerful in combinatorics. Since this proof affords the application of many of the techniques provided in this paper, we give a detailed exposition of it here.

Then we present a new view of toric varieties. Namely, we show that toric varieties are homemorphic to homotopy limits over the face poset of the fan defining the variety. This leads to a spectral sequence to compute the homology of toric varieties. Explicit results follow quite directly in the case of toric varieties associated with simplicial fans.

More briefly we cover two applications for which details are contained in other papers. We describe a new result on the homotopy type of the order complex of the poset $S_p(G)$ of nontrivial p-subgroups of a finite group G [17]. Finally, we review results obtained by homotopy limit methods on the topology of subspace arrangements in [30].

2 Set-up and Basic Tools

In the following all *posets* are finite partially ordered sets. A basic operation we will use is the construction of subposets like $P_{\leq p} := \{p' \in P : p' \leq p\}$, $P_{<p} := \{p' \in P : p' < p\}$, etc. Also, the *order complex* of P is the (abstract) simplicial complex $\Delta(P)$ given by the collection of chains in P. By $|\Delta|$ we denote the geometric realization of the complex Δ. (See Stanley [22, Chap. 3] for other basic concepts, notation, etc.) Whenever we talk about the topology of a poset, we refer to the order complex. Thus a contractible poset is a poset whose order complex is contractible, and a poset is a cone whenever the order complex is a cone, etc. All *topological spaces* will have the homotopy type of a finite CW-complex. (See Whitehead [28] for the basics about this category.)

The following lemma is extremely useful for our inductive proofs (in the next section, and elsewhere): it can be used for proving that a map is a homotopy equivalence if it is "glued together" from two homotopy equivalences.

Lemma 2.1 (GLUING LEMMA: see for example tom Dieck [25])
Let $X = A \cup B$ and $Y = C \cup D$ be spaces and $f : X \to Y$ a continuous map with the property $f(A) \subseteq C$ and $f(B) \subseteq D$. Assume that the pairs $(A \cap B, A)$, $(A \cap B, B)$, $(C \cap D, C)$ and $(C \cap D, D)$ have the homotopy extension property.

If $f|A : A \to C$, $f|B : B \to D$ and $f|(A \cap B) : A \cap B \to C \cap D$ are all homotopy equivalences, then the map $f : X \to Y$ is a homotopy equivalence.

A *join semilattice* is a poset P such that every finite subset has unique minimal upper bound. Such a P has a unique maximal element $\hat{1}$ (note that P is finite). The *crosscut complex* $\Gamma(P)$ of a join semilattice is the simplicial complex of all nonempty subsets of $\min(P)$ that have an upper bound in $P \backslash \hat{1}$.

Proposition 2.2 (CROSSCUT THEOREM: see Björner [3, (10.8)])
For every join semilattice, the crosscut complex $\Gamma(P)$ is homotopy equivalent to the order complex of $P \backslash \hat{1}$.

3 Homotopy Comparison Lemmas

In our setting, a *P-diagram* $\mathcal{D}$ is an assignment of spaces D_p to the elements $p \in P$, and of maps $d_{pp'} : D_p \to D_{p'}$ to the order relations $p \geq p'$, in such a way that d_{pp} is the identity, and $d_{pp'} \circ d_{p'p''} = d_{pp''}$ for $p \geq p' \geq p''$.

If $\mathcal{D}$ is a P-diagram, and $P' \subseteq P$ is a subposet of P (that is, a subset with the induced partial order), then we write $\mathcal{D}[P']$ for the *induced subdiagram*: the P'-diagram whose spaces and maps are given by the spaces and maps in $\mathcal{D}$.

The *limit* $\varinjlim \mathcal{D}$ of a P-diagram $\mathcal{D}$ is constructed from the disjoint union $\biguplus_{p \in P} D_p$ by identification of $x \in D_p$ with $d_{pp'}(x) \in D_{p'}$, for all $p \geq p'$ and all $x \in D_p$.

For example, "subspace arrangements" give rise to "inclusion diagrams." The following definition provides suitable generality for our purposes.

Definition 3.1 *An* arrangement *is a finite collection $\mathcal{A}$ of closed subspaces in some ambient space U, such that any nonempty intersection of subspaces in $\mathcal{A}$ is a (finite) union of subspaces in $\mathcal{A}$.*

Then there is a natural associated diagram, whose poset P is in bijection with $\mathcal{A}$, the order on P is by reverse *inclusion, the spaces A_p are the elements of $\mathcal{A}$ corresponding to $p \in P$, and the maps are the inclusion maps $A_p \hookrightarrow A_{p'}$ for $p \geq p'$.*

In general, a subspace diagram *will mean the inclusion diagram of an arrangement. With this, one sees that the limit of a subspace diagram is exactly the union $\bigcup \mathcal{A} = \bigcup_{p \in P} A_p$.*

(An assumption one needs for the PROJECTION LEMMA below is that the inclusion maps $A_p \hookrightarrow A_{p'}$ are closed cofibrations, i.e., that these maps possess the homotopy lifting property. Equivalently, the subspace $A_{p'} \times \{0\} \cup A_p \times I$ is a retract of $A_{p'} \times I$, or that $(A_{p'}, A_p)$ is an

NDR-pair, see [28, I.5]. This assumption is always satisfied in a situation where everything is smooth, or where one is dealing with inclusions of subcomplexes in a triangulable arrangement.)

