Homology Confinement for Signed Posets and Other Simplicial Complexes

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Abstract

P. Hanlon has recently proved that the homology of a signed poset, whose associated poset is Cohen-Macaulay of rank n, is confined to the interval [n/2,n]. In this note we give a simple explanation, by Alexander duality, to this phenomenon when the associated poset is acyclic Gorenstein. The results are extended to general group actions on simplicial complexes.

1 Introduction

Signed posets, as defined by Reiner [8], form a generalization of the concept of a partially ordered set. A homology theory for them, in terms of the simplicial homology of a certain subcomplex of the chain complex of an associated poset, has been defined by Fischer [4]. This homology has some interesting connections to the combinatorics of the signed poset. Fischer also defined a signed poset to be EL-labelable if the associated poset is EL-labelable, but noted that this condition fails to imply the Cohen-Macaulay property, namely: An EL-labelable signed poset may have nonzero homology in degrees below the rank of the associated poset. Motivated by Fischer's examples, Phil Hanlon [5] has recently shown that, nevertheless, if the associated poset is Cohen-Macaulay of rank n (as in the EL-labelable case) then the homology of the signed poset vanishes for degrees i outside the interval $\lfloor n/2 \rfloor \leq i \leq n$. In this note we intend to provide a simple topological explanation for this homology

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confinement phenomenon in the (special but important) case that the associated poset is Gorenstein* (acyclic Gorenstein). This explanation provides, in such a case, an alternative proof of Hanlon's result which avoids the machinery of spectral sequences. We also extend these results to more general group actions on simplicial complexes.

2 A Prototypical Result

Let us first recall a few definitions. An n-dimensional simplicial complex Δ is **Cohen-Macaulay** if the (reduced) homology and local homology groups at each point of the underlying topological space $X := |\Delta|$ vanish except, possibly, at the top dimension:

$$\tilde{H}_i(X) = H_i(X, X \setminus \{x\}) = 0 \qquad (\forall i \neq n, \forall x \in X). \tag{1}$$

Δ is **Gorenstein*** (or **acyclic Gorenstein**) if, moreover, its top-dimensional global and local homology groups are those of an n-sphere:

$$\tilde{H}_n(X) = H_n(X, X \setminus \{x\}) = Z \qquad (\forall x \in X). \tag{2}$$

In other words, an n-dimensional Gorenstein* complex is simultaneously a homology n-sphere and a homology n-manifold. Analogous definitions apply for coefficient groups other than the integers. The above definitions can be rephrased entirely in terms of simplicial homology (of the complex and of links of simplices in it), and there are also equivalent definitions in terms of algebraic properties of the Stanley-Reisner face ring of Δ [9]. The ubiquity of these concepts may be demonstrated by the fact that every shellable simplicial complex is Cohen-Macaulay, and the boundary complex of a simplicial convex polytope is Gorenstein*.

Recall also that a subcomplex Δ_1 of a simplicial complex Δ is **full** if every simplex of Δ whose vertices are in Δ_1 is itself in Δ_1. (For 1-dimensional complexes, this is the concept of **induced subgraph**.) Given Δ, a full subcomplex is determined by its set of vertices, and the collection of all simplices disjoint from the simplices of any subcomplex form the full subcomplex on the complementary set of vertices.

Theorem 1 *(Alexander duality for Gorenstein* complexes)*
Let Δ be a simplicial complex, and let Δ_1 and Δ_2 be full subcomplexes of Δ on complementary nonempty vertex sets. If Δ is n-dimensional Gorenstein then:*

$$\tilde{H}_i(\Delta_1) \cong \tilde{H}^{n-i-1}(\Delta_2) \qquad (\forall i). \tag{3}$$

Proof:
Under the given hypotheses, the underlying topological space $|\Delta_2|$ is a deformation retract of the complement $|\Delta| \setminus |\Delta_1|$ of $|\Delta_1|$ (cf. [7, Lemma 70.1]). Our claim is therefore a version of Alexander duality [7, Theorem 71.1] for complexes which are homology n-spheres and homology n-manifolds instead of being genuine topological

n-spheres. Careful examination of the proof of Alexander duality in [7, pp. 424–425], using Poincaré and Lefschetz duality, shows that it applies to this more general setting as well. □

