Special Power Series Solutions of Linear Differential Equations

Sergei A. Abramov[*]
Computer Center of
the Russian Academy of Science,
Vavilova 40, Moscow 117967, Russia.
`abramov@ccas.ru`

Marko Petkovšek[†]
Department of Mathematics,
University of Ljubljana,
Jadranska 19, 61111 Ljubljana, Slovenia.
`marko.petkovsek@mat.uni-lj.si`

Summary

We characterize generalized hypergeometric series that solve a linear differential equation with polynomial coefficients at an ordinary point of the equation, and show that these solutions remain hypergeometric at any other ordinary point. Therefore to find all generalized hypergeometric series solutions, it suffices to look at a finite number of points: all the singular points, and a single, arbitrarily chosen ordinary point.

We also show that at a point $x = a$ we can have power series solutions with:

- polynomial coefficient sequence – only if the equation is singular at $a + 1$,
- non-polynomial rational coefficient sequence – only if the equation is singular at a.

1 Introduction

We consider ordinary linear differential equations $Ly = 0$ where

$$L = p_r(x)\frac{d^r}{dx^r} + \cdots + p_1(x)\frac{d}{dx} + p_0(x) \tag{1}$$

is a linear differential operator with polynomial coefficients $p_0(x), p_1(x), \ldots, p_r(x) \in \mathbb{Q}[x]$, with $p_r(x) \not\equiv 0$. Our goal is to find all $a \in \mathbb{C}$ and all formal power series

$$y_a(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{2}$$

which satisfy $Ly_a = 0$, and whose coefficients a_n have a "nice" explicit representation in terms of n.

Technically, we cannot apply L to y_a because y_a is a power series in $x - a$, while the coefficients of L are power series in x. Therefore we *define* that $Ly_a = 0$ iff $L_a y = 0$ where

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{3}$$

$$L_a = p_r(x+a)\frac{d^r}{dx^r} + \cdots + p_1(x+a)\frac{d}{dx} + p_0(x+a). \tag{4}$$

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It is well known that the power series coefficients of a solution of a linear differential equation with polynomial coefficients satisfy a linear recurrence with polynomial coefficients. We will need this recurrence, so we reproduce it here. Following [5], we denote the falling and rising factorial powers by

$$x^{\underline{k}} = \begin{cases} 1, & k = 0, \\ x(x-1)\cdots(x-k+1), & k > 0, \end{cases} \qquad x^{\overline{k}} = \begin{cases} 1, & k = 0, \\ x(x+1)\cdots(x+k-1), & k > 0, \end{cases}$$

respectively. Write

$$p_j(x+a) = \sum_{i=0}^{d} c_{ij}(a)x^i, \quad (0 \le j \le r), \tag{5}$$

where not all $c_{dj}(a)$ are zero, not all $c_{ir}(a)$ are zero, and $c_{ij}(a) = 0$ unless $0 \le i \le d$ and $0 \le j \le r$. Let

$$b = \max_{0 \le j \le r} (\deg p_j(x) - j)\,.$$

If (3) satisfies (5), then the coefficients a_n satisfy

$$\sum_{k=0}^{r+b} a_{n+k} \sum_{j} (n+k)^{\underline{j}}\, c_{j-k+b,j}(a) = 0, \qquad \textit{for all integer } n, \tag{6}$$

where by convention, $a_n = 0$ for $n < 0$ (cf. [7, Eqn. 9]). This is a linear recurrence with polynomial coefficients in n.

So the problem of finding "nice" power series solutions (2) of $Ly_a = 0$ splits into two subproblems:

P1 Find all candidate values of a for which $L_a y = 0$ may have solutions of the form (3) with "nice" a_n.

P2 Find "nice" solutions a_n of the corresponding recurrence (6).

Once **P1** has been solved and the candidate expansion points a have been found, the algorithms of [2], [1], and [6], resp., can be used at each a (assuming there are finitely many of them) to find all polynomial, rational, resp. hypergeometric solutions of the corresponding recurrence (6). In particular, a detailed description of an algorithm to find all hypergeometric series solutions of (1) given the expansion point a is presented in [7]. This solves **P2**.

A short discussion of **P1** in the case of hypergeometric coefficients is given in [7, Sec. 3.2], but a completely satisfactory solution has not been provided yet. In this paper, we show how to find all $a \in \mathbb{C}$ and all solutions (2) of $Ly = 0$ for which there exists:

1. a polynomial $p(x)$ such that $a_n = p(n)$ for all large enough n (Section 3),
2. a rational function $r(x)$ such that $a_n = r(n)$ for all $n \ge 0$ (Section 4),
3. a rational function $R(x)$ such that $a_{n+1} = R(n)a_n$ for all large enough n (Section 5).

