An Algorithm for Computing Plethysm Coefficients

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Abstract: The plethysm of two Schur functions can be expressed as a sum of Schur functions with nonnegative integer coefficients. Current algorithms for computing plethysms are designed to compute the whole expansion. However, in some applications only a few coefficients are of interest. In this work, we develop an algorithm for calculating individual plethysm coefficients. We also give a simple result concerning the zero coefficients which is obtained from the combinatorial properties of the Kostka and the inverse Kostka numbers

Résumé:
Le pléthysme de deux fonctions de Schur peut s'exprimer comme une combinaison linéaire de fonctions de Schur à coefficients entiers positifs. Les algorithmes actuellement connus calculent le développement complet. Toutefois, pour certaines applications, on ne s'intéresse qu'à un petit nombre de coefficients. Dans ce travail, nous développons un algorithme permettant le calcul indidividuel des coefficients. Nous donnons aussi un résultat simple concernant les coefficients nuls, obtenu à partir des propriétés combinatoires de la matrice de Kostka et de son inverse.

1 Introduction

Given two Schur functions $s_\lambda(x)$ and $s_\mu(x)$ where $x = (x_1, x_2, \cdots,)$, λ and μ are partitions of weight m and n, respectively, the plethysm $s_\lambda[s_\mu(x)]$ is the symmetric function obtained by substituting the monomials in $s_\mu(x)$ for the variables of $s_\lambda(x)$. D.E. Littlewood introduced this operation more than 50 years ago in connection with the representation theory of the matrix groups and showed that $s_\lambda[s_\mu(x)] = \sum_{\gamma \vdash mn} c^\gamma_{\lambda,\mu} s_\gamma(x)$ with nonnegative integer coefficients. Current algorithms [1] for computing the expansion of $s_\lambda[s_\mu(x)]$ make use of the plethysm of $s_\lambda(x)$ with the power sum symmetric function $p_k(x)$, and involve multiplications of Schur functions. The expansion coefficients are determined at the end of the algorithm, when similar terms are collected and combined. An obvious drawback of this type of algorithm is that one can not calculate a single coefficient without going through the whole expansion. In particular, one cannot know the zero coefficients in advance.

In this paper, we derive a formula for the coefficient $c^\gamma_{\lambda,\mu}$ in the expansion of $s_\lambda[s_\mu(x)]$ by using the transition matrices among various bases of the ring Λ^n of symmetric functions of homogeneous degree n. As it turned out, the formula involves three types of objects: the characters of the symmetric group, the Kostka numbers and the inverse Kostka numbers of the nested shapes. We use the combinatorial properties of these objects to derive a simple result on the zero coefficients. We introduce *bubble sequences* and *bubble movements* and describe an algorithm for computing the nested inverse Kostka numbers. Finally, we give an example of computing the plethysm coefficient.

Throughout this paper, by *partition* we mean a weakly *increasing* sequence of positive integers. The common notation λ', $\ell(\lambda)$, and $|\lambda|$ are used for the *conjugate, length* and *weight* of a partition λ respectively, and the notation $\lambda \vdash n$ indicates that λ is a partition of n. For an integer k we define $k\lambda = (k\lambda_1, k\lambda_2, \cdots, k\lambda_\ell)$. Let λ/μ denote a skew partition, where $\lambda \supset \mu$. Define the *column width* of a skew partition $c(\lambda/\mu)$ to be the longest row in the Young diagram of λ/μ, and the *row width* $r(\lambda/\mu) = c(\lambda'/\mu')$. When $\mu = \emptyset$, $c(\lambda)$ and $r(\lambda)$ are equal to the total number of columns and rows covered by λ, respectively. For $F(x) \in \Lambda^n$ and $\gamma \vdash n$, we use $< F(x), s_\gamma(x) >$ to denote the coefficient of $s_\gamma(x)$ in the expansion of $F(x)$.

Recall from the theory of symmetric functions (cf. [4]) that the ring Λ^n of symmetric functions has five well-known bases: the Schur functions $s_\lambda(x)$; the monomial symmetric functions $m_\lambda(x)$; the elementary symmetric functions $e_\lambda(x)$; the homogeneous symmetric functions $h_\lambda(x)$; and the power sum symmetric functions $p_\lambda(x)$; where $\lambda \vdash n$ and $x = (x_1, x_2, \cdots,)$. These bases are connected via the transition matrices. In particular, we have

$$s_{\gamma/\sigma}(x) = \sum_{\tau \vdash |\gamma/\sigma|} K^{-1}_{\tau,\gamma/\sigma} h_\tau(x), \tag{1}$$

