# Affine Lie algebras and reduced Schur functions

Susumu ARIKI, Tatsuhiro NAKAJIMA and Hiro-Fumi YAMADA

**Abstract**

This article is a summary of the works [ANY1,2] and [NY2]. We discuss Schur functions with reduced variables arising in the basic representations of certain affine Lie algebras. They are called reduced Schur functions. For our purpose it is convenient to employ the notion of cores and quotients of the Young diagrams. We give a description of the linear relations among reduced Schur functions, which gives an information of the decomposition matrices of the symmetric groups and the Hecke algebras as well.

## §1 The basic $A_1^{(1)}$-module

In 1978, Lepowsky and Wilson [LW] constructed the basic representation of the affine Lie algebra $A_1^{(1)}$ by making use of the vertex operator. It is realized on the space of polynomials of infinitely many variables $V^{(2)} = \mathbb{C}[t_1, t_3, t_5, \ldots]$ as follows. For any odd natural number $j$, let $a_j = \frac{\partial}{\partial t_j}$ and $a_{-j} = jt_j$. Then $\{a_j \ (j \in \mathbb{Z}, \text{odd}), Id\}$ span the infinite dimensional Heisenberg algebra acting on $V^{(2)}$. Let $p$ be an indeterminate and put

$$\xi(t,p) = \sum_{j\geq 1,odd} t_j p^j = \sum_{j\geq 1,odd} \frac{a_{-j}}{j} p^j,$$

$$\xi(\tilde{\partial}, p^{-1}) = \sum_{j\geq 1,odd} \frac{1}{j}\frac{\partial}{\partial t_j} p^{-j} = \sum_{j\geq 1,odd} \frac{a_j}{j} p^{-j}.$$

The vertex operator is defined by

$$X(p) = -\frac{1}{2} e^{2\xi(t,p)} e^{-2\xi(\tilde{\partial},p^{-1})}.$$

Expanding $X(p)$ as a formal Laurent series of $p$:

$$X(p) = \sum_{k\in} X_k p^{-k},$$

we have differential operators $X_k$ ($k \in \mathbb{Z}$) acting on $V^{(2)}$. It is proved that the operators $a_j$ ($j \in \mathbb{Z}$,odd), $X_k$ ($k \in \mathbb{Z}$) and identity give the basic representation of $A_1^{(1)}$.

The set of weights $P$ is of the basic representation is known (see [K]) as

$$P = \left\{ \Lambda_0 - q\delta + p\alpha_1 ; p, q \in \mathbb{Z}, q \geq p^2 \right\}.$$

Here $\Lambda_0$ is the highest weight and $\delta$ denotes the fundamental imaginary root, i.e., $\delta = \alpha_0 + \alpha_1$, where $\alpha_0$ and $\alpha_1$ are simple roots. A weight $\Lambda$ on the parabola $q = p^2$ is said to be maximal in the sense that $\Lambda + \delta$ is no longer a weight. The weights on each parabola $q = p^2 + n$ ($n \in \mathbb{N}$) consist a single Weyl group orbit. The Weyl-Kac character formula ([K]) tells us that the weight multiplicity on this parabola equals $p(n)$, the number of partitions of $n$.

## §2 2-reduced Schur functions

Our first problem is to write down the weight vectors of the basic $A_1^{(1)}$-module $V^{(2)}$. To this end we recall the Schur functions ([M]). For any Young diagram $\lambda$ of size $N$, the Schur function indexed by $\lambda$ is defined by

$$S_\lambda(t) = \sum_{\nu_1 + 2\nu_2 + \cdots = N} \chi_\lambda(\nu) \frac{t_1^{\nu_1} t_2^{\nu_2} \cdots}{\nu_1! \nu_2! \cdots},$$

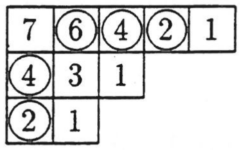
where $\chi_\lambda(\nu)$ is the character value of the irreducible representation $\lambda$ of the symmetric group $\mathfrak{S}_N$, evaluated at the conjugacy class of cycle type $\nu = (1^{\nu_1} 2^{\nu_2} \cdots N^{\nu_N})$. The Schur function $S_\lambda(t)$ is obviously a weighted homogeneous ($\deg t_j = j$) polynomial of degree $N$. We also define the *2-reduced Schur functions*, which play an essential role in our argument:

$$S_\lambda^{(2)}(t) = S_\lambda(t)|_{t_2 = t_4 = \cdots = 0} \in V^{(2)}.$$

According to Murnaghan-Nakayama's formula ([J]), $S_\lambda(t) = S_\lambda^{(2)}(t)$ if $\lambda$ does not have a 2-hook. Such a $\lambda$ is called a 2-core. It is easily seen that $\kappa_r := (r, r-1, \ldots, 2, 1)$ ($r \in \mathbb{N}$) exhaust all of 2-cores ($\kappa_0 = \emptyset$). In our realization the maximal weight vectors are given by $S_{\kappa_r}(t)$ for $r = 0, 1, 2, \dots$ ([DJKM]). We denote by $\Lambda_r$ the maximal weight whose weight vector is $S_{\kappa_r}(t)$.