Of course, the first two problems are special cases of the last one, but they are sufficiently interesting to warrant individual treatment. We also show that existence of a power series solution with rational coefficients implies existence of a solution with rational logarithmic derivative.

2 Preliminaries

Let L be as in (1), and $a \in \mathbb{C}$. If $p_r(a) = 0$ then L is *singular* at $x = a$, and a is a *singular point* of L. Otherwise a is an *ordinary point* of L.

Let $\vartheta = x\frac{d}{dx}$. The following well-known result will be useful:

Lemma 1 *Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series, and $p(x)$ a polynomial. Then*

$$p(\vartheta)y(x) = p(\vartheta)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} p(n)a_n x^n.$$

Call a sequence $(a_n)_{n=0}^{\infty}$ *hypergeometric* if there is a rational function $R(x)$ such that $a_{n+1} = R(n)a_n$ for all *large enough n*. If a_n is hypergeometric then $R(x)$ is uniquely determined and we call it the *consecutive-term ratio* of a_n. Obviously, a rational sequence is hypergeometric, and the product of hypergeometric sequences is hypergeometric.

Two hypergeometric sequences a_n and b_n are *similar* if there is a rational function $r(x)$ such that $a_n = r(n)b_n$ for all large enough n. A linear combination of pairwise similar hypergeometric terms is obviously hypergeometric. Also, if a_n is hypergeometric and k a fixed integer, then a_{n+k} is similar to a_n.

A formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is called a *(generalized) hypergeometric series* if the sequence of coefficients $(a_n)_{n=0}^{\infty}$ is hypergeometric.

Lemma 2 *Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a hypergeometric series, and $p(x)$ a polynomial. Then $p(x)y(x)$ is a hypergeometric series.*

Proof: Let $p(x) = \sum_{k=0}^{d} c_k x^k$ and $p(x)y(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$p(x)y(x) = \sum_{k=0}^{d}\sum_{n=0}^{\infty} a_n c_k x^{n+k} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\min\{n,d\}} c_k a_{n-k},$$

so $b_n = \sum_{k=0}^{d} c_k a_{n-k}$ for $n \geq d$. This is a linear combination of pairwise similar hypergeometric terms, hence it is hypergeometric. □

If $f(x)$ and $g(x)$ are two formal power series such that $f(x) - g(x)$ is a polynomial, we write $f(x) \sim g(x)$. In particular, $f(x) \sim 0$ iff $f(x)$ is a polynomial.

3 Polynomial coefficients

Let $a_n = p(n)$ for some polynomial $p(x)$ and for all large enough n. Then, as it is well known, a_n satisfies a linear recurrence with constant coefficients, and its generating function (3) is a rational function of x, of the form

$$y(x) \sim \sum_{n=0}^{\infty} p(n)x^n = p(\vartheta)\sum_{n=0}^{\infty} x^n = p(\vartheta)\frac{1}{1-x} = \frac{P(x)}{(1-x)^{s+1}} \quad (7)$$

where P is a polynomial, $P(1) \neq 0$, and $\deg P = s = \deg p$. Since $y(x)$ is singular at $x = 1$, so is L_a, hence L is singular at $x = a + 1$. – Thus we have

Theorem 1 *Let L be a linear differential operator with polynomial coefficients, and a_n a polynomial function of n. If a series $y_a(x) \sim \sum_{n=0}^{\infty} a_n(x-a)^n$ satisfies $Ly_a = 0$, then L is singular at $x = a + 1$.*

Therefore to find solutions (2) of (1) with polynomial coefficients a_n, it suffices to consider all the roots of $p_r(x+1) = 0$ as candidate expansion points a, and to use the algorithm of [2] at each of them to find polynomial solutions of the corresponding recurrence (6).

4 Rational coefficients

Next we look for rational solutions a_n of (6). More precisely, now we require that there is a rational function $r(x)$ such that $a_n = r(n)$ for *all* $n \geq 0$. In particular, this means that $r(x)$ can have no nonnegative integer poles.