$$m_\alpha(x) = \sum_{\gamma} K^{-1}_{\alpha,\gamma} s_\gamma(x), \tag{2}$$

$$s_\lambda(x) = \sum_{\sigma \vdash |\lambda|} \frac{\chi^\lambda_\sigma}{z_\sigma} p_\sigma(x), \tag{3}$$

and

$$s_\mu(x) = \sum_{\alpha \vdash |\mu|} K_{\mu,\alpha} m_\alpha(x), \tag{4}$$

where χ^λ_σ is the character of the symmetric group S_n, and $z_\alpha = \prod_{i \geq 1} i^{n_i(\alpha)} n_i(\alpha)!$ with $n_i(\alpha)$ being the number of parts of α equal to i; $K_{\mu,\alpha}$ is the Kostka number; $K^{-1}_{\alpha,\gamma}$ and $K^{-1}_{\tau,\gamma/\sigma}$ are the inverse Kostka numbers.

Let $u(x), v(x)$ and $w(x)$ be symmetric functions. We have the following properties for plethysm (cf [3] and [4]):

Distributivity:

$$(u+v)[w] = u[w] + v[w], \tag{5}$$

and

$$(uv)[w] = u[w]v[w]. \tag{6}$$

Commutativity with power sum symmetric function:

$$u[p_k] = p_k[u]. \tag{7}$$

Conjugation:

$$< s_\lambda[s_\mu(x)], s_\gamma(x) >= \begin{cases} < s_\lambda[s_{\mu'}(x)], s_{\gamma'}(x) > & \text{if } |\mu| \text{ even} \\ < s_{\lambda'}[s_{\mu'}(x)], s_{\gamma'}(x) > & \text{if } |\mu| \text{ odd.} \end{cases} \tag{8}$$

These results will be used in deriving the formula for plethysm coefficients.

2 A Formula for the Plethysm Coefficients

Let σ, β and γ be partitions such that $|\gamma| = |\sigma||\beta|$. The coefficient of $s_\gamma(x)$ in the expansion of the product of a Schur function with a monomial symmetric function is

$$\begin{aligned} < s_\sigma(x)m_\beta(x), s_\gamma(x) > &= < m_\beta(x), s_{\gamma/\sigma}(x) > \\ &= < m_\beta(x), \sum_\tau K^{-1}_{\tau,\gamma/\sigma} h_\tau(x) > \quad \text{by } (1) \\ &= K^{-1}_{\beta,\gamma/\sigma}, \end{aligned} \tag{9}$$

since $< m_\beta(x), h_\tau(x) >= 1$ if $\beta = \tau$, 0 otherwise (cf [4]). Thus, the coefficient of $s_\gamma(x)$ in the product of two monomial symmetric functions is

$$\begin{aligned} < m_\alpha(x)m_\beta(x), s_\gamma(x) > &= < \sum_\sigma K^{-1}_{\alpha,\sigma} s_\sigma(x) m_\beta(x), s_\gamma(x) > \quad \text{by } (2) \\ &= < \sum_\sigma K^{-1}_{\alpha,\sigma} \sum_\tau K^{-1}_{\beta,\tau/\sigma} s_\tau(x), s_\gamma(x) > \quad \text{by } (9) \\ &= \sum_{\gamma \supset \sigma} K^{-1}_{\alpha,\sigma} K^{-1}_{\beta,\gamma/\sigma}. \end{aligned} \tag{10}$$

It is clear that this result can be generalized to more than two monomial symmetric functions by induction.

Proposition 2.1 *Let $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$ be ℓ partitions, and $|\gamma| = \sum_{i=1}^{\ell} |\alpha^{(i)}|$. Then,*

$$< m_{\alpha^{(1)}}(x) m_{\alpha^{(2)}}(x) \cdots m_{\alpha^{(\ell)}}(x), s_\gamma(x) >= N_{\gamma;\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(\ell)}} \tag{11}$$

with

$$N_{\gamma;\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(\ell)}} = \sum_{\gamma^{(1)},\gamma^{(2)},\ldots,\gamma^{(\ell)}} \prod_{i=1}^{\ell} K^{-1}_{\alpha^{(i)},\gamma^{(i-1)}/\gamma^{(i)}} \tag{12}$$

where the summation is over all sequences of nested partitions $\gamma = \gamma^{(0)} \supset \gamma^{(1)} \supset \gamma^{(2)} \supset \cdots \supset \gamma^{(\ell)} = \emptyset$ *satisfying* $|\gamma^{(i-1)}/\gamma^{(i)}| = |\alpha^{(i)}|$ *for* $1 \leq i \leq \ell$.

We shall refer to $N_{\gamma;\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(\ell)}}$ as the *nested inverse Kostka number of shape* γ, *type* $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell)}$. Using (11), we obtain the following formula for the coefficient of $s_\gamma(x)$ in the expansion of plethysm of two Schur functions.