The *homotopy limit* $\underrightarrow{\mathsf{holim}}\,\mathcal{D}$ of a P-diagram $\mathcal{D}$ can be constructed from the disjoint union

$$\biguplus_{p\in P} \Delta(P_{\leq p}) \times D_p,$$

by "making the obvious identifications," as follows. For each order relation $p \geq p'$, there is an injection $i : \Delta(P_{\leq p'}) \hookrightarrow \Delta(P_{\leq p})$, and a map $d_{pp'} : D_p \to D_{p'}$. Using the map $\mathrm{id} \times d_{pp'}$, we identify the subspaces

$$\Delta(P_{\leq p'}) \times D_p \quad \subseteq \quad \Delta(P_{\leq p}) \times D_p$$

and

$$\Delta(P_{\leq p'}) \times D_{p'} \quad \subseteq \quad \Delta(P_{\leq p'}) \times D_{p'}.$$

If "$\equiv$" is the equivalence relation generated by these identifications, then the homotopy limit of the P-diagram $\mathcal{D}$ is the space

$$\underrightarrow{\mathsf{holim}}\,\mathcal{D} \;:=\; \Big(\biguplus_{p\in P} \Delta(P_{\leq p}) \times D_p\Big)\Big/ \equiv .$$

An inductive construction of the homotopy limit is as follows: start with a disjoint union $\biguplus_{p\in P} D_p$. Then, for every pair $p > p'$, glue to it a copy of $D_p \times I$, to create a mapping cylinder of the map $d_{pp'}$. Then, for every triple $p > p' > p''$, glue into this a copy of $D_p \times \Delta(p > p' > p'')$, and so on. The "usual" spectral sequence for the (co)homology of homotopy limits follows this inductive construction, i.e., it uses a filtration induced by the skeleta of $\Delta(P)$.

We may observe that for every P-diagram $\mathcal{D}$ there is a canonical partial order on $\mathbf{P} := \biguplus_{p\in P} D_p$ defined by setting $x \geq x'$ for $x \in D_p$ and $x' \in D_{p'}$ if and only if $p \geq p'$ and $d_{pp'}(x) = x'$. With this definition there is a canonical bijection $\Delta(\mathbf{P}) \longleftrightarrow \underrightarrow{\mathsf{holim}}(\mathcal{D})$; however, the topology on the two spaces is different: in fact, with the usual topology on simplicial complexes the subspaces $D_p \subseteq \Delta(\mathbf{P})$ get a discrete topology.

Examples 3.2 Here are some trivial examples for the construction of homotopy limits.

(i) If D_p is a one-point space for all $p \in P$, then $\underrightarrow{\mathsf{holim}}\,\mathcal{D}$ is (isomorphic to) the order complex of P.

(ii) If $P = \{p, p'\}$ has two points, and $p' > p$, then $\underrightarrow{\mathsf{holim}}\,\mathcal{D} = D_{p'} \cup_f D_p$ is the mapping cylinder of the map $f = d_{p'p} : D_{p'} \to D_p$.
(The mapping cylinder is homotopy equivalent to the image space D_p, which is the limit of the diagram.)

(iii) If $\mathcal{D}$ is a diagram such that all the spaces D_p are identical, and the maps are identity maps, then $\underrightarrow{\mathsf{holim}}\,\mathcal{D} \cong \Delta(P) \times D_p$.
(The limit of such a diagram is D_p, if P is connected.)

If $\mathcal{D}$ is a P-diagram and $\mathcal{E}$ is a Q-diagram, then a *map of diagrams* $\phi : \mathcal{D} \to \mathcal{E}$ consists of an order preserving map of posets $f : P \to Q$, together with a collection of maps $\phi_p : D_p \to E_{f(p)}$, which have to be compatible in the sense that for $p > p'$, the maps $e_{f(p)f(p')} \circ \phi_p$ and $\phi_{p'} \circ d_{pp'}$: $D_p \to E_{f(p')}$ have to coincide, that is, the square

$$\begin{array}{ccc} D_p & \xrightarrow{\phi_p} & E_{f(p)} \\ {\scriptstyle d_{pp'}}\big\downarrow & & \big\downarrow{\scriptstyle e_{f(p)f(p')}} \\ D_{p'} & \xrightarrow{\phi_{p'}} & E_{f(p')} \end{array}$$

has to commute for all $p \geq p'$. (Formally, this amounts to the requirement that Φ be a natural transformation between the functors $\mathcal{D}$ and $\mathcal{E}$.)

We now start our presentation of "tools" with the observation that every map of diagrams induces a map of their homotopy limits in a natural way.

Lemma 3.3 (MAPS)
Let $\phi : \mathcal{D} \to \mathcal{E}$ be a map of diagrams. Then ϕ induces a natural map

$$\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}.$$

Lemma 3.4 (EMBEDDING LEMMA)
Let $\Phi : \mathcal{D} \to \mathcal{E}$ be a map of P-diagrams associated with id : $P \to P$. *If $\Phi_p : D_p \hookrightarrow E_p$ is a closed embedding for every $p \in P$, then Φ induces a closed embedding*

$$\widehat{\Phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \hookrightarrow \underrightarrow{\mathrm{holim}}\, \mathcal{E}.$$

In particular, if $\mathcal{D}$ is a P-diagram and $Q \subseteq P$ is a subposet, then $\underrightarrow{\mathrm{holim}}\, \mathcal{D}[Q] \hookrightarrow \underrightarrow{\mathrm{holim}}\, \mathcal{D}$ is a closed embedding.

