Tensor Representations for Orthosymplectic Lie Superalgebras and Hook Schur Functions

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0. Introduction

In his thesis [Sc1] and a subsequent paper [Sc2] Schur proved that there is an action of the symmetric group S_k on the tensor space $V^{\otimes k}$ ($V = \mathbb{C}^n$) by place permutations which commutes with the action of the complex general linear group $GL(n, \mathbb{C})$ on $V^{\otimes k}$. This fundamental result, often referred to as Schur-Weyl duality, links in a critical way the combinatorics and representation theories of the symmetric and general linear groups. Brauer [Br] generalized this picture to show that there is an action of an algebra $B_k(n)$, which is now called the Brauer algebra, on $V^{\otimes k}$ which commutes with the action of the orthogonal group $O(n, \mathbb{C})$. For the symplectic group $Sp(n, \mathbb{C})$ (n even) a similar result holds: the centralizer algebra of the $Sp(n, \mathbb{C})$-action on $V^{\otimes k}$ is given by the action of the Brauer algebra $B_k(-n)$.

When V is $\mathbb{Z}_2$-graded, the action of the symmetric group S_k on $V^{\otimes k}$ by "graded" place permutations determines the centralizer of the general linear Lie superalgebra on that space. Berele and Regev [BR] and Sergeev [Se] exploited this action to study certain modules for the superalgebra and their characters, which are hook Schur functions. In [FM], Fischman and Montgomery generalized their work to cotriangular Hopf algebras which arise from enveloping algebras of general linear Lie color algebras.

In this paper we describe an orthogonal-symplectic version of the superalgebra theory. More specifically, we

(1) briefly discuss the orthosymplectic Lie color algebras $\mathfrak{spo}(V, \beta)$; (The orthosymplectic Lie superalgebras are just a particular example of such a Lie color algebra $\mathfrak{spo}(V, \beta)$.)

(2) describe an action of the Brauer algebra on tensor space which commutes with the action of the orthosymplectic Lie color algebra $\mathfrak{spo}(V, \beta)$;

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(3) use the action of the Brauer algebra to construct a family of $\mathfrak{spo}(V,\beta)$-maximal vectors of $V^{\otimes k}$ and produce a decomposition of $V^{\otimes k}$ into the direct sum of submodules T^λ naturally indexed by partitions;

(4) present expressions for the characters of the $\mathfrak{spo}(V,\beta)$-modules T^λ and give a combinatorial description of these characters in terms of tableaux; and

(5) describe a Robinson-Schensted-Knuth type of insertion scheme for the tableaux which models the decomposition of the tensor space into the modules T^λ.

The detailed proofs of the results described here can be found in our paper [BLR].

Some remarks on the results

(i) To our knowledge orthosymplectic Lie color algebras were first introduced in [B], which discusses the Brauer algebra action but does not prove that this action commutes with $\mathfrak{spo}(V,\beta)$. The notion of an orthosymplectic Lie color algebra allows us to give a uniform proof that the Brauer algebra action on tensor space commutes with the action of the orthogonal Lie algebra, the symplectic Lie algebra, the orthosymplectic Lie superalgebra as well as more general group graded orthosymplectic algebras. In [BLR] these results are derived in the context of braided monoidal categories, which provides a convenient framework for studying commuting actions. It is shown in [BLR] that there is a centralizing action of an algebra (which has a diagram-type basis like the Brauer algebra) on tensor space for any kind of "Lie like" algebra or quantum group for which the category of finite-dimensional modules has a braided monoidal structure and a special isomorphism between V and V^*.

(ii) Our derivation of the maximal vectors in the orthosymplectic case extends the work in [BBL], which computes all the maximal vectors for the orthogonal and symplectic Lie algebras. The modules T^λ which we are considering are the same as the ones studied by Bars and Balantekin [BB]. They gave Jacobi-Trudi type formulas for their characters, but did not derive a combinatorial description of the characters in terms of tableaux nor did they provide an insertion scheme to model the decomposition of $V^{\otimes k}$ into the T^λ's. In their paper [BB], Bars and Balantekin seem to indicate that the modules T^λ are irreducible, but this is not clear to us, either from their work or from ours. In fact, R.C. King in personal communication has told us that he has found explicit examples of T^λ which are not irreducible.

(iii) There are other papers, notably [FJ] and [CK], which describe how to index representations of the orthosymplectic Lie superalgebra by partitions, but to our knowledge none of them has given an interpretation for their characters in terms of tableaux. The main ingredient for the tableau description is identity (8) in Theorem 4.5 below. The very similar identity in Theorem 4.5 (9) is given in the work of Cummins and King [CK]. This identity could be used in combination with results of Sundaram [Su2] to produce another combinatorial interpretation for these characters. See [Ki] for a survey of the use of tableaux in the study of representations of Lie superalgebras.

(iv) Berele and Regev [BR] provided a combinatorial description of the characters of the modules for the general linear Lie superalgebra $\mathfrak{gl}(m,n)$ which appear in the decomposition of tensor space by describing them as hook Schur functions, (that is, hybrid Schur functions which correspond to tableaux which have both a column-strict and a row-strict part). In a similar fashion we give a combinatorial description of the characters of the modules T^λ as hybrid symplectic-ordinary Schur functions corresponding to tableaux which have both a symplectic and a row-strict part. The symplectic part is a symplectic tableau of the kind first introduced by King [Ki1] to describe the characters of symplectic groups.