Theorem 2.2 *Suppose* $|\gamma| = |\lambda||\mu|$. *The coefficient of* $s_\gamma(x)$ *in the expansion of* $s_\lambda[s_\mu(x)]$ *is*

$$< s_\lambda[s_\mu(x)], s_\gamma(x) >= \sum_{\sigma \vdash |\lambda|} \frac{\chi^\lambda_\sigma}{z_\sigma} \sum_{\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(\ell(\sigma))}} \prod_{i=1}^{\ell(\sigma)} K_{\mu,\alpha^{(i)}} N_{\gamma;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\ldots,\sigma_\ell\alpha^{(\ell(\sigma))}}, \tag{13}$$

where $\alpha^{(i)}$ *is a partition of weight* $|\mu|$ *for* $1 < i \leq \ell(\sigma)$, *and* $N_{\gamma;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\ldots,\sigma_\ell\alpha^{(\ell(\sigma))}}$ *is the nested inverse Kostka number of shape* γ, *type* $\sigma_1\alpha^{(1)}, \sigma_2\alpha^{(2)}, \cdots, \sigma_\ell\alpha^{(\ell(\sigma))}$.

Proof: By (3) and distributivity (5), we have

$$s_\lambda[s_\mu(x)] = \sum_{\sigma \vdash |\lambda|} \frac{\chi^\lambda_\sigma}{z_\sigma} p_\sigma[s_\mu(x)], \tag{14}$$

where

$$\begin{aligned} p_\sigma[s_\mu(x)] &= \prod_{i=1}^{\ell(\sigma)} p_{\sigma_i}[s_\mu(x)] \qquad \text{by (6)} \\ &= \prod_{i=1}^{\ell(\sigma)} s_\mu[p_{\sigma_i}(x)] \qquad \text{by (7)} \\ &= \prod_{i=1}^{\ell(\sigma)} \sum_{\alpha^{(i)} \vdash |\mu|} K_{\mu,\alpha^{(i)}} m_{\alpha^{(i)}}[p_{\sigma_i}(x)] \qquad \text{by (4) and (5)} \\ &= \prod_{i=1}^{\ell(\sigma)} \sum_{\alpha^{(i)} \vdash |\mu|} K_{\mu,\alpha^{(i)}} m_{\sigma_i\alpha^{(i)}}(x) \end{aligned} \tag{15}$$

where we have used the fact that $m_{\alpha^{(i)}}[p_{\sigma_i}(x)] = m_{\sigma_i\alpha^{(i)}}(x)$. The theorem follows by substituting (15) into (14), switching the order of the product and the summation and expressing $\prod_{i=1}^{\ell(\sigma)} m_{\sigma_i\alpha^{(i)}}(x)$ in terms of Schur functions using (11).

3 Characterization of the Zero Coefficients

We first briefly review the combinatorial interpretations of the Kostka and the inverse Kostka numbers.

Suppose α is a partition of n and λ/μ is a skew partition of the same weight. A *column-strict tableau of shape* λ/μ, *content* α is a filling of the Young diagram of shape λ/μ with numbers, $1, 2, \cdots, n$ such that exactly α_i copies of i are used for $1 \leq i \leq n$, and the numbers must be non-decreasing in each row and strictly increasing in each column. The total number of such tableaux is then equal to the Kostka number $K_{\lambda/\mu,\alpha}$.

The combinatorial interpretation for the inverse Kostka numbers was given by Egecioglu and Remmel (see [2]) in terms of *special rim-hook tabloids.* Recall that a *rim hook* (or border strip) h of a skew partition λ/μ is a consecutive sequence of cells along the North-Eastern boundary of the Young diagram of λ/μ such that any two consecutive cells of h shares an edge and the removal of the cells of h from λ/μ results in a diagram corresponding to another skew partition. Denote the number of rows and columns covered by h as $r(h)$ and $c(h)$, respectively, and let $|h|$ be the total number of cells contained in h, called the *length* of h. We call the rim hook h *special* if it contains the first cell of the first non-empty row of λ/μ. A *special rim hook tabloid of shape* λ/μ, *type* α is a sequence of partitions $H = (\lambda \supset \lambda^{(0)} \supset \lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(k)} = \mu)$ such that for $1 \leq i \leq k$, where $k \leq \ell(\lambda)$, $h_i = \lambda^{(i-1)}/\lambda^{(i)}$ is a special rim-hook of $\lambda^{(i-1)}/\mu$ and $order(|h_1|, |h_2|, \cdots, |h_k|) = \alpha$, where $order(i_1, i_2, \cdots, i_\ell)$ denotes the rearrangement of the sequence $(i_1, i_2, \cdots, i_\ell)$ in increasing order. The *row-sign* of H is defined by

$$\omega_r(H) = \prod_{h_i \in H} \omega_r(h_i), \quad \text{where} \quad \omega_r(h_i) = (-1)^{r(h_i)-1}. \tag{16}$$