Lemma 3.5 (CONE LEMMA: Bousfield & Kan [6, XII.3.1(iii)])
Let P be a poset with least element $\hat{0}$. If $\mathcal{D}$ is a P-diagram, then the inclusion $\{\hat{0}\} \subseteq P$ induces a homotopy equivalence

$$D_{\hat{0}} \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{D}.$$

Lemma 3.6 (PROJECTION LEMMA: Segal [21], Bousfield & Kan [6, XII.3.1(iv)])
Let $\mathcal{A}$ be an arrangement with intersection poset P, let $\mathcal{D}$ be the corresponding P-diagram. Then the natural collapsing map $\xi : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \to \underrightarrow{\mathrm{lim}}\, \mathcal{D}$ is a homotopy equivalence.

(See [30, Lemma 1.6] for a simple inductive argument.)

Lemma 3.7 (WEDGE LEMMA: Ziegler & Živaljević [30, Lemma 1.8])
Let P be a poset with maximal element $\hat{1}$. Let $\mathcal{D}$ be a P-diagram so that there exist points $c_{p'} \in D_{p'}$ for all $p' < \hat{1}$ such that $d_{pp'}$ is the constant map $d_{pp'} : x \mapsto c_{p'}$, for $p > p'$. Then

$$\underrightarrow{\mathrm{holim}}\, \mathcal{D} \simeq \bigvee_{p \in P} \big(\Delta(P_{<p}) * D_p\big),$$

where the wedge is formed by identifying $c_p \in \Delta(P_{<p}) * D_p$ with $p \in \Delta(P_{<\hat{1}}) * D_{\hat{1}}$, for $p < \hat{1}$.

Theorem 3.8 (UPPER FIBER LEMMA)
Let $\mathcal{D}$ be a P-diagram and let $\mathcal{E}$ be a Q-diagram. Assume $\phi : \mathcal{D} \to \mathcal{E}$ is a map of diagrams. If ϕ induces a homotopy equivalence

$$\widehat{\phi} : \underrightarrow{\mathrm{holim}}(\mathcal{D}[p \in P : f(p) \geq q]) \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{E}[Q_{\geq q}] \quad \textit{for all } q \in Q,$$

then ϕ induces a homotopy equivalence $\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}$.

Note that by the CONE LEMMA the condition of the UPPER FIBER LEMMA implies that $\underrightarrow{\mathrm{holim}}(\mathcal{D}[p \in P : f(p) \geq q])$ is homotopy equivalent to E_q for all $p \in P$. Thus, in particular, the UPPER FIBER LEMMA implies the following HOMOTOPY LEMMA, for which a simple inductive proof was described in [30, Lemma 1.7].

As a special case, when all the spaces D_p and E_q are one-point spaces, the UPPER FIBER LEMMA contains the QUILLEN FIBER THEOREM (in the case of upper ideals $f^{-1}(P_{\geq p})$).

Corollary 3.9 (HOMOTOPY LEMMA: tom Dieck [25], Bousfield & Kan [6, XII.4.2], Vogt [27])
Let $\mathcal{D}$ and $\mathcal{E}$ both be P-diagrams, and assume that there is a map of diagrams $\phi : \mathcal{D} \to \mathcal{E}$ that corresponds to the identity map $\mathrm{id} : P \to P$ on the posets. If $\phi_p : D_p \to E_p$ is a homotopy equivalence for each $p \in P$, then ϕ induces a homotopy equivalence $\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}$.

(The fact that a similar HOMOTOPY LEMMA is not available for ordinary limits $\underrightarrow{\lim}\, \mathcal{D}$ is one of the main reasons to work with homotopy limits instead.)

Corollary 3.10 (MAPPING CYLINDER LEMMA)
Let $\mathcal{D}$ be a P-diagram and let $\mathcal{E}$ be a Q-diagram. Assume $\phi : \mathcal{D} \to \mathcal{E}$ is a map of diagrams, with $f : P \to Q$ the accompanying map of posets. We define a partial order $P \oplus_f Q$ on the disjoint union $P \uplus Q$ by

$$s \geq s' \quad :\Longleftrightarrow \quad \begin{cases} s, s' \in P & \textit{and} \quad s \geq s' \textit{ in } P, \textit{ or} \\ s, s' \in Q & \textit{and} \quad s \geq s' \textit{ in } Q, \textit{ or} \\ s \in P, s' \in Q & \textit{and} \quad f(s) \geq s' \textit{ in } Q. \end{cases}$$

Let $\mathcal{D} \oplus_\phi \mathcal{E}$ be the $P \oplus_f Q$-diagram defined by

$$(\mathcal{D} \oplus_\phi \mathcal{E})_s := \begin{cases} E_s \textit{ if } s \in Q, \\ D_s \textit{ if } s \in P. \end{cases}$$

The maps of $\mathcal{D} \oplus_\phi \mathcal{E}$ are induced by the maps of the diagram $\mathcal{D}$ and $\mathcal{E}$ and the map ϕ. Then the natural inclusion $\underrightarrow{\mathrm{holim}}\, \mathcal{E} \hookrightarrow \underrightarrow{\mathrm{holim}}\, \mathcal{D} \oplus_\phi \mathcal{E}$ given by the EMBEDDING LEMMA *is a homotopy equivalence.*

Corollary 3.11 (DIRECT IMAGE LEMMA: Dwyer & Kan [10, 9.8])
Let $\mathcal{D}$ be a P-diagram, and let $f : P \to Q$ be a poset morphism. Define a Q-diagram $f_\natural \mathcal{D}$ by setting $f_\natural D_q := \underrightarrow{\mathrm{holim}}\, \mathcal{D}[f^{-1}(Q_{\geq q})]$, and taking for $f_\natural d_{qq'}$ the inclusion maps $\underrightarrow{\mathrm{holim}}\, \mathcal{D}[f^{-1}(Q_{\geq q})] \hookrightarrow \underrightarrow{\mathrm{holim}}\, \mathcal{D}[f^{-1}(Q_{\geq q'})]$.