Theorem 2 *Let Δ be a simplicial complex with a (simplicial) involution σ. Let Δ_1 be the subcomplex of Δ consisting of all simplices not containing more than one vertex from each σ-orbit. If Δ is n-dimensional Gorenstein*, then*

$$\tilde{H}_i(\Delta_1) = 0 \qquad (i < \lfloor n/2 \rfloor). \tag{4}$$

Proof:
We would like to find a "complementary" subcomplex Δ_2 in the sense of the preceding theorem. Unfortunately, Δ_1 is not a full subcomplex of Δ, since an edge of Δ whose vertices are interchanged by σ does not belong to Δ_1 although its vertices do. This problem can be overcome by forming the *stellar subdivision* of Δ with respect to these edges with interchanged endpoints. This refined complex Δ' is intermediate between Δ and its first barycentric subdivision $\mathrm{sd}(\Delta)$, and its vertices correspond to either the original vertices of Δ or to the above-mentioned edges of Δ. Δ' has Δ_1 as the full subcomplex on the original set of vertices of Δ. Let Δ_2 be the full subcomplex on the complementary set of vertices of Δ'. The vertices in a simplex of Δ_2 correspond to disjoint edges of Δ, each with endpoints interchanged by σ, which are all contained in the same simplex of Δ. Since a simplex of Δ has at most $n+1$ vertices, it follows that

$$\dim \Delta_2 \leq \lfloor (n+1)/2 \rfloor - 1.$$

Therefore, by the preceding theorem,

$$n - i - 1 \geq \lfloor (n+1)/2 \rfloor \Longrightarrow \tilde{H}_i(\Delta_1) \cong \tilde{H}^{n-i-1}(\Delta_2) = 0.$$

This range of values of i, where the homology of Δ_1 must vanish, may be written equivalently

$$i < n - \lfloor (n+1)/2 \rfloor = \lfloor n/2 \rfloor,$$

as claimed. □

The main result of [5] is a formulation of Theorem 2 for Cohen-Macaulay self-dual posets. The proof uses spectral sequences, and proceeds by induction on the size of the poset. These arguments can be extended rather easily to the case of an arbitrary Cohen-Macaulay complex with an involution, by using a slightly different filtration to define the spectral sequence. The proof given above provides a transparent explanation of why such a result should be true, at least in the Gorenstein* case. Extensions and analogues of this result, to be discussed in the next section, may probably be proved in the general Cohen-Macaulay context (say by Hanlon's methods) after having been "identified" in the Gorenstein* case.

Note also that, in the specific application to signed posets, the duality involution σ on the associated poset may have fixed edges but cannot have fixed vertices, since the poset associated to a signed poset has no self-dual elements. We shall come back to this point in the following section.

3 Extensions

We shall now use Theorem 1 to obtain various extensions and analogues of Theorem 2. The common setup for the various cases will be a simplicial complex Δ with a simplicial action of a finite group G. The complex Δ will be assumed to be n-dimensional Gorenstein*, and will contain two "complementary" complexes Δ_1 and Δ_2. As in Theorem 2, Δ_1 and Δ_2 will be full subcomplexes of a suitable refinement Δ' of Δ, on complementary vertex sets.

3.1 No Edge in An Orbit

Assume that G is an arbitrary finite group which acts on Δ. Let Δ_1 be the set of all simplices in Δ which contain at most one vertex from each G-orbit. Let Δ' be the subdivision of Δ obtained by introducing a new vertex for each simplex of Δ which is contained in a G-orbit and has more than one vertex. Two vertices of Δ in the same G-orbit cannot be the endpoints of an edge of Δ', and therefore Δ_1 is (homeomorphic to) the full subcomplex of Δ' on the original vertex set of Δ. Consider the complementary full subcomplex Δ_2 of Δ'. Each vertex of Δ_2 can be associated to one of the G-orbits of vertices in Δ. A k-simplex of Δ_2 all of whose vertices are associated to the same G-orbit corresponds to a *chain* of non-singleton Δ-simplices with vertices in that orbit. The maximal Δ-simplex in this chain has therefore dimension at least $k+1$. More generally, from each nonempty simplex s of Δ_2 one can construct a simplex of Δ of larger dimension, where the dimension difference is (at least) the number of G-orbits intersected by s. These larger simplices are characterized by the property that they never intersect a G-orbit in a single vertex. Since a maximal simplex in Δ contains $n+1$ vertices, it is easy to deduce that if

$$n+1 = gq + r \qquad (1 \le r \le g) \tag{5}$$

then each maximal Δ-simplex intersects at least $q+1$ distinct G-orbits. Therefore

$$\dim \Delta - \dim \Delta_2 \ge q + 1 = \lfloor n/g \rfloor + 1, \tag{6}$$

and, by Theorem 1, the homology group $\tilde{H}_i(\Delta_1)$ can be nonzero only if

$$i \ge n - 1 - \dim \Delta_2 \ge q = \lfloor n/g \rfloor. \tag{7}$$

This extends Theorem 2 (which is the special case $g = 2$), and answers, for Gorenstein* complexes, the question posed by Hanlon at the end of his paper [5]. As mentioned above, one may attempt to prove an analogous result for Cohen-Macaulay complexes using the methods of [5].