Let $SRHT(\lambda/\mu, \alpha)$ denote the set of all special rim hook tabloids of shape λ/μ, type α. Then,

$$K^{-1}_{\alpha,\lambda/\mu} = \sum_{T \in SRHT(\alpha,\lambda/\mu)} \omega_r(T). \tag{17}$$

It is clear from the above combinatorial definition that when $c(\lambda/\mu)$ is bigger than $c(\alpha)$, it is impossible fill the diagram of shape λ/μ with special rim hooks of type α, since the hooks are not long enough. In this case, $SRHT(\lambda/\mu, \alpha) = \emptyset$ and the inverse Kostka number is zero. We have

$$\text{if} \quad K^{-1}_{\alpha,\lambda/\mu} \neq 0 \quad \text{then} \quad c(\lambda/\mu) \leq c(\alpha). \tag{18}$$

This leads to the following result on the zero plethysm coefficients.

Theorem 3.1 *Suppose $|\gamma| = |\lambda||\mu|$. Let $c(\gamma)$ and $r(\gamma)$ denote column and row widths of γ, respectively. Then, the coefficient of $s_\gamma(x)$ in the expansion of $s_\lambda[s_\mu(x)]$ is zero if*

$$c(\gamma) > |\lambda|c(\mu),$$

or,

$$r(\gamma) > |\lambda|r(\mu).$$

Proof: Suppose $< s_\lambda[s_\mu(x)], s_\gamma(x) > \neq 0$. By (13), there must exist some partitions σ and $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell(\sigma))}$ such that $N_{\gamma;\sigma_1\alpha^{(1)},\sigma_2\alpha^{(2)},\cdots,\sigma_{\ell(\sigma)}\alpha^{(\ell(\sigma))}} \neq 0$ and $K_{\mu,\alpha^{(i)}} \neq 0$ for $1 \leq i \leq \ell(\sigma)$. It follows from (12) and (18) that the largest possible column width of the i-th segment is

$$c(\gamma^{(i-1)}/\gamma^{(i)}) \leq \sigma_i c(\alpha^{(i)}). \tag{19}$$

On the other hand, it is well-known that in order for $K_{\mu,\alpha^{(i)}} \neq 0$, we must have $c(\alpha^{(i)}) \leq c(\mu)$. Thus, the largest possible column width of γ is

$$\begin{aligned} c(\gamma) &= \sum_{i=1}^{\ell(\sigma)} c(\gamma^{(i-1)}/\gamma^{(i)}) \\ &\leq \sum_{i=1}^{\ell(\sigma)} \sigma_i c(\alpha^{(i)}) \\ &\leq \sum_{i=1}^{\ell(\sigma)} \sigma_i c(\mu) \\ &= |\sigma| c(\mu) \\ &= |\lambda| c(\mu), \quad \text{since } |\sigma| = |\mu|. \end{aligned}$$

Now, by applying the above result to the RHS of (8) and using the fact that $|\lambda| = |\lambda'|$, we have that the largest possible column width of γ' is

$$c(\gamma') \leq |\lambda|c(\mu'),$$

or, equivalently,

$$r(\gamma) \leq |\lambda|r(\mu),$$

since $c(\gamma') = r(\gamma)$ for any partition λ.

4 Bubble Sequence and the Algorithm

In order to make use of (13) to compute the plethysm coefficients, we introduce the idea of *bubble sequence* and develop an algorithm to evaluate the nested inverse Kostka numbers. We remark that the nested inverse Kostka number $N_{\gamma;\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(\ell)}}$ is related to the character, the Kostka number and the inverse Kostka numbers as follows.

(a) When $\alpha^{(1)} = \alpha$, and $\alpha^{(i)} = \emptyset$ for $2 \le i \le \ell$,

$$N_{\gamma;\alpha^{(1)}} = K^{-1}_{\alpha,\gamma}. \tag{20}$$

(b) When each $\alpha^{(i)}$ has only one part, say $\alpha^{(i)} = (\alpha_i)$ for $1 \le i \le \ell$,

$$N_{\gamma;(\alpha_1),(\alpha_2),(\alpha_\ell)} = \chi^{\gamma}_{\alpha}. \tag{21}$$

(c) When $\alpha^{(i)} = (1^{\alpha_i})$ for $1 \le i \le \ell$,

$$N_{\gamma;(1^{\alpha_1}),(1^{\alpha_2}),\cdots,(1^{\alpha_\ell})} = K_{\gamma,\alpha'}. \tag{22}$$

Hence an algorithm for computing $N_{\gamma;\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(\ell)}}$ can be used for calculating all three objects.