Open questions

(a) There is an extensive literature of papers by Bernstein and Leites [BLe], [Le1-Le2], Kac [K1-K2], van der Jeugt, Hughes, King, and Thierry-Mieg [JHKT1-JHKT3], Penkov and Serganova [PS1-PS3], [P] [Sr], Kac and Wakimoto [KW], and others which studies representations of Lie superalgebras using Lie theoretic and geometric methods. These approaches also yield character formulas, the most general of which is the Weyl-Kac character formula. We have not made any effort to understand our character formulas in this other setting, although the formulas must be equal in many cases. Even for the superalgebra $\mathfrak{gl}(m,n)$, the connection between the results of [BR] and [Se], the Sergeev-Pragacz character formula, and the Weyl-Kac character formula needs to be better understood. King and others have done some work in this direction, (see [Ki2]). The relationship between the centralizer approach to the representation theory of Lie superalgebras and the approach via Kac modules and typical weights also needs to be better explained.

(b) In determining the character of the module T^λ we have shown that it is equal to the polynomial $sc_\lambda(x_1,x_1^{-1},\ldots,x_r,x_r^{-1},y_1,y_1^{-1},\ldots,y_s,y_s^{-1},1)$ which appears as the coefficient of the Schur function $s_\lambda(z_1,\ldots,z_{r+s})$ in the identity

$$\begin{aligned}\prod_{1\le i<j\le r+s}(1-z_iz_j)&\frac{\prod_{j=1}^{r+s}(1+z_j)\prod_{j=1}^{r+s}\prod_{i=1}^{s}(1+y_iz_j)(1+y_i^{-1}z_j)}{\prod_{j=1}^{r+s}\prod_{i=1}^{r}(1-x_iz_j)(1-x_i^{-1}z_j)}\\ &=\sum_\lambda sc_\lambda(x_1,x_1^{-1},\ldots,x_r,x_r^{-1},y_1,y_1^{-1},\ldots,y_s,y_s^{-1},1)s_\lambda(z_1,\ldots,z_{r+s}).\end{aligned}\tag{0.1}$$

There are two classical identities of Littlewood [Li] and Weyl [We] for the characters of the symplectic group $Sp(2r,\mathbf{C})$ and the special orthogonal group $SO(2s+1,\mathbf{C})$ (see [Su1,Su2]):

$$\begin{aligned}\prod_{1\le i<j\le r}(1-z_iz_j)&\frac{1}{\prod_{j=1}^{r}\prod_{i=1}^{r}(1-x_iz_j)(1-x_i^{-1}z_j)}\\ &=\sum_{\mu,\ \ell(\mu)\le r} sp_\mu(x_1,x_1^{-1},\ldots,x_r,x_r^{-1})s_\mu(z_1,\ldots,z_r),\ \text{and}\end{aligned}\tag{0.2}$$

$$\begin{aligned}\prod_{1\le i<j\le s}(1-z_iz_j)&\prod_{j=1}^{s}(1+z_j)\prod_{j=1}^{s}\prod_{i=1}^{s}(1+y_iz_j)(1+y_i^{-1}z_j)\\ &=\sum_{\nu,\ \ell(\nu)\le s} so_{\nu'}(y_1,y_1^{-1},\ldots,y_s,y_s^{-1},1)s_\nu(z_1,\ldots,z_s),\end{aligned}\tag{0.3}$$

where sp_μ is the character of the symplectic group $Sp(2r,\mathbf{C})$ labeled by the partition μ having no more than r parts; and so_ν is the character of the special orthogonal group $SO(2s+1,\mathbf{C})$ labeled by the conjugate ν' of the partition ν which has no more than s parts. When the orthogonal part is zero, the numerator in (0.1) is 1, and (0.1) gives the classical identity in (0.2). If instead the symplectic portion is zero, identity (0.1) reduces to (0.3). Is there a combinatorial interpretation for sc_λ which expresses it as a hybrid object built from symplectic and orthogonal characters? Our combinatorial description of sc_λ is as a hybrid symplectic-general linear character rather than a hybrid symplectic-orthogonal character.

(c) We have not included proofs of the analogues of the Schur-Weyl duality in [BLR], i.e. we have not shown that the actions of the Brauer algebra and the Lie color algebra $\mathfrak{spo}(V,\beta)$ each generate the full centralizer of the action of the other. We have succeeded in proving various parts of both halves of the duality in the orthosymplectic - Brauer algebra setting, but do not have a proof that works in all cases.

1. LIE COLOR ALGEBRAS AND $\mathfrak{spo}(V,\beta)$

Let κ denote a field of characteristic zero. Let G be a finite abelian group with identity element 1_G. A *symmetric bicharacter* on G is a map $\beta: G \times G \to \kappa^*$ into the multiplicative group of the field such that

(1) $\beta(ab,c) = \beta(a,c)\beta(b,c)$,
(2) $\beta(a,bc) = \beta(a,b)\beta(a,c)$, and
(3) $\beta(a,b)\beta(b,a) = 1$ for all $a,b \in G$.