Then there is a homotopy equivalence $\underrightarrow{\mathrm{holim}}\, f_\natural \mathcal{D} \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{D}$.

The following, quite innocent-looking, lemma contains the key idea to the combinatorial formulas for "Grassmannian" arrangements in Section 6. It uses the notion of a *homotopy* between P-diagrams $\mathcal{D}$ and $\mathcal{E}$: a P-diagram $\mathcal{H}$ with diagram maps $\phi : \mathcal{D} \to \mathcal{H}$ and $\psi : \mathcal{E} \to \mathcal{H}$ such that the maps ϕ_p and ψ_p are homotopy equivalences. This is motivated by the special case where we would have compatible maps $f_p : H_p \to [0,1]$ such that $f_p^{-1}(0) = D_p$ and $f_p^{-1}(1) = E_p$ for all $p \in P$, and such that the inclusions $D_p, E_p \hookrightarrow H_p$ are homotopy equivalences.

Corollary 3.12 (HOMOTOPY BETWEEN DIAGRAMS)
If there is a homotopy between P-diagrams $\mathcal{D}$ and $\mathcal{E}$, then $\underrightarrow{\mathrm{holim}}\, \mathcal{D} \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{E}$.

From this result, we can see e.g. that mapping cones are well defined (for homotopy classes of maps, up to homotopy!). In particular, in Example 3.2(v) one can insert any contractible space for $D_{p''}$ and still obtain the homotopy type of the mapping cone.

Theorem 3.13 (INVERSE IMAGE LEMMA: Dwyer & Kan [10, 9.4])
Let $f : P \to Q$ be a poset morphism and let $\mathcal{E}$ be a Q-diagram. Define the P-diagram $f^\sharp\mathcal{E}$ by $(f^\sharp E)_p := E_{f(p)}$ and $f^\sharp e_{pp'} = e_{f(p)f(p')}$.

If for all $q \in Q$, $f^{-1}(Q_{\leq q})$ is contractible, then f induces a homotopy equivalence

$$\underrightarrow{\mathrm{holim}}\, f^\sharp\mathcal{E} \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}.$$

The condition "$f^{-1}(Q_{\leq q})$ is contractible" in the INVERSE IMAGE LEMMMA *cannot* be replaced by the order dual condition "$f^{-1}(Q_{\geq q})$ is contractible."

Let $\mathcal{D}$ be the P-diagram and let $\mathcal{E}$ be the Q-diagram in our Figure. Then $D_a = D_b = E_{f(a)}$ is a point and $D_c = E_{f(c)}$ is a circle. All maps are the constant maps to the point $D_a = D_b = E_{f(a)}$. The homotopy limit of the P-diagram $\mathcal{D}$ is a 2-sphere and the homotopy limit of the Q-diagram $\mathcal{E}$ is a cone over a circle. Obviously, $f^{-1}(Q_{\geq q})$ is contractible for all $q \in Q$, but $\underrightarrow{\mathrm{holim}}\, \mathcal{D} \not\simeq \underrightarrow{\mathrm{holim}}\, \mathcal{E}$.

By a lemma of Babson [2] (see [23, Lemma 3.2]) the conditions of the INVERSE IMAGE LEMMA are satisfied if (a) the inverse image $f^{-1}(q)$ is contractible for all $q \in Q$, and (b) $P_{\geq p} \cap f^{-1}(q)$ is contractible for all $p \in P$ and $q \in Q$ with $f(p) < q$.

Proposition 3.14 (ORDER HOMOTOPY THEOREM)
Let $\mathcal{D}$ be a P-diagram, and let $f : P \to P$ be a poset map which is decreasing *(that is, $f(p) \leq p$ for all $p \in P$).*

Then the inclusion $i : f(P) \hookrightarrow P$ induces a homotopy equivalence

$$\underrightarrow{\mathrm{holim}}\, \mathcal{D}[f(P)] \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{D}.$$

The ORDER HOMOTOPY THEOREM is definitely false if we use *increasing* maps: see the case of a single mapping cylinder.

Note that the CONE LEMMA is a very special case of the ORDER HOMOTOPY THEOREM. We just remark that for any function satisfying $f(f(p)) = f(p) \leq p$ for all $p \in P$ the homotopy limit $\underrightarrow{\mathrm{holim}}\, \mathcal{D}[f(P)]$ is in fact a deformation retract of $\underrightarrow{\mathrm{holim}}\, \mathcal{D}$.

Another simple consequence of the ORDER HOMOTOPY THEOREM is that one can remove "join irreducibles" from a diagram without changing the homotopy type of the homotopy limit. That is, if Q is obtained from P by deleting an element that covers exactly one element, then the homotopy limits of a P-diagram $\mathcal{D}$ and of its deletion $\mathcal{D}[Q]$ are homotopy equivalent.