A *bubble sequence* of length ℓ is defined to be a sequence of ℓ distinct nonnegative integers $B = (b_1, b_2, \cdots, b_\ell)$. We shall refer to the elements of B as *bubbles,* and the integers i such that $i \notin B$ and $0 \le i \le b_{max}$, the largest part of B, as *holes.* Associated with each partition λ, we have a bubble sequence $B_\lambda = (\hat\lambda_1, \hat\lambda_2, \cdots, \hat\lambda_i, \cdots, \hat\lambda_\ell)$, where $\hat\lambda_i = \lambda_i + i - 1$ for $1 \le i \le \ell$. Conversely, corresponding to each *ordered* bubble sequence $B = (b_1 < b_2 < \cdots < b_\ell)$, we have a unique partition $\lambda = (b_1, b_2, \cdots, b_i, \cdots, b_\ell) - (0, 1, \cdots i-1, \cdots, \ell - 1)$. We can construct an *(uneven) staircase diagram* from B as follows. Start from the point $P(0,0)$. For $0 \le i \le b_{\max}$, we move a pen (without lifting it off the paper) one unit to the right if $i \notin B$; one unit down if $i \in B$, until we reach a terminating position Q. This (uneven) staircase forms the North-Eastern boundary of the diagram. Now, draw a vertical line through P and a horizontal line through Q to complete the Western and the Southern boundary of the diagram. For example, the staircase diagram of the bubble sequence $B = (0, 1, 4, 8, 13)$ is

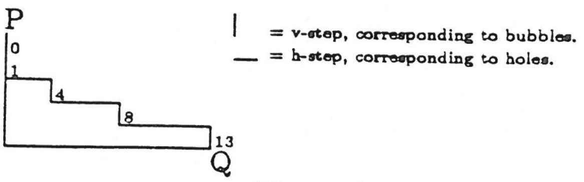

Figure 1

Note that the vertical steps (labelled) in Figure 1 correspond to the bubbles $0, 1, 4, 8, 13$ of B while the horizontal steps (not labelled) correspond to the holes $2, 3, 5, 6, 7, 9, 10, 11, 12$

of B. The partition corresponding to B is $\lambda = (0,1,4,8,13) - (0,1,2,3,4) = (0,0,2,5,9)$, whose Young diagram can be uniquely obtained from the above staircase diagram.

Let $1 \le j \le \ell$, and $n \ge 0$. We define a *bubble movement* $M(j,n)$ on a bubble sequence $B = (b_1, b_2, \cdots, b_\ell)$ by subtracting n from b_j:

$$M(j,n)B = (b_1, b_2, \cdots, b_j - n, \cdots, b_\ell). \tag{23}$$

Clearly, $M(j,0)$ is an identity operation. We call a bubble movement *legal* if the resulting sequence is again a bubble sequence. It is clear that the effect of a bubble movement is to destroy the bubble b_j (by turning it into a hole) and create a new bubble to fill the hole $b_j - n$. In the following discussion, we will assume that all bubble movements are legal. We will see that a bubble movement on B_λ corresponds to removal of a rim hook from λ, and vice versa.

Suppose $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_i, \cdots, \lambda_j, \cdots, \lambda_\ell)$ and μ is a partition obtained from λ by removing a rim hook h of length $|h|$ which starts at the i-th row and ends at the j-th row of λ, as shown in Figure 2.

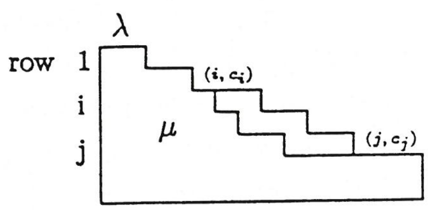

Figure 2

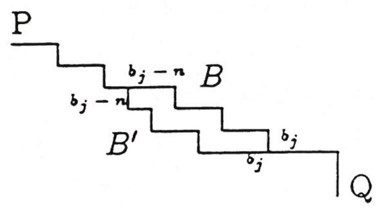

Figure 3

Let the coordinates of the starting and ending cell of h be (i, c_i) and (j, c_j), respectively. Then, c_i and c_j must satisfy

$$\lambda_{i-1} + 1 \le c_i \le \lambda_i, \quad \text{and} \quad c_j = \lambda_j. \tag{24}$$

It is clear from Figure 2 that the numbers of rows and columns covered by h are $r(h) = j - i + 1$ and $c(h) = \lambda_j - c_i + 1$, respectively and the length of h is

$$\begin{aligned} |h| &= r(h) + c(h) - 1 \\ &= j - i + \lambda_j - c_i + 1. \end{aligned} \tag{25}$$