A κ-vector space V is *G-graded* if it is the direct sum $V = \bigoplus_{a\in G} V_a$ of subspaces indexed by the elements of G. If $v \in V_a$ for some $a \in G$, then v is *homogeneous* of degree a.

Assume β is a fixed symmetric bicharacter on a group G. A *Lie color algebra* $(\mathfrak{g}, G, \beta)$ is a G-graded vector space $\mathfrak{g} = \bigoplus_{a\in G} \mathfrak{g}_a$ with a κ-bilinear bracket $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(1) $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{ab}$, for all $a,b \in G$,
(2) $[x,y] = -\beta(b,a)[y,x]$, and
(3) $[x,[y,z]] = [[x,y],z] + \beta(b,a)[y,[x,z]]$, for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, and all $z \in \mathfrak{g}$.

When the group is the cyclic group of order 2, $G = \{\pm 1\} = \{(-1)^a \mid a = 0,1\}$, and $\beta((-1)^a,(-1)^b) = (-1)^{ab}$, then $\mathfrak{g}$ is a *Lie superalgebra*.

Let $\mathfrak{gl}(V,\beta) = \mathrm{End}(V)$ denote the κ-vector space of κ-linear maps from V to V with the G-grading assigned by $\mathfrak{gl}(V,\beta)_a = \{x \in \mathrm{End}(V) \mid xV_b \subseteq V_{ab} \text{ for all } b \in G\}$ and with the bracket $[x,y] = xy - \beta(b,a)yx$ for all $x \in \mathfrak{gl}(V,\beta)_a, y \in \mathfrak{gl}(V,\beta)_b$. Then $\mathfrak{gl}(V,\beta) = \bigoplus_{a\in G} \mathfrak{gl}(V,\beta)_a$ with this multiplication is a Lie color algebra, the so-called *general linear Lie color algebra*.

A *β-skew-symmetric bilinear form* is a κ-bilinear map $\langle,\rangle: V \times V \to \kappa$ such that

(1) the form $\langle,\rangle$ on V is nondegenerate,
(2) $\langle V_a, V_b\rangle = 0$ if $a \neq b^{-1}$, and
(3) $\langle v,w\rangle = -\beta(b,a)\langle w,v\rangle$, for all $v \in V_a$, $w \in V_b$.

For each $a \in G$, define

$$\mathfrak{spo}(V,\beta)_a = \{x \in \mathfrak{gl}(V,\beta)_a \mid \langle xu,v\rangle + \beta(b,a)\langle u,xv\rangle = 0 \text{ for all } u \in V_b \text{ and } v \in V\}.$$

Then $\mathfrak{spo}(V,\beta) = \bigoplus_{a\in G} \mathfrak{spo}(V,\beta)_a$ is a simple Lie color subalgebra of the Lie color algebra $\mathfrak{gl}(V,\beta)$.

2. THE BRAUER ALGEBRA ACTION ON TENSOR SPACE

A *k-diagram* is a graph with two rows of k vertices each, one above the other, and k edges such that each vertex is incident to precisely one edge. Let $\eta \in \kappa$. The product of two k-diagrams d_1 and d_2 is given by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains k paths and some number γ of closed loops. Let d be the k-diagram with edges given by the paths in the graph (with the loops removed). Then the product d_1d_2 is given by $d_1d_2 = \eta^\gamma d$. The *Brauer algebra* $B_k(\eta)$ is the κ-span of the k-diagrams. The κ-linear extension of the diagram multiplication makes $B_k(\eta)$ into an associative κ-algebra.

Let $\mathfrak{g} = \mathfrak{spo}(V, \beta)$, the *orthosymplectic Lie color algebra*. The bilinear form affords a $\mathfrak{g}$-module isomorphism $F: V \longrightarrow V^*$, $v \longmapsto \langle v, \cdot\rangle$ between V and its dual space V^*. Let $B = \{v_1, \ldots, v_N\}$ be a homogeneous basis of V, i.e. for each $1 \le i \le N$, $v_i \in V_{c_i}$ for some $c_i \in G$. Let $\{v^1, \ldots, v^N\}$ be the dual basis in V^*. Assume

$$F_B = (F_{i,j})_{1\le i,j\le N}, \qquad \text{where} \quad \langle v_i, v_j\rangle = F_{i,j}, \tag{2.1}$$

is the matrix of the form $\langle , \rangle$ with respect to the basis B. Sometimes we will write F_{v_i,v_j} instead of $F_{i,j}$. Let $F_B^{-1} = (F_{i,j}^{-1})_{1\le i,j\le N}$ be the inverse of the matrix F_B. Then

$$F(v_i) = \sum_{j=1}^{N} F_{i,j} v^j, \qquad \text{and} \qquad F^{-1}(v^j) = \sum_{i=1}^{N} F_{j,i}^{-1} v_i. \tag{2.2}$$

Let d be a k-diagram. Label the top vertices (left to right) with a sequence $\underline{a} = (a_1, a_2, \ldots, a_k)$ of basis elements $a_i \in B$ and the bottom vertices (left to right) with a sequence $\underline{b} = (b_1, b_2, \ldots, b_k)$ of basis elements $b_i \in B$. Assign a weight to each edge and each crossing of this labeled k-diagram according to the following:

(1) If a horizontal edge (a, a') on the top has a to the left of a', assign the weight $F_{a,a'}$ to it.