Lemma 3.15 (LOWER FIBER LEMMA)
Let $\mathcal{D}$ be a P-diagram and $\mathcal{E}$ a Q-diagram. Let $\phi : \mathcal{D} \to \mathcal{E}$ be a morphism of diagrams. If for all $q \in Q$ the induced map

$$\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D}[f^{-1}(Q_{\leq q})] \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}[Q_{\leq q}]$$

is a homotopy equivalence, then $\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \to \underrightarrow{\mathrm{holim}}\, \mathcal{E}$ is a homotopy equivalence.

We will call a P-diagram $\mathcal{D}$ a *poset diagram* if the spaces D_p are order complexes, $D_p = \Delta(Q_p)$ for posets Q_p, and the maps $d_{pp'} : D_p \to D_{p'}$ are simplicial maps induced by poset morphisms $f_{pp'} : Q_p \to Q_{p'}$.

Let $\mathcal{D}$ be a poset diagram. Then we define the poset $P\underrightarrow{\lim}\,\mathcal{D}$ on the set $\biguplus_{p \in P} Q_p \times \{p\} = \{(q,p) : p \in P, q \in Q_p\}$ by setting $(q,p) \geq (q',p')$ if and only if $p \geq p'$ and $f_{pp'}(q) \geq q'$.

Proposition 3.16 (SIMPLICIAL MODEL LEMMA)
For every poset diagram $\mathcal{D}$ over a finite poset P, there is a natural homotopy equivalence

$$\Psi : \mathrm{ho}\underrightarrow{\lim}\,\mathcal{D} \;\to\; \Delta(P\underrightarrow{\lim}\,\mathcal{D}).$$

The PROJECTION LEMMA studies the effect of collapsing the order complexes $\Delta(P_{\leq p})$ in the construction of the homotopy limit of a P-diagram $\mathcal{D}$ Now we are going to study the effect of collapsing the spaces D_p, $p \in P$. First we observe that this collapsing induces a map $\mathrm{ho}\underrightarrow{\lim}\,\mathcal{D} \to \Delta(P)$ for a P-diagram $\mathcal{D}$. This map turns out to be a quasifibration, a concept introduced by Dold & Thom [8], under certain strong conditions. A map $f : X \to Y$ between topological spaces is a *quasifibration* if for any $y \in Y$, $x \in f^{-1}(y)$ the induced maps $f_* : \pi_i(X, f^{-1}(y), x) \to \pi_i(Y, y)$ are isomorphism for all $i \geq 0$.

Proposition 3.17 (QUASIFIBRATION LEMMA) [19],[9, (9.10)])
If $\mathcal{D}$ is a P-diagram such that all the maps $d_{pp'}$ are homotopy equivalences then the natural projection map $\mathrm{ho}\underrightarrow{\lim}\,\mathcal{D} \longrightarrow \Delta(P)$ is a quasi-fibration. In particular, if $\Delta(P)$ is contractible, then for every $p \in P$ the inclusion of D_p into $\mathrm{ho}\underrightarrow{\lim}\,\mathcal{D}$ induces a homotopy equivalence.

4 Björner's generalized homotopy-complementation formula

Björner's generalized homotopy-complementation formula [4] is an effective tool to compute the homotopy type of a simplicial complex Δ in the case when a large, contractible induced subcomplex $\Delta_{\overline{A}}$ is known, whose connections to the rest of the complex are not too complicated.

In the following, we provide a "homotopy limits" proof of Björner's result, thereby demonstrating the applicability of some of our lemmas.

Let $\Delta \subseteq 2^S$ be a finite (abstract) simplicial complex with vertex set S, let $A \subseteq S$ be a subset of its vertex set, and denote by $\overline{A}$ the complement $S \backslash A$ of A.

Let $\Delta_A := \{\sigma \in \Delta : \sigma \subseteq A\} = \Delta \cap 2^A$ be the induced subcomplex on A, and similarly for $\Delta_{\overline{A}}$, the induced complex on $\overline{A}$. In the following, the key assumption we will make is that $\Delta_{\overline{A}}$ is contractible.

Theorem 4.1 (GENERALIZED HOMOTOPY-COMPLEMENTATION FORMULA: Björner [4])
For any simplicial complex $\Delta \subseteq 2^S$ and $A \subseteq S$, define a new simplicial complex T_A by taking the union of all the simplicial complexes

$$(p \uplus \sigma) * (\mathrm{star}_\Delta(\sigma) \cap \Delta_{\overline{A}})$$

where p is an additional point $p \notin S$, and $*$ denotes the join of two complexes.

If $\Delta_{\overline{A}}$ is contractible, then Δ and T_A are homotopy equivalent.

We note that there are alternative ways to describe the construction of T_A. For example, one can (as Björner does in his manuscript [4]) start from a wedge (or a disjoint union) of the spaces $E_{(\sigma,\sigma)} = (p \uplus \sigma) * (\mathrm{star}_\Delta(\sigma) \cap \Delta_{\overline{A}})$, and then check that all the identifications of the limit $\underrightarrow{\lim}\,\mathcal{E}$

are generated by identifying, for $\sigma \subset \tau$, the identical subcomplexes $(p \uplus \sigma) * (\mathrm{star}_\Delta(\tau) \cap \Delta_{\overline{A}})$ in $E_{(\sigma,\sigma)}$ and in $E_{(\tau,\tau)}$.