It is also clear from the Figure 2 that μ is related to λ by

$$\mu_k = \begin{cases} \lambda_k & \text{for} \quad 1 \le k \le i-1 \\ c_i - 1 & \text{for} \quad k = i \\ \lambda_{k-1} - 1 & \text{for} \quad i < k \le j \\ \lambda_k & \text{for} \quad j < k \le \ell. \end{cases} \tag{26}$$

The effect of removing the rim hook h from λ can be better seen from the bubble sequence. Let $B_\mu = \mu + (0, 1, \cdots, \ell - 1) = (\hat{\mu}_1, \hat{\mu}_2, \cdots, \hat{\mu}_k \cdots, \hat{\mu}_\ell)$. By (26) and (25), $\hat{\mu}_k$ satisfy

$$\hat{\mu}_k = \begin{cases} \lambda_k + k - 1 & \text{for} \quad 1 \leq k \leq i-1 \\ c_i + i - 2 = \lambda_j + j - 1 - |h| & \text{for} \quad k = i \\ \lambda_{k-1} + k - 2 & \text{for} \quad i < k \leq j \\ \lambda_k + k - 1 & \text{for} \quad j < k \leq \ell, \end{cases} \tag{27}$$

and it follows from (23) that

$$B_\mu = order(M(j, |h|)B_\lambda). \tag{28}$$

Thus we have shown that removing a rim hook from λ is equivalent to applying a legal bubble movement on the sequence B_λ. It is not hard to show the converse is also true. Suppose B is an ordered bubble sequence and $B' = order(M(j, n)B)$. If we draw the staircase diagrams corresponding to the bubble sequences B' and B starting from the same point, then the region formed between them is exactly a rim hook, as it can be seen from Figure 3. This is due to the fact that the staircase diagram of B' contains exactly the same steps as B, except its $(b_j - n)$-th step is vertical instead of horizontal, and its b_j-th step is horizontal instead of vertical, as shown in Figure 3. The following results can be easily verified by looking at Figure 2 and 3.

Proposition 4.1 *Suppose $B_\lambda = (b_1, b_2, \cdots, b_\ell)$ and $B_\mu = order(M(j, n)B_\lambda)$ such that*

$$B_\mu = (b_1 < b_2 < \cdots b_{i-1} \overset{i}{<} b_j - n < b_i < \cdots < b_{j-1} \overset{j}{<} b_{j+1} < \cdots < b_\ell). \tag{29}$$

Then, $\lambda/\mu = h$ is a rim hook of λ such that

(a) *The length of h is $|h| = n$.*
(b) *The starting row of h is $s(h) = i$.*
(c) *The finishing row of h is $f(h) = j$.*
(d) *The number of rows covered by h is $r(h) = j - i + 1 = inv(M(j, n)B_\lambda) + 1$, where $inv(i_1, i_2, \cdots, i_\ell)$ denotes the number of inversions in the sequence:*

$$inv(i_1, i_2, \cdots, i_\ell) = \sum_{k=1}^{\ell} inv(i_k) \quad \text{where } inv(i_k) = \#\{i_m : i_m > i_k \text{ and } m < k\} \tag{30}$$

(e) *The row-sign of h is $\omega_r(h) = (-1)^{inv(M(j,n)B_\lambda)}$.*
(f) *The number of columns covered by h is $c(h) = n - (j - i)$.*

Now, consider the effect of applying successive bubble movements on a bubble sequence.

Proposition 4.2 *Suppose* $1 \le j_1 \ne j_2 \le \ell$, *and* $n_1, n_2 > 0$. *Let* $B_\lambda, B_{\lambda^{(1)}}$ *and* $B_{\lambda^{(2)}}$ *be bubble sequences corresponding to partitions* $\lambda, \lambda^{(1)}$ *and* $\lambda^{(2)}$ *respectively, such that* $B_{\lambda^{(1)}} = order(M(j_1, n_1)B_\lambda)$ *and* $B_{\lambda^{(2)}} = order(M(j_2, n_2)B_{\lambda^{(1)}})$, *which we write as*

$$B_\lambda \stackrel{M(j_1,n_1)}{\longrightarrow} B_{\lambda^{(1)}} \stackrel{M(j_2,n_2)}{\longrightarrow} B_{\lambda^{(2)}}. \tag{31}$$

Then, the sequence of partitions $H = (\lambda \supset \lambda^{(1)} \supset \lambda^{(2)})$ *is a special rim hook tabloid of shape* $\lambda/\lambda^{(2)}$, *type* (n_1, n_2) *iff* $b_{j_1} - n_1 < b_{j_2} - n_2$. *Further, the row-sign of* H *is*

$$\omega_r(H) = (-1)^{inv(M(j_2,n_2)M(j_1,n_1)B_\lambda)} \tag{32}$$

where $M(j_2, n_2)M(j_1, n_1)B_\lambda = M(j_2, n_2)(M(j_1, n_1)B_\lambda)$, *namely, the action is* from right to left.