(2) If a horizontal edge (b, b') on the bottom has b to the left of b', weight it by $F_{b,b'}^{-1}$.

(3) Weight each vertical edge (a, b) by $\delta_{a,b}$ (Kronecker delta),

(4) Weight each crossing by $-\beta(\ell_1, \ell_2)$, where ℓ_1 is a vertex adjacent to the first edge, and ℓ_2 is a vertex adjacent to the second edge in the crossing. Of the four vertices adjacent to the two edges that cross, ℓ_1 and ℓ_2 should be chosen to be the last two vertices (in order) when counting off the vertices in a counterclockwise fashion beginning from the bottom left corner of the diagram. (Our convention is $\beta(v, w) = \beta(c, c')$ when $v \in V_c, w \in V_{c'}$.)

The *weight* of the labeled k-diagram, which we denote $d_{\underline{a},\underline{b}}$, is the product of the weights over all the edges and crossings.

For a k-diagram d, we define

$$(a_1 \otimes \cdots \otimes a_k)\Psi_d = \sum_{b_1,\ldots,b_k\in B} d_{\underline{a},\underline{b}} b_1 \otimes \cdots \otimes b_k,$$

where $d_{\underline{a},\underline{b}}$ is the weight of the k-diagram d with top vertices labeled by $a_1, \ldots, a_k$ and bottom vertices labeled by $b_1, \ldots, b_k$.

Theorem 2.3. *The map Ψ extends to a homomorphism*

$$\Psi: B_k(n - m) \to \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes k}, V^{\otimes k})$$

of algebras where $m = \dim V_{(0)}$, $V_{(0)} = \sum_{c\in G,\beta(c,c)=1} V_c$, and $n = \dim V_{(1)}$, $V_{(1)} = \sum_{c\in G,\beta(c,c)=-1} V_c$.

3. MAXIMAL VECTORS

Let $\dim V_{(0)} = m = 2r$ and $\dim V_{(1)} = n = 2s$ or $2s+1$. Assume

$$B_0 = \{t_1, t_1^*, t_2, t_2^*, \dots, t_r, t_r^*\} \quad B_1 = \{u_1, u_1^*, u_2, u_2^*, \dots, u_s, u_s^*, (u_{s+1})\}, \quad \text{and } B = B_0 \cup B_1 \tag{3.1}$$

are bases of $V_{(0)}$, $V_{(1)}$, and V, respectively, such that

$$\langle v, w\rangle = \langle v^*, w^*\rangle = 0, \quad \text{and} \quad \langle v, w^*\rangle = -\beta(w^*, v)\langle w^*, v\rangle = \delta_{v,w},$$

for all $v, w \in \{t_1, t_2, \dots, t_r, u_1, u_2, \dots, u_s, (u_{s+1})\}$. It is to be understood that u_{s+1} occurs only when $n = 2s+1$ and in that case $u_{s+1}^* = u_{s+1}$. We extend the definition of $*$ so that $(t_i^*)^* = t_i$ and $(u_j^*)^* = u_j$ for all $1 \le i \le r$ and all $1 \le j \le s$. Note for all $v \in B$ that $v^* \in V_{a^{-1}}$ whenever $v \in V_a$. The matrix units $E_{v,w}$, for $v, w \in B$, determine a homogeneous basis of $\mathfrak{gl}(V,\beta)$ with $E_{v,w} \in \mathfrak{gl}(V,\beta)_{ab^{-1}}$ whenever $v \in V_a$ and $w \in V_b$. The elements $h_v = E_{v,v} - E_{v^*,v^*}, v \in \{t_1, \dots, t_r, u_1, \dots, u_s\}$ belong to $\mathfrak{spo}(V,\beta)_{1_G}$, and they span the space $\mathfrak{h}$ of diagonal matrices in $\mathfrak{spo}(V,\beta)$. Let $\{\epsilon_v\}$ be the dual basis in $\mathfrak{h}^*$ and for convenience set $\epsilon_i = \epsilon_{t_i}$ and $\delta_j = \epsilon_{u_j}$ for $1 \le i \le r$ and $1 \le j \le s$. Define elements $x_i \in \mathfrak{spo}(V,\beta)$, $i = 1, \dots, r+s$, by

$$\begin{aligned}
x_i &= E_{t_i,t_{i+1}} - \beta(t_i, t_{i+1})E_{t_{i+1}^*,t_i^*}, && 1 \le i \le r-1,\\
x_r &= E_{t_r,u_1} + \beta(t_r, u_1)E_{u_1^*,t_r^*},\\
x_{r+j} &= E_{u_j,u_{j+1}} + \beta(u_j, u_{j+1})E_{u_{j+1}^*,u_j^*}, && 1 \le j \le s-1,\\
x_{r+s} &= \begin{cases} E_{u_s,u_{s+1}} + \beta(u_s, u_{s+1})E_{u_{s+1}^*,u_s^*}, & \text{if } n = 2s+1,\\ E_{u_s,u_{s-1}^*} + \beta(u_s^*, u_{s-1})E_{u_{s-1},u_s^*}, & \text{if } n = 2s.\end{cases}
\end{aligned} \tag{3.2}$$

The elements x_i, x_i^t, $(i = 1, \dots, r+s)$, where t is the ordinary transpose, generate $\mathfrak{spo}(V,\beta)$ as a Lie color algebra.