Also, there are countless variations possible, corresponding to different coverings of the complex Δ. The beauty of Björner's set-up is that his transformations of Δ end up with a subspace diagram, and thus with a limit instead of a homotopy limit, which leads to an effective model for $\Delta/\Delta_{\overline{A}}$.

5 Toric Varieties

In this section we give a representation of the topological space underlying a toric variety (see Danilov [7], Fulton [13], and Ewald [11] for general background on toric varieties) as the homotopy limit of a diagram. For this we recall a description, due to MacPherson (see Yavin & Fischli [29], [12]), of a toric variety. A decomposition of $\mathbb{R}^n$ into a complex of closed, convex, polyhedral cones with apex $\mathbf{0}$ is called a *complete fan.* If all cones in Σ are generated by lattice points in $\mathbb{Z}^n$, then Σ is called *rational.* Assume that Σ is a complete and rational fan in $\mathbb{R}^n$. Then let $\mathcal{P}$ be a cell decomposition of the unit ball in $\mathbb{R}^n$ that is dual to the one induced by Σ. For $\sigma \in \Sigma$ we denote by $\check{\sigma}$ the cell in $\mathcal{P}$ that corresponds to σ. Thus $\check{\sigma}$ is a cell of dimension $n - \dim(\sigma)$.

We identify the n-torus $\mathcal{T}^n$ with the image of the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. For all cones $\sigma \in \Sigma$ the image of σ under this projection is a subtorus $\pi(\sigma) = \pi(\mathrm{span}_{\mathbb{R}}(\sigma)) = \mathcal{T}_\sigma$ of $\mathcal{T}$. Since σ is rational, this is a closed subtorus of dimension $\dim(\sigma)$. Thus the quotient $\mathcal{T}^n/\mathcal{T}_\sigma$ is a real torus of dimension $n - \dim(\sigma)$.

The *toric variety* X_Σ is obtained from $\mathcal{P} \times \mathcal{T}^n$ by modding out $(\check{\sigma})^\circ \times \mathcal{T}^n$ by the action of $\mathcal{T}_\sigma$ on $\mathcal{T}$ for each $\sigma \in \Sigma$. This leads to a nice (compact, Hausdorff) quotient space since we mod out by larger tori on $\partial\sigma \times \mathcal{T}^n$. In particular, we see that the toric variety X_Σ has a well-defined map $\Pi : X_\Sigma \to \mathcal{P}$, for which the fiber over any interior point of $\check{\sigma}$ is isomorphic to $\mathcal{T}/\mathcal{T}_\sigma$.

Let P_Σ be the poset whose elements are in bijection with the cones in Σ and whose order relation is defined by the reversed inclusion of the cones in Σ. Thus P_Σ is the poset of non-empty faces of $\mathcal{P}$, ordered by inclusion. In particular, P has a largest element $\hat{1}$ corresponding to the 0-dimensional cone $\{\mathbf{0}\}$. We construct a diagram $\mathcal{D}_\Sigma$ over the poset P_Σ as follows. For $\sigma \in \Sigma$, set $D_\sigma = \mathcal{T}/\mathcal{T}_\sigma$. Topologically, D_σ is a $n - \dim(\sigma)$ torus. The map $d_{\tau,\sigma}$ for $\tau \subseteq \sigma$ is the map induced by the projection $\mathcal{T}/\mathcal{T}_\tau \to \mathcal{T}/\mathcal{T}_\sigma$.

Proposition 5.1 *Let Σ be a complete and rational fan in $\mathbb{R}^n$. Then the toric variety X_Σ is homeomorphic to the homotopy limit of the diagram $\mathcal{D}_\Sigma$ associated with Σ:*

$$\underrightarrow{\mathrm{holim}}\, \mathcal{D}_\Sigma \cong X_\Sigma.$$

The resolution of singularities of a toric variety also fits our homotopy limit framework. Namely, let Σ', Σ be two complete, rational fans in $\mathbb{R}^n$ such that Σ' is a refinement of Σ (i.e., for every open cone τ'° in Σ' there is an open cone τ° in Σ such that $\tau'^\circ \subseteq \tau^\circ$). Thus there is an induced map $f : P_{\Sigma'} \to P_\Sigma$. Also assume that τ' is a cone in Σ' whose interior is contained in the interior of the cone τ of Σ. Then the inclusion $\tau' \hookrightarrow \tau'$ induces a surjective map $\phi_{\tau'\tau} : \mathcal{T}/\mathcal{T}_{\tau'} \to \mathcal{T}/\mathcal{T}_\tau$. Is is easily seen that ϕ induces a map of diagrams. Hence there is an induced map $\widehat{\phi} : \underrightarrow{\mathrm{holim}}\, \mathcal{D}_{\Sigma'} \cong X_{\Sigma'} \to \underrightarrow{\mathrm{holim}}\, \mathcal{D}_\Sigma \cong X_\Sigma$. The map $\widehat{\phi}$ is surjective since f and all $\phi_{\tau'\tau}$ are surjective.