Proof: Let $h_1 = \lambda/\lambda^{(1)}$ and $h_2 = \lambda^{(1)}/\lambda^{(2)}$, respectively. Then, by Prop.4.1, h_1 is a rim hook of λ with hook length n_1, and h_2 is a rim hook of $\lambda^{(1)}$ with hook length n_2, since we have assumed that the bubble movements are legal at each stage. Thus, (h_1, h_2) is a rim hook tabloid of shape $\lambda/\lambda_{(2)}$. When $b_{j_1} - n_1 < b_{j_2} - n_2$, the starting row of h_1 is above that of h_2, according to Prop. 4.1 (b). Hence the rim hook tabloid (h_1, h_2) is special. The row-sign of H is

$$\omega_r(H) = (-1)^{inv(M(j_1,n_1)B_\lambda)}(-1)^{inv(M(j_2,n_2)B_{\lambda^{(1)}})} \text{ by Prop. 4.1 (e).} \tag{33}$$

where by (30),

$$inv(M(j_1, n_1)B_\lambda) = inv(b_{j_1} - n_1),$$
$$inv(M(j_2, n_2)B_{\lambda^{(1)}}) = inv(b_{j_2} - n_2),$$

and

$$inv(M(j_2, n_2)M(j_1, n_1)B_\lambda) = inv(M(j_1, n_1)B_\lambda) + inv(M(j_2, n_2)B_{\lambda^{(1)}})$$

since $inv(b_{j_2} - n_2)$ has the same value in the sequence $M(j_2, n_2)B_{\lambda^{(1)}}$ and $M(j_2, n_2)M(j_1, n_1)B_\lambda$, due to the fact that $j_1 < j_2$ and $b_{j_1} - n_1 < b_{j_1}$.

We can generalize the result to more than two bubble movements.

Proposition 4.3 *Let* λ, μ *and* α *be partitions such that* $|\lambda/\mu| = \alpha$. *Let* σ *be a permutation of* $(1, \cdots, \ell)$ *where* $\ell = \ell(\alpha)$, *and suppose we can find a sequence of legal bubble movements such that*

$$B_\lambda \stackrel{M(j_1,\alpha_{\sigma_1})}{\longrightarrow} B_{\lambda^{(1)}} \stackrel{M(j_2,\alpha_{\sigma_2})}{\longrightarrow} B_{\lambda^{(2)}} \cdots \stackrel{M(j_\ell,\alpha_{\sigma_\ell})}{\longrightarrow} B_{\lambda^{(\ell)}=\mu}, \tag{34}$$

where $j_1, j_2, \cdots j_\ell$ are distinct positive integers. Then, the sequence of partitions $H_\sigma = (\lambda \supset \lambda^{(1)} \supset \lambda^{(2)} \ldots \lambda^{(\ell)} = \mu)$ is a special rim hook tabloid of shape λ/μ type α, iff $j_1 - n_1 < j_2 - n_2 < \cdots < j_\ell - n_\ell$. Furthermore, the row-sign of H_σ is

$$\omega_r(H_\sigma) = (-1)^{inv(\prod_{i=1}^{\ell} M(j_i, \alpha_{\sigma_i})B_\lambda)} = (-1)^{\epsilon(\sigma)} \tag{35}$$

where $\epsilon(\sigma)$ is the parity of the permutation σ, and

$$K^{-1}_{\alpha,\lambda/\mu} = \sum_{\sigma} \omega_r(H_\sigma), \tag{36}$$

where the summation is over all possible permutations σ such that (34) *is valid.*

Based on Prop. 4.3, we have the following procedure for computing $N_{\gamma;\alpha^{(1)}\alpha^{(2)}\ldots\alpha^{(\ell)}}$. It is understood here that $|\gamma| = \sum_{i=1}^{\ell} |\alpha^{(i)}|$, and $\ell(\gamma) \leq \sum_{i=1}^{\ell} c(\alpha^{(i)})$, for otherwise the coefficient is zero. Let $B_\gamma = (\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_n)$ where n is the length of γ, and $\hat{\gamma}_i = \gamma + i - 1$ for each i. The procedure consists of ℓ stages. At stage k, where $1 \leq k \leq \ell$, we compute $K^{-1}_{\alpha^{(k)},\gamma^{(k-1)}/\gamma^{(k)}}$ for all possible shapes $\gamma^{(k-1)}/\gamma^{(k)}$ by moving bubbles.