Suppose that $c_{p,q}$ denotes the diagram in $B_k(n-m)$ with a horizontal edge connecting the pth and qth nodes on both the top and bottom, and with every other top node connected to the one directly below it. Let $\underline{p} = \{p_1, \dots, p_j\}$ and $\underline{q} = \{q_1, \dots, q_j\}$ be disjoint ordered subsets of $\mathcal{K} = \{1, \dots, k\}$, and assume that $(\underline{p}, \underline{q}) = \{(p_1, q_1), \dots, (p_j, q_j)\}$. Denote by $\mathcal{P}(j)$ the set of all such $(\underline{p}, \underline{q})$. We assume $c_{\emptyset,\emptyset}$ is the identity diagram, and for each $(\underline{p}, \underline{q}) \in \mathcal{P}(j)$, we let $c_{\underline{p},\underline{q}} = c_{p_1,q_1} \cdots c_{p_j,q_j}$ for $j = 1, \dots, \lfloor k/2 \rfloor$. Let $\mathcal{P} = \cup_j \mathcal{P}(j)$ where $\mathcal{P}(0) = \{(\emptyset, \emptyset)\}$.

Theorem 3.3. *Suppose $\lambda = (\lambda_1, \dots, \lambda_\ell, 0, \dots)$ is an (r,s)-hook shape partition (i.e. $\lambda_{r+1} \le s$) of $k - 2j$ for some j such that $0 \le j \le \lfloor k/2 \rfloor$. Assume $(\underline{p}, \underline{q}) \in \mathcal{P}(j)$ and fix a standard tableau τ of shape λ with entries in $(\underline{p} \cup \underline{q})^c$. Let $\tau^{(1)}$ and $\tau^{(2)}$ be the corresponding subtableaux of τ of shapes $\lambda^{(1)} = (\lambda_1, \dots, \lambda_r)$ and $\lambda^{(2)} = (\lambda_{r+1}, \dots, \lambda_\ell)' = (\lambda'_{r+1}, \dots, \lambda'_{r+s})$ respectively, where $'$ denotes the conjugate partition. Let $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_r\epsilon_r + \lambda'_{r+1}\delta_1 + \cdots + \lambda'_{r+s}\delta_s \in \mathfrak{h}^*$ denote the weight determined by λ. Then $\theta = \beta_{\tau,\underline{p},\underline{q}} c_{\underline{p},\underline{q}} y_\tau$ is a maximal vector of weight λ where y_τ is the Young symmetrizer corresponding to τ and $\beta_{\tau,\underline{p},\underline{q}} = w_1 \otimes \cdots \otimes w_k$ is the simple tensor defined by*

$$w_i = \begin{cases} t_1 & \text{if } i \in \underline{p}\\ t_1^* & \text{if } i \in \underline{q}\\ t_j & \text{if } i \in (\underline{p} \cup \underline{q})^c \text{ and } i \text{ is in } j\text{th row of } \tau^{(1)}\\ u_j & \text{if } i \in (\underline{p} \cup \underline{q})^c \text{ and } i \text{ is in } j\text{th row of } \tau^{(2)}.\end{cases}$$

If $n = 2s$ *and* $\ell(\lambda^{(2)}) = s$, *then* $\theta^\circ = \beta^\circ_{\tau,\underline{p},\underline{q}} c_{\underline{p},\underline{q}} y_\tau$ *is a maximal vector of weight* $\lambda^\circ = \lambda - 2\lambda'_{r+s}\delta_s$ *where* $\beta^\circ_{\tau,\underline{p},\underline{q}}$ *is the simple tensor obtained from* $\beta_{\tau,\underline{p},\underline{q}}$ *by replacing each factor* u_s *with* u^*_s.

Theorem 3.4. *Suppose that* $r + s \geq k$. *Then*

$$\{\beta_{\tau,\underline{p},\underline{q}} c_{\underline{p},\underline{q}} y_\tau \mid (\underline{p},\underline{q}) \in \mathcal{P},\ \tau \in \mathcal{HST}_{r,s}((\underline{p} \cup \underline{q})^c)\}$$

is a linearly independent set of maximal vectors where $\mathcal{HST}_{r,s}((\underline{p} \cup \underline{q})^c)$ *is the set of all standard tableaux with* (r,s)*-hook shape and entries in* $(\underline{p} \cup \underline{q})^c$.

In these theorems the Young symmetrizer y_τ is constructed from the row and column groups of τ as in [BL]. What is meant by a maximal vector is a common eigenvector for $\mathfrak{h}$ with eigenvalue given by $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_r\epsilon_r + \lambda'_{r+1}\delta_1 + \cdots + \lambda'_{r+s}\delta_s \in \mathfrak{h}^*$ which is killed by all the elements x_i, $i = 1, \ldots, r+s$ in (3.2). An irreducible $\mathfrak{spo}(V,\beta)$-module has a unique (up to scalar multiple) maximal vector, so finding the maximal vectors helps locate the irreducible components of $V^{\otimes k}$.