Proposition 5.2 *Let Σ' be complete rational fan which is the refinement of the complete rational fan Σ. Then there is a surjective map $\widehat{\phi} : X_{\Sigma'} \to X_\Sigma$.*

It is well known that X_Σ is non-singular if and only if Σ is simplicial (i.e., all cones are simplicial) and unimodular (i.e., all full-dimensional cones are equivalent to $\{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$ under unimodular transformations from $GL(\mathbb{R}^n, \mathbb{Z})$). For an example that shows that $\widetilde{H}^*(X_\Sigma)$ is not a combinatorial invariant of Σ in general see [16]. It is also well known that for any complete rational fan Σ there is a simplicial and unimodular complete rational fan Σ' which refines Σ. In this case $\widehat{\phi}$ is a resolution of singularities.

One can also use our results to investigate the (co)homology of a toric variety. For this we set up a spectral sequence introduced by Segal [21], which uses the filtration of $\underrightarrow{\mathrm{holim}}\,\mathcal{D}$ by the s-skeleta of the order complexes. For a simplicial complex Δ we denote by Δ^s its s-skeleton.

Assume $\mathcal{D}$ is a P-diagram for a poset P. Then we denote by $\underrightarrow{\mathrm{holim}}\,\mathcal{D}^s$ the image of $\biguplus_{p\in P} D_p \times \Delta(P_{\leq p})^s$ in $\underrightarrow{\mathrm{holim}}\,\mathcal{D}$. The filtration $\underrightarrow{\mathrm{holim}}\,\mathcal{D}^0 \subseteq \underrightarrow{\mathrm{holim}}\,\mathcal{D}^1 \subseteq \cdots \underrightarrow{\mathrm{holim}}\,\mathcal{D}$ defines a spectral sequence with termination $\widetilde{H}_*(\underrightarrow{\mathrm{holim}}\,\mathcal{D})$ in the E^2-term and $E^1_{st} = \widetilde{H}_{s+t}(\underrightarrow{\mathrm{holim}}\,\mathcal{D}^s, \underrightarrow{\mathrm{holim}}\,\mathcal{D}^{s-1})$. Following Segal's arguments on finds that E^1_{st} is given by $\bigoplus_{\sigma_0<\cdots<\sigma_s\in\Delta(P)^n} \widetilde{H}_t(\mathcal{T}/\mathcal{T}_{\sigma_s})$. Now assume $(\sigma_0 < \cdots < \sigma_s) \times c$ is a $(s+t)$-cell in $\underrightarrow{\mathrm{holim}}\,\mathcal{D}^s$. Then the differential of the cell complex $\underrightarrow{\mathrm{holim}}\,\mathcal{D}$ is given by $\partial(\sigma_0 < \cdots < \sigma_s) \times c =$

$$\sum_{i=0}^{s-1}(-1)^i(\sigma_0 < \cdots < \widehat{\sigma_i} < \cdots < \sigma_s) \times c \, +$$

$$+\,(-1)^{s-1}(\sigma_0 < \cdots < \sigma_{s-1}) \times d_{\sigma_s\sigma_{s-1}}(c) + (-1)^n(\sigma_0 < \cdots < \sigma_s) \times \partial c.$$

Thus the differential $d^1_{st} : E^1_{st} \to E^1_{s-1,t}$ applied to the cell $(\sigma_0 < \cdots < \sigma_s) \times c$ equals $\sum_{i=0}^{s-1}(-1)^i(\sigma_0 < \cdots < \widehat{\sigma_i} < \cdots < \sigma_s) \times c + (-1)^{s-1}(\sigma_0 < \cdots < \sigma_{s-1}) \times d_{\sigma_s\sigma_{s-1}}(c)$, where c is a cycle in $H_t(\mathcal{T}/\mathcal{T}_{\sigma_s})$.

From this it is easily seen that our spectral sequence is isomorphic to the deRahm-Hodge spectral sequence applied by Danilov [7, Chap 3, §12] to compute the cohomology of a toric variety.

6 Subspace Arrangements

Arrangements of affine subspaces in $\mathbb{R}^n$ also allow an application of the homotopy limit method. Let $\mathcal{A}$ be a finite set of affine subspaces in $\mathbb{R}^n$. Let us denote by $\widehat{\mathcal{A}}$ the corresponding arrangement of spheres in the one-point compactification S^n of $\mathbb{R}^n$. Under our assumptions intersections of spheres in $\widehat{\mathcal{A}}$ are again spheres. The following result can be deduced from the PROJECTION, HOMOTOPY and the WEDGE LEMMA.

Theorem 6.1 (Ziegler & Živaljević [30])
Let $\mathcal{A}$ be a finite set of affine subspaces in $\mathbb{R}^n$. Let $\widehat{U_\mathcal{A}}$ be the one-point compactification of the set-theoretic union of the subspaces in $\mathcal{A}$ and let P be the intersection poset of $\mathcal{A}$. Then

$$\widehat{U_\mathcal{A}} \;\simeq\; \bigvee_{p\in P} S^{\dim(p)} * \Delta(P_{<p}).$$