For a nonconstant irreducible polynomial $f(n)$ and a nonzero polynomial $g(n)$, denote

$$\deg(f(n); g(n)) = \max\{k \geq 0;\ f(n)^k \mid g(n)\}.$$

Proposition 1 *Let $a_n = p(n)/q(n)$ be a rational function of n with p and q relatively prime, and $f(n)$ any nonconstant, irreducible polynomial. If there are polynomials $p_0(n), p_1(n), \ldots, p_s(n)$ such that*

$$\sum_{j=0}^{s} p_j(n) a_{n+j} = 0, \quad p_0, p_s \neq 0, \tag{8}$$

then

$$\deg(f(n);\, q(n)) \leq \sum_{i=0}^{\infty} \deg(f(n+i);\, p_s(n-s)).$$

For a proof, cf. [1].

Assume that L is not singular at $x = a$, hence that $p_r(a) = c_{0,r}(a) \neq 0$. Then the leading term of (6) is the one with $k = r + b$, and its leading coefficient is

$$p_{r+b}(n) = \sum_j (n + r + b)^{\underline{j}} c_{j-r,j}(a) = c_{0,r}(a)(n + r + b)^{\underline{r}}.$$

The order of (6) in this case is $s = r + b$, so $p_s(n - s) = c_{0,r}(a) n^{\underline{r}}$. By Proposition 1, the denominator of a_n can only have irreducible factors of the form $n - k$ where k is a nonnegative integer. But for the series (3) to make sense, a_n can have no nonnegative integer poles. Hence a_n is a polynomial in n. We conclude that (4) can have non-polynomial rational solutions only when $p_r(a) = 0$.

Therefore to find solutions (2) of (1) with non-polynomial rational coefficients a_n, it suffices to consider the singular points of (1) as candidate expansion points a, and to use the algorithm of [1] at each of them to find rational solutions of the corresponding recurrence (6).

In [7], a function is called *d'Alembertian* if it can be written as $f_1(x) \int f_2(x) \int \cdots \int f_k(x) dx \ldots dx\, dx$ where the f_i have rational logarithmic derivatives. We want to show now that a power series with rational coefficients is a d'Alembertian function. Let $f(x) = \sum_{n=0}^{\infty} x^n/(n - \alpha)^k$ where α is not a nonnegative integer. Then

$$\vartheta^k x^{-\alpha} f(x) = \vartheta^k \sum_{n=0}^{\infty} \frac{x^{n-\alpha}}{(n-\alpha)^k} = \sum_{n=0}^{\infty} x^{n-\alpha} = \frac{x^{-\alpha}}{1-x}, \tag{9}$$

so

$$f(x) = x^{\alpha} \vartheta^{-k} \frac{x^{-\alpha}}{1-x} = x^{\alpha} \int \frac{1}{x} \cdots \int \frac{1}{x} \int \frac{x^{-\alpha-1}}{1-x} dx\, dx \ldots dx \tag{10}$$

(with k integral signs). This is clearly d'Alembertian. Now, if a_n is a rational function of n, its partial fraction decomposition

$$a_n = p(n) + \sum_{i=0}^{s} \sum_{j=0}^{t_j} \frac{\beta_{ij}}{(n - \alpha_i)^j}$$

together with (7), (10) and the fact that D'Alembertian functions form a ring, shows that (3) is d'Alembertian as well. But if $L_a y = 0$ has a d'Alembertian solution then it also has a solution with rational logarithmic derivative [3, Theorem 4], and so does $Ly = 0$. Thus we have

Theorem 2 *Let L be a linear differential operator with polynomial coefficients, and a_n a non-polynomial rational function of n. If the series $y_a(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ satisfies $L y_a = 0$, then L is singular at $x = a$, and the equation $Ly = 0$ has a solution with rational logarithmic derivative.*

Example 1 *The equation*

$$2x(x-1)y''(x) + (7x - 3)y'(x) + 2y(x) = 0 \tag{11}$$

is singular at $x = 0$ and $x = 1$. Let's find power series solutions at $x = 0$. Recurrence (6) in this case is

$$(n+1)(2n+3)a_{n+1} - (n+2)(2n+1)a_n = 0 \tag{12}$$

and is satisfied by the rational sequence $a_n = 2(n+1)/(2n+1)$ (and by any constant multiple of it). Thus (11) has a power series solution with rational coefficients