4. THE $\mathfrak{spo}(V,\beta)$-MODULES T^λ AND THEIR CHARACTERS

Suppose that $|n - m| \geq k$. Then by the work of Wenzl [W] we know that the Brauer algebra $B_k(n-m)$ is semisimple and has simple summands indexed by the partitions in the set

$$\widehat{B}_k = \{\lambda \vdash k - 2h \mid h = 0, 1, \ldots, \lfloor k/2 \rfloor\}.$$

More specifically, there is an isomorphism

$$\Gamma\colon B_k(n-m) \longrightarrow \bigoplus_{\lambda \in \widehat{B}_k} M_{d_\lambda}(\kappa),$$

where $M_{d_\lambda}(\kappa)$ denotes the full matrix algebra of $d_\lambda \times d_\lambda$ matrices with entries in κ.

For each $\lambda \in \widehat{B}_k$ and $1 \leq P, Q \leq d_\lambda$, let $E^\lambda_{P,Q}$ denote the matrix unit in the λth block of $\bigoplus_{\lambda \in \widehat{B}_k} M_{d_\lambda}(\kappa)$ which has a 1 as its (P,Q) entry and zeros everywhere else. Suppose $e^\lambda_{P,Q} = \Gamma^{-1}(E^\lambda_{P,Q})$ and define

$$T^{\lambda'} = V^{\otimes k} e^\lambda_{P,Q}, \tag{4.1}$$

where λ' denotes partition conjugate to λ. The space $T^{\lambda'}$ is an $\mathfrak{spo}(V,\beta)$-module since the action of $\mathfrak{spo}(V,\beta)$ on $V^{\otimes k}$ commutes with the action of $e^\lambda_{P,Q} \in B_k(n-m)$; however, the module $T^{\lambda'}$ may be (0).

Let us recall the definitions of certain symmetric functions. If $Y = \{y_1, \ldots, y_q\}$ is a set of commuting variables ordered by $y_1 < y_2 < \ldots < y_q$, then a *column-strict tableau* of shape λ/μ is a filling of the boxes in the Ferrers diagram of λ with y_j's such that the y's are weakly increasing (left to right) along rows and strictly increasing down columns. Given a column-strict tableau T of shape λ define y^T to be the product over all boxes of λ of the elements y_i in the boxes. Then the *skew Schur function* is given by

$$s_{\lambda/\mu}(Y) = \sum_T y^T,$$

where the sum is over all column-strict tableaux T of shape λ/μ. The (ordinary) Schur function $s_\lambda(Y)$ is the skew Schur function $s_{\lambda/\mu}(Y)$ with $\mu = \emptyset$.

The *hook Schur functions* defined in [BR] and [Se] (see also [M], I §3, Ex. 23-24 and I §5, Ex. 23) describe the characters of the irreducible $\mathfrak{gl}(m,n)$-modules appearing in $V^{\otimes k}$ (and also the $\mathfrak{gl}(V,\beta)$-irreducibles in $V^{\otimes k}$ see [Mo]). Assume as in (3.1) that $B_0 = \{t_1, t_1^*, t_2, t_2^*, \ldots, t_r, t_r^*\}$ $(m = 2r)$, $B_1 = \{u_1, u_1^*, u_2, u_2^*, \ldots, u_s, u_s^*, (u_{s+1})\}$ $(n = 2s$ or $2s+1)$, and $B = B_0 \cup B_1$ are bases of $V_{(0)}$, $V_{(1)}$ and V, respectively. Order the variables z_b, $b \in B = B_0 \cup B_1$ by

$$z_{t_1} < z_{t_1^*} < \cdots < z_{t_r} < z_{t_r^*} < z_{u_1} < z_{u_1^*} < \cdots < z_{u_s} < z_{u_s^*} < (z_{u_{s+1}}).$$

A *bitableau* of shape λ is a filling of the Ferrers diagram of λ with elements of B such that

(1) the portion of the diagram filled with z_t's is the diagram of a partition $\mu \subseteq \lambda$;

(2) the z_t's are weakly increasing (left to right) along rows and strictly increasing down columns; and

(3) the z_u's are weakly increasing down the columns and strictly increasing across the rows.

Given a bitableau T of shape λ, let z^T be the product of the elements z_b in all the boxes of λ. Then the *hook Schur function* is given by

$$s_\lambda(\tilde{Z}) = \sum_T z^T,$$

where the sum is over all bitableaux T of shape λ.

Let $Z_0 = \{z_b \mid b \in B_0\}$. A *symplectic tableau* of shape λ is a filling of the boxes in the Ferrers diagram of λ with z_b's, $b \in B_0$, such that

(1) the z_b's are weakly increasing (left to right) along rows and strictly increasing down columns; and

(2) the elements z_{t_i} and $z_{t_i^*}$ never appear in a row with number greater than i.

For a symplectic tableau T of shape λ define z^T to be the product of the elements z_b in all the boxes of λ. The *symplectic Schur function* is the sum,

$$sp_\lambda(Z_0) = \sum_T z^T, \tag{4.2}$$

over all the symplectic tableaux T of shape λ.

Assume $q = \mathrm{Card}(Y)$ is sufficiently large, i.e. $q >> r$, and define functions $sc_\lambda(Z_0)$ by the identity

$$\sum_\lambda sc_\lambda(Z_0)s_\lambda(Y) = \frac{\prod_{1\le i<j\le q}(1-y_iy_j)}{\prod_{b\in B_0}\prod_{j=1}^q(1-z_by_j)}. \tag{4.3}$$

When λ is a partition such that the number of parts $\ell(\lambda) \le r$, then $sc_\lambda(Z_0) = sp_\lambda(Z_0)$, and these polynomials describe the characters of the symplectic group $Sp(2r,\mathbb{C})$. The combinatorics of these functions is discussed in [Su1].