An equivalent result can be found in Vassiliev [26, III. §6. Theorem 1]. In Vassiliev's formulation the spaces $\Delta(P_{<p})$ are replaced by quotients of simplices by crosscut complexes, the

spaces $K(p)$ in his notation. More precisely, for an arbitrary subspace V corresponding to some point $p = p_V$ in the intersection lattice P of $\mathcal{A}$, let $V_1, \ldots, V_t$ be the subspaces in $\mathcal{A}$ such that V_i contains V as a subspace. Let $\Sigma(p)$ be the simplex which is spanned, in the abstract sense, by the vertices $V_1, \ldots, V_t$. Vassiliev calls a face τ of $\Sigma(p)$ *marginal* if V is **not** the intersection of the subspaces corresponding to the vertices of τ. Thus the marginal faces are the simplices in the crosscut complex $\Gamma(P_{\leq p})$ of $P_{\leq p}$. By the CROSSCUT THEOREM the complex of marginal faces is homotopy equivalent to $\Delta(P_{<p})$. In Vassiliev's formula the spaces $S^{\dim(p)} * \Delta(P_{<p})$ are replaced by $S^{\dim(p)-1} * \Sigma(p)/\Gamma(P_{\leq p})$. Let us analyse $\Sigma(p)/\Gamma(P_{\leq p})$. If $\Gamma(P_{\leq p})$ is the full simplex $\Sigma(p)$, then $\Sigma(p)/\Gamma(P_{\leq p})$ and by the CROSSCUT THEOREM also $\Delta(P_{<p})$ are contractible. In particular, $S^{\dim(p)} * \Delta(P_{<p})$ and $S^{\dim(p)-1} * \Sigma(p)/\Gamma(P_{\leq p})$ are contractible. If $\Gamma(P_{\leq p})$ is some non-empty part of the boundary of $\Sigma(p)$ then $\Sigma(p)/\Gamma(P_{\leq p})$ is the suspension of $\Gamma(P_{\leq p})$. Thus again the CROSSCUT THEOREM shows that $S^{\dim(p)} * \Delta(P_{<p})$ and $S^{\dim(p)-1} * \Sigma(p)/\Gamma(P_{\leq p})$ are homotopic. If $\Gamma(P_{\leq p})$ is empty, then we have to "interpret" $\Sigma(p)/\Gamma(P_{\leq p})$ as the suspension of the empty space, which is in our definition of the join with a two point space. Then the homotopy equivalence also follows in this case.

By Alexander duality on S^n we infer from Theorem 6.1 the following formula of Goresky & MacPherson [14].

Theorem 6.2 (Goresky & MacPherson [14])
Let $\mathcal{A}$ be a finite set of affine subspaces in $\mathbb{R}^n$. Let $M_{\mathcal{A}}$ be the complement $S^n - \widehat{U_{\mathcal{A}}}$ and let P be the intersection poset of $\mathcal{A}$. Then

$$\widetilde{H}^i(M_{\mathcal{A}}, \mathbb{Z}) \cong \bigoplus_{p \in P} \widetilde{H}_{\operatorname{codim}(p)-i-2}(\Delta(P_{<p}), \mathbb{Z}),$$

where $\operatorname{codim}(p)$ *denotes the real codimension of the subspace corresponding to p.*

Analogous results for arrangements of spheres and projective spaces can be found in Ziegler & Živaljević [30].

7 Subgroup Complexes

The order complex of the poset $S_p(G) = \{P \leq Q : |P| = p^i \neq 1\}$ of non-trivial p-subgroups of a finite group G has received considerable interest over that past few years (see for example [1]). It was already observed by Quillen [20] that $S_p(G)$ is homotopy equivalent to the poset $A_p(G)$ of non-trivial elementary abelian p-subgroups of G. In [17] the authors consider the covering of $\Delta(A_p(G))$ by the subcomplexes $\Delta(A_p(NA))$ for a fixed solvable normal p'-subgroup N and maximal elementary abelian subgroup A of G. Then they use the following facts :

(a) Intersections of the spaces of type $\Delta(A_p(NA))$ are again of the type $\Delta(A_p(ND))$ for some elementary abelian p-subgroup D of G.

(b) For a solvable normal p'-group N and an elementary abelian p-subgroup A the complex $\Delta(A_p(NA))$ are homotopic to a wedge of spheres of dimension $\operatorname{rank}(D) - 1$.

Observation (a) follows by basic group theoretical argumentation. Assertion (b) is much less obvious. It was established by Quillen [20, Theorem 11.2], but also follows by applications of the homotopy limit methods (see [17, Theorem (A)]). Using facts (a) and (b), the PROJECTION, HOMOTOPY and the WEDGE LEMMA the following wedge decomposition of $\Delta(A_p(G))$ for finite solvable groups G with non-trivial normal p'-group is proved in [17].

Theorem 7.1 (Pulkus & Welker [17, Theorem (B)])
Let G be a finite group and let p be a prime. Let N be a solvable normal p'-subgroup. Let CN/N be the intersection of all maximal elementary abelian p-subgroups of G/N. For $AN/N \in A_p(G/N)$ let $c_{AN/N}$ be an arbitrary but fixed point in $\Delta(A_p(G/N)_{>AN/N})$. Then $\Delta(S_p(G))$ is homotopy equivalent to

$$\bigvee_{AN/N \in A_p(G/N)_{>CN/N} \cup \{CN/N\}} \Delta(A_p(NA)) * \Delta(A_p(G/N)_{>AN/N}).$$

where the wedge is formed by identifying, for $AN/N > N/N$,

$$\begin{array}{rl} \textit{the point} & c_{AN/N} \in \Delta(A_p(G/N)_{>AN/N}) * A_p(NA) \\ \textit{with the point} & AN/N \in \Delta(A_p(G/N)) * \Delta(A_p(N1)). \end{array}$$

In particular, if A is a maximal elementary abelian group of rank r in G, then $\Delta(A_p(NA)) * \Delta(A_p(G/N)_{>AN/N})$ is homotopic to a wedge of $(r-1)$-spheres.

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