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{2(n+1)}{2n+1} x^n = \frac{1}{1-x} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n+1/2} \\ &= \frac{1}{1-x} + \frac{1}{2\sqrt{x}} \int \frac{dx}{\sqrt{x}(1-x)} = \frac{1}{1-x} + \frac{1}{2\sqrt{x}} \log \frac{1+\sqrt{x}}{1-\sqrt{x}}, \end{aligned}$$

using (10) and a suitable constant of integration. This is a d'Alembertian function. But (11) is also satisfied by $g(x) = 1/\sqrt{x}$, which has rational logarithmic derivative. □

We want to point out that using Theorem 2, we may not be able to find solutions where a_n does not equal $r(n)$ for some values of n. For instance, equation $(1-x)y'' - y' = 0$ has solution $y(x) = -\log(1-x) = \sum_{n=1}^{\infty} x^n/n$ with non-polynomial rational coefficients, although the equation is not singular at $x = 0$. This is because $a_0 = 0$ while $r(n)$ has a pole at $n = 0$. Such solutions are covered in the next section.

5 Hypergeometric coefficients

It turns out that to find power series solutions with hypergeometric coefficients, instead of (2) and (3) it is more convenient to write

$$y_a(x) = \sum_{n=0}^{\infty} b_n \frac{(x-a)^n}{n!} \tag{13}$$

and

$$y(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \tag{14}$$

respectively, where

$$b_n = a_n n!$$

is hypergeometric iff a_n is. Then instead of (6) we have

$$\sum_{k=0}^{r+b} b_{n+k} q_k(n) = 0, \qquad \textit{for all large enough } n, \tag{15}$$

where $q_k(n) = \sum_j (n+b)^{\underline{j}} c_{j,j+k-b}(a)$. Since $j+k-b \leq r$, it follows that $\deg q_k(n) \leq r+b-k$. In particular, $q_{r+b}(n)$ is constant.

Theorem 3 *Let $x = a$ be an ordinary point of L, and (13) a hypergeometric series solution of $Ly = 0$. Then for all large enough n,*

$$b_{n+1} = \zeta A(n) \frac{C(n+1)}{C(n)} b_n,$$

where $\zeta \in C \setminus \{0\}$ is a nonzero constant, A and C are polynomials, and $\deg A \leq 1$.

Proof: If b_n is eventually zero, this is trivially true. Otherwise b_n is eventually nonzero (because it satisfies a homogeneous first-order recurrence with rational coefficients). Let $R(n)$ be the rational function equal to b_{n+1}/b_n for all large enough n. We look for $R(n)$ in the form

$$R(n) = \zeta \frac{A(n)}{B(n)} \frac{C(n+1)}{C(n)} \tag{16}$$

where ζ is a nonzero constant, A, B, C are monic polynomials, $\gcd(A(n), B(n+k)) = 1$ for all nonnegative integers k, and $\gcd(A(n), C(n)) = \gcd(B(n), C(n+1)) = 1$ as well.

By [6, Theorem 5.1], $B(n)$ divides the leading coefficient of recurrence (15) which is

$$q_{r+b}(n) = \sum_j (n+b)^{\underline{j}} c_{j,j+r} = c_{0,r} = p_r(a) \neq 0,$$

a nonzero constant. So $B(n) = 1$. By that same theorem, ζ is a nonzero root of the algebraic equation

$$\sum_{k=0}^{r+b} \alpha_k \zeta^k = 0, \tag{17}$$

where α_k is the coefficient of n^M in $P_k(n) = q_k(n)\prod_{j=0}^{k-1} A(n+j)$, and $M = \max_{0 \le k \le r+b} \deg P_k$. Write $\delta = \deg A$. Since $\deg q_{r+b} = 0$ and $\deg q_k \le r+b-k$ for $k < r+b$, it follows that $\deg P_{r+b} = (r+b)\delta$ and $\deg P_k \le r+b-k(1-\delta)$ for $k < r+b$. If $\delta > 1$ then

$$\deg P_{r+b} - \deg P_k \ge (r+b)\delta - (r+b-k(1-\delta)) = (\delta-1)(r+b-k) > 0,$$

so $\deg P_k < \deg P_{r+b}$ for $k < r+b$. Therefore $M = r+b$ and all the α's are zero except α_{r+b}. Hence (17) has no nonzero roots, and (15) has no hypergeometric solution with $\delta > 1$. It follows that $\deg A = \delta \le 1$. □

Theorem 4 *Let $x = 0$ be an ordinary point of L, and $y(x) = \sum_{n=0}^{\infty} b_n x^n$ a hypergeometric series solution of $Ly = 0$. Then $y(x)$ has positive radius of convergence, and is of the form*

a) $y(x) \sim p(x)e^{\zeta x}$, or

b) $y(x) \sim p(x)(1-\zeta x)^\alpha$, or

c) $y(x) \sim p(x)/(1-\zeta x)^s + q(x)\log(1-\zeta x)$,

where $p(x), q(x)$ are polynomials, $q(x) \not\equiv 0$, $\zeta \in C \setminus \{0\}$, $\alpha \in C$, and s is a positive integer.