Analogously, if $Z_1 = \{z_b \mid b \in B_1\}$ and $q >> s$, there are functions $sb_\lambda(Z_1)$ defined by the identity

$$\sum_\lambda sb_\lambda(Z_1)s_\lambda(Y) = \frac{1}{\prod_{b\in B_1}\prod_{j=1}^q(1-z_by_j)}\prod_{1\le i\le j\le q}(1-y_iy_j). \tag{4.4}$$

When $\mathrm{Card}(B_1)$ is odd and λ is a partition such that $\ell(\lambda) \le s$, then the polynomials $sb_\lambda(Z_1)$ describe the characters of the orthogonal group $SO(2s+1,\mathbb{C})$ and its Lie algebra $\mathfrak{so}(2s+1)$.

Theorem 4.5. *Let $\tilde{Z}$ be the set of variables $\{z_b\}$ indexed by the elements $b \in B = B_0 \cup B_1$. Then the following identities give equivalent definitions of polynomials $sc_\lambda(\tilde{Z})$:*

(1) $$sc_\lambda(\tilde{Z}) = \sum_{\mu \subseteq \lambda} \left(\sum_{\pi} (-1)^{|\pi|/2} c^\lambda_{\mu,\pi} \right) s_\mu(\tilde{Z}),$$

where $s_\mu(\tilde{Z})$ denotes the hook Schur function labeled by μ, and $c^\lambda_{\mu,\pi}$ is the Littlewood-Richardson coefficient. The inner sum is over all partitions of the form $\pi = (r_1 - 1, \ldots, r_p - 1 \mid r_1, \ldots, r_p)$, $r_1 \leq q-1$, in Frobenius notation.

(2) $$sc_\lambda(\tilde{Z}) = \sum_{\tau \subseteq \lambda'} \left(\sum_{\rho} (-1)^{|\rho|/2} c^{\lambda'}_{\tau,\rho} \right) s_{\tau'}(\tilde{Z}),$$

where the inner sum is over all partitions of the form $\rho = (r_1 + 1, \ldots, r_p + 1 \mid r_1, \ldots, r_p)$, $r_1 \leq q-1$, in Frobenius notation.

(3) $$\sum_\lambda sc_\lambda(\tilde{Z}) s_\lambda(Y) = \frac{\prod_{b \in B_1} \prod_{j=1}^q (1 + z_b y_j)}{\prod_{b \in B_0} \prod_{j=1}^q (1 - z_b y_j)} \prod_{1 \leq i < j \leq q} (1 - y_i y_j).$$

(4) $$\sum_\lambda sc_\lambda(\tilde{Z}) s_{\lambda'}(Y) = \frac{\prod_{b \in B_0} \prod_{j=1}^q (1 + z_b y_j)}{\prod_{b \in B_1} \prod_{j=1}^q (1 - z_b y_j)} \prod_{1 \leq i \leq j \leq q} (1 - y_i y_j).$$

(5) $$sc_\lambda(\tilde{Z}) = \tfrac{1}{2} \det \left(h_{\lambda_i - i - j + 2}(\tilde{Z}) + h_{\lambda_i - i + j}(\tilde{Z}) \right).$$

where $h_\ell(\tilde{Z}) = s_{(\ell)}(\tilde{Z})$, and (ℓ) is the partition of ℓ with just one part.

(6) $$sc_\lambda(\tilde{Z}) = \tfrac{1}{2} \det \begin{pmatrix} (e_{\lambda'_i - i - j + 2}(\tilde{Z}) - e_{\lambda'_i - i - j}(\tilde{Z})) \\ \qquad + (e_{\lambda'_i - i + j}(\tilde{Z}) - e_{\lambda'_i - i + j - 2}(\tilde{Z})) \end{pmatrix},$$

where $e_\ell(\tilde{Z}) = s_{(1^\ell)}(\tilde{Z})$, and (1^ℓ) is the partition of ℓ with all parts equal to 1.

(7) $$sc_\lambda(\tilde{Z}) = \det \left(e_{\lambda'_i - i + j}(\tilde{Z}) - e_{\lambda'_i - i - j}(\tilde{Z}) \right).$$

(8) $$sc_\lambda(\tilde{Z}) = \sum_{\mu \subseteq \lambda} sc_\mu(Z_0) s_{\lambda'/\mu'}(Z_1),$$

where $s_{\lambda'/\mu'}(Z_1)$ is the skew Schur function in the variables z_b, $b \in B_1$.

(9) $$sc_\lambda(\tilde{Z}) = \sum_{\mu \subseteq \lambda} sb_{\mu'}(Z_1) s_{\lambda/\mu}(Z_0),$$

where $s_{\lambda/\mu}(Z_0)$ is the skew Schur function in the variables z_b, $b \in B_0$.