3.2 No Complete Orbit

Let G be again a finite group acting on Δ, but let Δ_1 consist of all simplices in Δ which do not contain a complete G-orbit. In this case, let Δ' be the subdivision of

Δ obtained by introducing a new vertex for each simplex of Δ which is a complete G-orbit. Again, Δ_1 is the full subcomplex of Δ' on the original set of vertices of Δ. The complementary full subcomplex Δ_2 consists of all the points fixed under G in the underlying space $|\Delta|$, where the G-action is extended from Δ to $|\Delta|$ in a piecewise linear fashion (i.e., linearly in each simplex). We can conclude that $\tilde{H}_i(\Delta_1)$ can be nonzero only if

$$i \geq \dim \Delta - \dim |\Delta|^G - 1. \tag{8}$$

This bound can be meaningful when additional assumptions are imposed. For example, if G acts *freely* on the *vertices* of Δ (i.e., all G-orbits of vertices have size $g := |G|$), then obviously

$$\dim |\Delta|^G + 1 \leq \lfloor (n+1)/g \rfloor \tag{9}$$

and the bound is

$$i \geq n - \lfloor (n+1)/g \rfloor. \tag{10}$$

For $g = 2$, this is the case of an involution having no fixed vertices in Δ. Thus we arrive again at Theorem 2, this time with a restriction on the type of action. By the remark at the end of Section 2, this restriction is always present in the motivating example of signed posets.

In general, the fixed-point set $|\Delta_2| = |\Delta|^G$ in this case is independent of the specific triangulation of $|\Delta|$ by Δ. It is interesting to cite in this context the classical result of P. A. Smith (see [3, p. 129]). Here all homology is taken with coefficients in the field of order p.

Theorem 3 *If G is a p-group (p prime) acting on a space X which is a Z_p homology n-sphere, then X^G is a Z_p homology r-sphere for some $-1 \leq r \leq n$ (where $r = -1$ means $X^G = \emptyset$). Moreover, if $p \neq 2$ then $n - r$ is even.*

3.3 Half Orbits

As an intermediate case between the preceding two extremes, let G be the cyclic group of order 4 acting on Δ, and let Δ_1 be the set of all Δ-simplices which contain at most half of the vertices of each G-orbit which is not a fixed vertex. In other words, such a simplex contains at most two of the vertices in an orbit of size 4, and at most one of the vertices in an orbit of size 2 (with no restriction on fixed vertices). Let Δ' be the subdivision of Δ obtained by introducing a new vertex for each simplex s of Δ contained in a G-orbit Gv such that either $|Gv| = 4$ and $3 \leq |s| \leq 4$, or $|Gv| = |s| = 2$. Again, Δ_1 is the full subcomplex of Δ' on the original vertex set of Δ, and the complementary full subcomplex Δ_2 has the following property: A nonempty simplex of Δ_2 contained in a G-orbit is either a single vertex or an edge, and gives rise to a simplex of Δ with at least twice as many vertices. Therefore

$$\dim \Delta_2 + 1 \leq \lfloor (n+1)/2 \rfloor, \tag{11}$$

and nonzero homology of Δ_1 may exist only for

$$i \geq n - \lfloor (n+1)/2 \rfloor = \lfloor n/2 \rfloor. \tag{12}$$

4 Final Remarks

It may be possible to extend the techniques of this paper beyond the context of Gorenstein* complexes. A central ingredient in such an extension is an appropriate generalization of Alexander duality, even in the modest form of zero/nonzero dichotomy for homology groups. This, in turn, seems to depend on a hypothetical extension of Lefschetz duality to (locally) Cohen-Macaulay complexes which are not necessarily homology manifolds. In view of the proof [7, Sections 64 and 70] of Lefschetz duality, it seems reasonable to restrict the complex to have no "boundary" (i.e., points where local homology vanishes in all dimensions). Perhaps such assumptions will suffice to produce some "inequalities" where one has "equalities" (i.e., group isomorphisms) in current Lefschetz duality.

Finally, we should mention here some work that has been done on aspects of Cohen-Macaulay simplicial complexes with a group action which are more combinatorial, having to do with simplex-counting and h-vectors rather than with homology. See the seminal paper [10], as well as [1] and [2].

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