To describe the weight vectors of other weights we recall the 2-quotient of a Young diagram ([O]). The 2-quotient of $\lambda = (\lambda_1, \dots, \lambda_n)$ is a pair of Young diagrams $(\lambda^0, \lambda^1)$ defined as follows: let us insert hooklength in each square of $\lambda$. The parts of $\lambda^e$ $(e = 0, 1)$ are the number of even entries in $i$-th row if $\lambda_i - i \equiv e \pmod 2$. This procedure gives a pair of Young diagrams.

**Example.** $\lambda = (5, 3, 2)$



$\lambda_i - i \pmod 2$

0

1

1

The circled entries of the first row form $\lambda^0 = (3)$ and of the second and the third rows $\lambda^1 = (1, 1)$.

For any Young diagram $\lambda$ we uniquely determine a 2-core by removing 2-hooks successively as many as possible and call it the 2-core of $\lambda$. Any Young diagram $\lambda$ of size $N$ determines the triplet $(\lambda^c, \lambda^0, \lambda^1)$, where $\lambda^c$ is the 2-core of $\lambda$, $(\lambda^0, \lambda^1)$ is the 2-quotient of $\lambda$ and $|\lambda^c| + 2(|\lambda^0| + |\lambda^1|) = N$, and *vice versa.* We often identify $\lambda$ with $(\lambda^c, \lambda^0, \lambda^1)$ by this one-to-one correspondence.

By using the boson-fermion correspondence established by Date et al.[DJKM], we can see the following.

**Proposition 2.1.** *The 2-reduced Schur function $S_\lambda^{(2)}(t)$ is a weight vector of weight $\Lambda_r - n\delta$ if $\lambda = (\kappa_r, \lambda^0, \lambda^1)$ with $|\lambda^0| + |\lambda^1| = n$.*

The above weight vectors satisfy certain linear relations in general since $\mathrm{mult}(\Lambda_r - n\delta) = p(n)$. Therefore the next problem is to find a suitable basis for each weight space. The following theorem gives an answer.

**Proposition 2.2.** *The 2-reduced Schur functions*

$$\left\{ S_\lambda^{(2)}(t); \lambda = (\kappa_r, \emptyset, \lambda^1) \text{ with } |\lambda^1| = n \right\}$$

*are linearly independent and hence constitute a basis for the weight space of weight* $\Lambda_r - n\delta$.

Any weight vector $S_\lambda^{(2)}(t)$ can be expressed uniquely as a linear combination of the basis vectors obtained above. We now focus on the coefficients of these expressions. The 2-sign $\delta_2(\lambda) = \delta_2(\lambda^c, \lambda^0, \lambda^1)$ is defined as follows (see [O]). If the 2-core $\lambda^c$ is obtained from $\lambda$ by removing a sequence of 2-hooks, where $q$ of them are column 2-hooks and the others are row 2-hooks, then

$$\delta_2(\lambda^c, \lambda^0, \lambda^1) = (-1)^q.$$

It can be proved that $\delta_2(\lambda^c, \lambda^0, \lambda^1)$ does not depend on the choice of 2-hooks being removed. The following is our main result.

**Theorem 2.3.** *For such a Young diagram* $\lambda$ *that* $\lambda = (\lambda^c, \lambda^0, \lambda^1)$ *with* $n = |\lambda^0| + |\lambda^1|$, *we have*

$$S_\lambda^{(2)}(t) = (-1)^{|\lambda^0|}\delta_2(\lambda) \sum_{\mu^1} \delta_2(\mu) LR^{\mu^1}_{\lambda^{0\prime}\lambda^1} S_\mu^{(2)}(t),$$

*where the summation runs over all Young diagrams* $\mu^1$ *of size* $n$, *the Young diagram* $\mu$ *corresponds to* $(\lambda^c, \emptyset, \mu^1)$ *and* $\lambda^{0\prime}$ *denotes the transpose of* $\lambda^0$. *We also denote by* $LR$ *the Littlewood-Richardson coefficient.*

## §3 An application to the Hecke algebra at root of unity

By virtue of the formula in Theorem 2.3 we can derive the following identity satisfied by irreducible character values of the symmetric groups:

$$\chi_\lambda(\nu) = (-1)^{|\lambda^0|}\delta_2(\lambda) \sum_{\mu^1} \delta_2(\mu) LR^{\mu^1}_{\lambda^{0\prime}\lambda^1} \chi_\mu(\nu),$$

for the 2-regular classes $\nu$, i.e., $\nu_2 = \nu_4 = \cdots = 0$ if $\nu = (1^{\nu_1} 2^{\nu_2} \cdots)$. Denote by $C_n^{(2)}$ the 2-regular character matrix of the $\mathfrak{S}_n$, i.e., columns are indexed only by 2-regular classes. The formula gives the linear relations among rows of $C_n^{(2)}$. Let $D_n^{(2)}$ be the decomposition matrix of the Specht modules of $\mathfrak{S}_n$ for the prime 2 (see [JK]). Then we know that

$D_n^{(2)}B_n^{(2)} = C_n^{(2)}$, where $B_n^{(2)}$ is the matrix whose entries are the Brauer characters. Thus we are led to the linear relations among rows of $D_n^{(2)}$.