Assume now that $z_{b^\circ} = z_b^{-1}$ for $b \in B$, (in particular, $z_{u_{\circ+1}} = 1$). Then using bicharacters that involve weighted traces of diagrams from $B_k(n-m)$ together with Theorem 4.5 we prove

Theorem 4.6. *Assume $|n - m| \geq k$, and for $\lambda \vdash k - 2h$, $h = 0, 1, \ldots, \lfloor k/2 \rfloor$, let T^λ be the $\mathfrak{spo}(V, \beta)$-module $V^{\otimes k} e^{\lambda'}_{P,Q}$ (compare (4.1)). Then the character of T^λ is the function $sc_\lambda(\tilde{Z})$ defined in Theorem 4.5.*

Remark 4.7. It follows immediately from the definition of the skew-Schur functions, the definition of $sc_\lambda(Z_0)$, and Theorem 4.5 (8) that the functions $sc_\lambda(\tilde{Z})$ can be expressed as a sum over monomials corresponding to certain tableaux which have a symplectic part and a row-strict part. It is this interpretation which allows us to develop an insertion scheme modeling these functions.

5. An insertion scheme for spo-tableaux

An *up-down tableau of length k and shape λ* is a sequence $\Lambda = (\lambda^0, \lambda^1, \ldots, \lambda^k)$ of partitions such that $\lambda^0 = \emptyset$ and $\lambda^k = \lambda$ and λ^i is obtained from λ^{i-1} by either adding or removing a box for each $i = 1, \ldots, k$. An *up-down (r,n)-tableau* (or when r and n are fixed, simply an *up-down tableau*) is an up-down tableau $\Lambda = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^k = \lambda)$ of length k and shape λ such that each λ^i is an (r,n)-hook shape partition (i.e. $\lambda^i_{r+1} \le n$).

Let $B_0 = \{t_1, t_1^*, \ldots, t_r, t_r^*\}$ and $B_1 = \{u_1, \ldots, u_n\}$ and let $B = B_0 \cup B_1$. Here we do not need to distinguish between the cases that n is even or odd. Order B as follows:

$$B = \{t_1 < t_1^* < t_2 < t_2^* < \ldots < t_r < t_r^* < u_1 < \ldots < u_n\}.$$

An *$\mathfrak{spo}(m,n)$-standard tableau of shape λ* is a filling of the boxes in the Ferrers diagram of λ with entries from B such that

($\mathfrak{spo}$.1) the subtableau U of T consisting of all the boxes with entries from B_0 is a column-strict tableau of partition shape, and the entries in row i are $\ge t_i$ for each row in U,

($\mathfrak{spo}$.2) the skew tableau T/U is row-strict.

Let $\mathcal{W}_k$ be the set of words of length k in the alphabet B and $\mathcal{P}_k$ the set of pairs (T, Λ) consisting of an $\mathfrak{spo}(m,n)$-standard tableau T of shape λ and an up-down (r,n)-tableau $\Lambda = (\lambda^0, \lambda^1, \ldots, \lambda^k)$ of length k and shape λ. In [BLR] we prove that there is a bijection between $\mathcal{W}_k$ and $\mathcal{P}_k$ which can be described as follows:

Let T be an $\mathfrak{spo}(m,n)$-standard tableau, and assume $a \in B$. We define an algorithm consisting of a sequence of steps which inserts a into T to yield a tableau $(a \to T)$.

(1) Start with $b = a$ and $i = j = 1$.

(2) If $b \in B_0$, then insert b into the ith row of T as follows: If there is an entry in row i which is greater than b, then displace the leftmost such entry and insert b into its box except in the following case. If $b = t_i$ and there is an t_i^* in the ith row, then replace the leftmost t_i^* in the row with t_i and remove the entry in the $(i,1)$-position (which is necessarily a t_i) making it an empty box. If there is no entry in the ith row which is greater than b, then adjoin b to the end of the row.

If $b \in B_1$, then insert b into the jth column as follows: If there is an entry in the jth column which is greater than b, then displace the topmost such entry and insert b into its position. If there is no entry in the jth column which is greater than b, then adjoin b at the end of the column.

(3) Set b equal to the displaced entry and change i to $p+1$ and j to $q+1$ where (p,q) was the position of the displaced entry. Repeat step (2) until an entry is adjoined to the end of a row or a column, or an empty box is created.

(4) Let $(a \to T)'$ be the result of steps (1)-(3). Set $(a \to T) = (a \to T)'$ if $(a \to T)'$ is a tableau, and $(a \to T) = jeu((a \to T)')$ if $(a \to T)'$ is a tableau with an empty box where "*jeu*" is the "*jeu de taquin*".

For each k, define maps $\mathfrak{spo}_k \colon \mathcal{W}_k \to \mathcal{P}_k$ by

(1) $\mathfrak{spo}_0(w) = (\emptyset, (\emptyset))$, where w is the emptyword;

(2) If $w = w_1 \cdots w_k$ is a word of length k, then $\mathfrak{spo}_k(w) = (T^k, (\lambda^0, \lambda^1, \ldots, \lambda^k))$ where, if $\mathfrak{spo}_{k-1}(w_1 \cdots w_{k-1}) = (T^{k-1}, (\lambda^0, \lambda^1, \ldots, \lambda^{k-1}))$, then $T^k = (w_k \to T^{k-1})$ and λ^k is the underlying partition of T^k.

Theorem 5.1. *The map* $spo_k : \mathcal{W}_k \longrightarrow \mathcal{P}_k$ *is a bijection.*

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