Stage 1: Each stage has three steps.

- step (a) Start with the bubble sequence B_γ of distinct numbers and subtract the numbers $\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_{\ell(\alpha^{(1)})}^{(1)}$ from the sequence such that the following three conditions are satisfied:
  - (i) No more than one number can be subtracted from the same position.
  - (ii) The resulting sequence must contain distinct non-negative integers.
  - (iii) The largest part of each resulting sequence can not exceed $\ell(\gamma) - 1 + \sum_{i=2}^{\ell} c(\alpha^{(i)})$.
- step (b) Count the inversion number *inv* of each bubble sequence obtained in step (a). Rearrange each sequence in increasing order, and attach to it the coefficient $(-1)^{inv}$.
- step (c) At the end of this stage, combine the identical ordered bubble sequences by adding together their coefficients.

Stage k: From each ordered bubble sequence with non-zero coefficient obtained in stage $k-1$, subtract the numbers $\alpha_j^{(k)}$ for $1 \leq j \leq \ell(\alpha^{(k)})$ (in any order), such that conditions (i)-(ii) are satisfied, and the largest part of the resulting sequence is at most $\ell(\gamma) - 1 + \sum_{i=k+1}^{\ell} c(\alpha^{(i)})$. Complete step (b) as for stage 1, and multiply $(-1)^{inv}$ with the old coefficient associated with each sequence. Complete step (c) as for stage 1.

Repeat stage k for $k = 2, 3, \cdots$ until stage ℓ is completed. There is only one bubble sequence left (if any), namely, $(0, 1, 2, \cdots, \ell(\gamma) - 1)$ corresponding to the empty partition, and its coefficient is equal to the number $N_{\gamma,\alpha^{(1)},\alpha^{(2)},\cdots,\alpha^{(\ell)}}$. Otherwise, the answer is zero.

Example 4.4 To compute $< s_{(2^2)}[s_{(12)}(x)], s_{(1^2235)}(x) >$ using Theorem 2.2, we first note that $\chi^{2^2}_{13} = -1$, $\chi^{2^2}_{2^2} = 2$, $\chi^{2^2}_{1^22} = 0$, and $\chi^{2^2}_{1^4} = 2$ by simple constructions, and $z_{13} = 3$, $z_{2^2} = 8$, $z_{1^4} = 24$, respectively. It is also easy to check that $K_{12,12} = 1$, $K_{12,1^3} = 2$ and $K_{12,3} = 0$ Thus, $\sigma = 13, 2^2$ or 1^4 only, while $\alpha^{(i)} = 12$ or 1^3 only, in the summations. Calculate the nested inverse Kostka numbers using the algorithm described above, we have $N_{1^2235;36,12} = -1, N_{1^2235;36,1^3} = 0, N_{1^2235;3^3,12} = 0, N_{1^2235;24,24} = -2, N_{1^2235;24,2^3} = 1, N_{1^2235;12,12,12,12} = -22, N_{1^2235;12,12,12,1^3} = 6, N_{1^2235;12,12,1^3,1^3} = -5$ and $N_{11^2235;12,1^3,1^3,1^3} = 6$. Note that the terms $N_{1^2235;2^3,2^3}$ and $N_{1^2235;1^3,1^3,1^3,1^3}$ are automatically zero since they do not satisfy the condition $\sum_{i=1}^{\ell} \sigma_i c(\alpha^{(i)}) \geq 5 = \ell(\gamma)$. Using the fact that the value of $N_{\gamma;\alpha^{(1)},\alpha^{(2)},\cdots,\alpha^{(\ell)}}$ is unchanged under permutation of $\alpha^{(i)}$s, we have

$$
\begin{aligned}
& < s_{(2^2)}[s_{(12)}(x)], s_{(1^2235)}(x) > \\
= \; & -1/3(N_{1^2235;36,12} + 2N_{1^2235;36,1^3} + 2N_{1^2235;3^3,12}) + 2/8(N_{1^2235;24,24} + C(2,1)2N_{1^2235;24,2^3}) \\
& +2/24(N_{1^2235;12,12,12,12} + C(4,1)2N_{1^2235;12,12,12,1^3} + C(4,2)2^2N_{1^2235;12,12,1^3,1^3} \\
& +C(4,3)2^3N_{11^2235;12,1^3,1^3,1^3}) \\
= \; & -1/3(-1 + 2 \cdot 0 + 2 \cdot 0) + 2/8(-2 + 4 \cdot 1) + 2/24(-22 + 8 \cdot 6 + 24 \cdot (-5) + 32 \cdot 6) \\
= \; & 9,
\end{aligned}
$$

where $C(n, k)$ is the binomial coefficient.

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