Proof: If $y(x)$ is a polynomial, this is trivially true. Otherwise, Theorem 3 implies that for all large enough n, $b_{n+1}/b_n = \zeta A(n)C(n+1)/C(n)$ where either $A(n) = 1$, or $A(n) = n - \alpha$ for some constant α. We distinguish three cases according to the form of A and the nature of α.

Case a) $A(n) = 1$
In this case, $b(n+1)/b(n) = \zeta C(n+1)/C(n)$, so $b_n = \lambda C(n)\zeta^n$ where λ is a constant. Hence by Lemma 1,

$$y(x) \sim \lambda \sum_{n=0}^{\infty} C(n)\frac{(\zeta x)^n}{n!} = \lambda C(\vartheta)e^{\zeta x} = p(x)e^{\zeta x} \tag{18}$$

where $p(x)$ is some polynomial of degree $s = \deg C(n)$.

Case b) $A(n) = n - \alpha$, where α is not a nonnegative integer
In this case, $b(n+1)/b(n) = \zeta(n-\alpha)C(n+1)/C(n)$, so $b_n = \lambda C(n)(-\alpha)^{\overline{n}}\zeta^n$ where λ is a constant. Hence by Lemma 1,

$$y(x) \sim \lambda \sum_{n=0}^{\infty} C(n)\frac{(-\alpha)^{\overline{n}}}{n!}(\zeta x)^n = \lambda C(\vartheta)\sum_{n=0}^{\infty}\binom{\alpha}{n}(-\zeta x)^n = \lambda C(\vartheta)(1-\zeta x)^\alpha = p(x)(1-\zeta x)^{\alpha-s} \tag{19}$$

where $p(x)$ is some polynomial and $\deg p = s = \deg C$.

Case c) $A(n) = n - \alpha$, with α a nonnegative integer
Here we still have the solution

$$y(x) \sim \lambda C(\vartheta)(1-\zeta x)^\alpha$$

which in this case is simply a polynomial in x. However, now there is another hypergeometric solution of (15), namely

$$b_n = \lambda C(n)(n-\alpha-1)!\zeta^{n-\alpha-1}, \qquad for\ n \geq \alpha+1,$$

which gives, using Lemma 1,

$$\begin{aligned} y(x) &\sim \lambda \sum_{n=\alpha+1}^{\infty} C(n)\frac{(n-\alpha-1)!}{n!}\zeta^{n-\alpha-1}x^n \\ &= \lambda C(\vartheta)\sum_{n=0}^{\infty}\frac{\zeta^n x^{n+\alpha+1}}{(n+1)^{\overline{\alpha+1}}} \\ &= \lambda C(\vartheta)\int\int\cdots\int\frac{1}{1-\zeta x}dx\ldots dx\,dx \end{aligned}$$

where there are $\alpha+1$ integral signs. Since the nested integral of $1/(1-\zeta x)$ has the form $P(x)\log(1-\zeta x)+Q(x)$ where P and Q are polynomials of degree $\leq \alpha$, we have finally

$$y(x) \sim \frac{p(x)}{(1-\zeta x)^s} + q(x)\log(1-\zeta x) \tag{20}$$

where p, q are polynomials, $\deg p \leq \alpha + s$, $\deg q \leq \alpha$, and $s = \deg C$. In fact, a more careful analysis shows that $p(x)$ is divisible by $(1-\zeta x)^\alpha$ if $s > \alpha$, and that it can be taken to be zero if $s \leq \alpha$. □

Corollary 1 *Let $x = a$ be an ordinary point of L, and $y_a(x) = \sum_{n=0}^{\infty} a_n (x-a)^n/n!$ a hypergeometric series satisfying $Ly_a = 0$. Then $y_a(x)$ is analytic at $x = a$, can be analytically continued to any ordinary point b of L, and its power series expansion at $x = b$ is hypergeometric.*

Proof: By our definition, $Ly_a = 0$ means $L_a y = 0$ where $y(x) = \sum_{n=0}^{\infty} a_n x^n/n!$, and L_a is as in (4). By Theorem 4, $y(x)$ has positive radius of convergence and can be analytically continued everywhere, except to $x = 1/\zeta$ in cases b) and c). It follows that $y_a(x)$ is analytic at $x = a$, can be analytically continued everywhere, except to $x = a + 1/\zeta$ in cases b) and c), and that $y_a(x) = y(x-a)$.