We can lift this situation to the Hecke algebra $H_n(q)$ with $q=-1$. The Hecke algebra $H_n(q)$ is defined over $(q)$ by generators $T_1,\ldots,T_{n-1}$ and relations

$$(T_i+1)(T_i-q)=0 \ \ (1\le i\le n-1),$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ \ (1\le i\le n-2),$$
$$T_iT_j = T_jT_i \ \ (|i-j|\ge 2).$$

By the theory of Dipper and James [DJ] we know that the decomposition matrix $\widetilde{D_n^{(2)}}$ of the Specht modules of $H_n(q)$ for $q=-1$ is of the form

$$\widetilde{D_n^{(2)}} = D_n^{(2)}U_n,$$

where $U_n$ is a certain lower unitriangular matrix. Let $S^\lambda$ be the Specht module over $H_n(-1)$ corresponding to the Young diagram $\lambda$, and $[S^\lambda]$ be the element of the Grothendieck group of $H_n(-1)$-modules. Then we have the following:

$$[S^\lambda] = (-1)^{|\lambda^0|}\delta_2(\lambda)\sum_{\mu^1}\delta_2(\mu)LR^{\mu^1}_{\lambda^{0\prime}\lambda^1}[S^\mu].$$

## §4 Some other cases

The discussion in section 2 and 3 are valid to some other cases, namely $A^{(1)}_{r-1}$ and $A^{(2)}_{2t}$. Here we just state our formula without precise definitions. For the case of $A^{(1)}_{r-1}$ we have

**Theorem 4.1.** *Let $S^{(r)}_\lambda(t)$ be the r-reduced Schur function indexed by $\lambda$. For any Young diagram $\lambda$ we have*

$$S^{(r)}_\lambda(t) = (-1)^{|\lambda^0|}\delta_r(\lambda)\sum_{\mu,\nu_1,\ldots,\nu_{r-1}}\delta_r(\mu)LR^{\lambda^{0\prime}}_{\nu_1\cdots\nu_{r-1}}LR^{\mu^1}_{\nu_1\lambda^1}\cdots LR^{\mu^{r-1}}_{\nu_{r-1}\lambda^{r-1}}S^{(r)}_\mu(t),$$

*where summation runs over Young diagrams $\mu$ and $\nu_1,\ldots,\nu_{r-1}$ such that $|\mu|=|\lambda|$, $\mu^0=\emptyset$ and the core of $\mu$ coincides with that of $\lambda$.*

In the realization of the basic representation of $A_{2t}^{(2)}$ on the space of polynomials, the weight vectors are expressed in terms of Schur's $Q$-functions $Q_\lambda(t)$ ([NY1]). Accordingly we obtain the linear relations for $r$-reduced $Q$-functions, where $r = 2t+1$. We introduce the new coefficients $NY$ by

$$2^{-\ell(\lambda)}Q_\lambda(t)S_\mu(t) = \sum_\nu NY^\nu_{\lambda\mu}S_\nu(t).$$

**Theorem 4.2.** *Let $\lambda$ be a strict partition, $\lambda^c$ and $(\lambda^0, \lambda^1, \dots, \lambda^t)$ be its $r$-bar core and $r$-bar quotient, respectively. Then*

$$\begin{aligned} Q_\lambda^{(r)}(t) &= 2^{[(\ell(\lambda)+\ell(\lambda^0))/2]}(-1)^{|\lambda^0|}\delta_{\bar r}(\lambda) \\ &\times \sum_{\mu,\nu_1,\dots,\nu_t} 2^{-[\ell(\mu)/2]}\delta_{\bar r}(\mu)\widetilde{LR}^{\lambda^0}_{\nu_1\cdots\nu_t} NY^{\mu^1}_{\nu_1\lambda^1}\cdots NY^{\mu^t}_{\nu_t\lambda^t}Q_\mu^{(r)}(t), \end{aligned}$$

*where summation runs over the strict partitions $\nu_1, \dots, \nu_t$ and $\mu$ such that $|\mu| = |\lambda|$, $\mu^c = \lambda^c$ and $\mu^0 = \emptyset$. Here $\widetilde{LR}$ denotes the linearization coefficient of the product of $Q$-functions.*

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S. Ariki
Department of Mathematics,
Tokyo University of Mercantile Marine,
Koto-ku, Tokyo 135, Japan
ariki@tansei.cc.u-tokyo.ac.jp

T. Nakajima
Department of Economics,
Meikai University,
Urayasu-shi, Chiba 279, Japan
nakajima@phys.metro-u.ac.jp

H. -F. Yamada
Department of Mathematics,
Hokkaido University,
Sapporo 060, Japan
yamada@math.hokudai.ac.jp