Let b be any ordinary point of L. Following Theorem 4, we now distinguish three cases.

Case a) $y(x) \sim p(x)e^{\zeta x}$
Here $y_a(x+b) = y(x-a+b) \sim \bar{p}(x)e^{\zeta x}$ where $\bar{p}(x) = e^{\zeta(b-a)}p(x-a+b)$.

Case b) $y(x) \sim p(x)(1-\zeta x)^\alpha$
Here $y_a(x+b) = y(x-a+b) \sim \bar{p}(x)(1-\bar{\zeta}x)^\alpha$ where $\bar{p}(x) = (1-\zeta(b-a))^\alpha p(x-a+b)$ and $\bar{\zeta} = \zeta/(1-\zeta(b-a))$.

Case c) $y(x) \sim p(x)/(1-\zeta x)^s + q(x)\log(1-\zeta x)$
Here $y_a(x+b) = y(x-a+b) \sim \bar{p}(x)/(1-\bar{\zeta}x)^s + \bar{q}(x)\log(1-\bar{\zeta}x)$ where $\bar{p}(x) = p(x-a+b)/(1-\zeta(b-a))^s$, $\bar{\zeta} = \zeta/(1-\zeta(b-a))$, and $\bar{q}(x) = q(x-a+b)$.

In the latter two cases, $y(x)$ and hence L_a is singular at $x = 1/\zeta$, therefore L is singular at $x = a + 1/\zeta$, so $a + 1/\zeta \neq b$ as b is an ordinary point of L, and $\zeta(b-a) \neq 1$. In the first two cases, $y_a(x+b)$ is a polynomial multiple of a hypergeometric series, which by Lemma 2 is again a hypergeometric series. In the last case, $y_a(x+b)$ is the sum of two such series. But the coefficients of both $1/(1-\bar{\zeta}x)^s$ with positive integer s, and of $\log(1-\bar{\zeta}x)$ are rational functions of n, so the coefficients of $y_a(x+b)$ are rational as well.

In all three cases, there exists a hypergeometric sequence $(b_n)_{n=0}^{\infty}$ such that $y_a(x+b) = \sum_{n=0}^{\infty} b_n x^n/n!$ in a neighborhood of $x = 0$. So $y_a(x) = \sum_{n=0}^{\infty} b_n (x-b)^n/n!$ is hypergeometric. □

By Corollary 1, the following algorithm will find all solutions (2) of $Ly_a = 0$ with hypergeometric a_n:

1. For each singular point a of L, find all solutions $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of $L_a y = 0$ with hypergeometric a_n, using the algorithm of [7]. Then the corresponding $y_a(x) = y(x-a)$ give all the hypergeometric series solutions at $x = a$.

2. Pick any ordinary point a of L. Find all solutions $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of $L_a y = 0$ with hypergeometric a_n, using either the algorithm of [7], or, since these solutions are linear combinations of terms with rational logarithmic derivative, the algorithm of [4], or a custom-designed algorithm for finding solutions of the three types described in Theorem 4. Then the corresponding $y_a(x) = y(x-a)$ give all the hypergeometric series solutions at $x = a$. For any other ordinary point b of L, $y_b(x) = y(x - a + b)$ has a hypergeometric expansion at $x = 0$, and the corresponding $y_a(x) = y_b(x-b)$ give all the hypergeometric series solutions at $x = b$.

6 Concluding remarks

Although we worked with the field of complex numbers and with analytic functions here, analogous results showing that there is a one-to-one correspondence between formal hypergeometric series solutions at any two ordinary points of the equation, can be established for any field of characteristic zero. Instead of representing power series by analytic functions, the proof would use their minimal annihilating differential operators.

Analogous techniques can also be used to find "nice" series solutions $y_a(x)$ of linear difference and q-difference equations using polynomial series expansions of [2], such as

$$y_a(x) = \sum_{n=0}^{\infty} a_n \binom{x-a}{n} \tag{21}$$

for difference equations, and

$$y_a(x) = \sum_{n=0}^{\infty} a_n \frac{(a/x;\, q)_n}{(n_q)!} x^n \tag{22}$$

for q-difference equations.

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