VEXILLARY ELEMENTS IN THE HYPEROCTAHEDRAL GROUP

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1. Introduction

The vexillary permutations in the symmetric group have interesting connections with the number of reduced words, the Littlewood-Richardson rule, Stanley symmetric functions, Schubert polynomials and the Schubert calculus. Lascoux and Schützenberger [13] have shown that vexillary permutations are characterized by the property that they avoid any subsequence of length 4 with the same relative order as 2143. Macdonald has given a good overview of vexillary permutations in [15]. In this paper we propose a definition for vexillary elements in the hyperoctahedral group. We show that the vexillary elements can again be determined by pattern avoidance conditions.

We will begin by reviewing the history of the Stanley symmetric functions and establishing our notation. We have included several propositions from the literature that we will use in the proof of the main theorem. In Section 2 we will define the vexillary elements in the symmetric group and the hyperoctahedral group. Finally we state and prove that the vexillary elements are precisely those elements which avoid 18 different patterns of lengths 3 and 4. Due to the quantity of cases that need to be analyzed we have used a computer to verify a key lemma in the proof of the main theorem. The definition of vexillary can be extended to cover the root systems of type A, B, C, and D; in all four cases the definition is equivalent to avoiding certain patterns. We conclude with several open problems related to vexillary elements in the hyperoctahedral group.

Let S_n be the symmetric group whose elements are permutations written in one-line notation as $[w_1, w_2, \dots, w_n]$. S_n is generated by the adjacent transpositions σ_i for $1 \le i < n$, where σ_i interchanges positions i and $i+1$ when acting on the right, *i.e.*, $[\dots, w_i, w_{i+1}, \dots]\sigma_i = [\dots, w_{i+1}, w_i, \dots]$.

Let B_n be the hyperoctahedral group (or signed permutation group). The elements of B_n are permutations with a sign attached to every entry. We use the compact notation where a bar is written over an element with a negative sign. For example $[\bar{3}, 2, \bar{1}] \in B_3$. B_n is generated by the adjacent transpositions σ_i for $1 \le i < n$, as in S_n, along with σ_0 which acts on the right by changing the sign of the first element, *i.e.*, $[w_1, w_2, \dots, w_n]\sigma_0 = [\overline{w_1}, w_2, \dots, w_n]$.

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If w can be written as a product of the generators $\sigma_{a_1}\sigma_{a_2}\ldots\sigma_{a_p}$ and p is minimal then the concatenation of the indices $a_1a_2\ldots a_p$ is a *reduced word* for w, and p is the *length* of w, denoted $l(w)$. Let $R(w)$ be the set of all reduced words for w. The signed (or unsigned) permutations $[w_1,\ldots,w_n]$ and $[w_1,\ldots,w_n,n+1,n+2,\ldots]$ have the same set of reduced words. For our purposes it will useful to consider these signed permutations as the same in the infinite groups $S_\infty = \cup S_n$ or $B_\infty = \cup B_n$.

In [19], Stanley gave a formula for the number of reduced words for a permutation $w \in S_\infty$ in terms of f^λ the number of standard tableaux of shape λ, namely

$$\#R(w) = \sum \alpha_w^\lambda f^\lambda \tag{1.1}$$

where the sum is over all partitions λ of $l(w)$ and the coefficients α_w^λ are non-negative integers. Bijective proofs of (1.1) were given independently by Lascoux and Schützenberger [12] and Edelman and Greene [4]. Reiner and Shimozono [17] have given a new interpretation of the coefficients α_w^λ in terms of $D(w)$-peelable tableaux.

Stanley also conjectured that there should be an analog of (1.1) for B_n. This conjecture was proved independently by Haiman [7] and Kraśkiewicz [8] in the following form:

$$\#R(w) = \sum \beta_w^\lambda g^\lambda \tag{1.2}$$

where the sum is over all partitions of $l(w)$ with distinct parts, g^λ is the number of standard tableaux on the shifted shape λ, and the coefficients β_w^λ are non-negative integers.

The coefficients α_w^λ and β_w^λ in (1.1) and (1.2) can be used to define symmetric functions which originally appeared in [19]. Let s_λ be the Schur function of shape λ and let Q_λ be the Q-Schur function of shape λ. See [14] for definitions of these symmetric functions.

Definition . For $w \in S_n$ or B_n respectively, define the *Stanley symmetric function* by

$$\begin{aligned} G_w &= \sum \alpha_w^\lambda s_\lambda \\ F_w &= \sum \beta_w^\lambda Q_\lambda. \end{aligned} \tag{1.3}$$

The Stanley symmetric functions can also be defined using the nilCoxeter algebra of S_n and B_n respectively(see [5] and [6]). The relationship between Kraśkiewicz's proof of (1.2) and B_n Stanley symmetric functions are explored in [10]. See also [3, 11, 20] for other connections to Stanley symmetric functions. The functions F_w are usually referred to as the Stanley symmetric functions of type C because they are related to the root systems of type C. The Weyl group for the root systems of type B and C are isomorphic, so we can study the group B_n by studying either root system. We extend the results of the main theorem to the root systems of type B and D at the end of Section 2.

We will use these symmetric functions to define vexillary elements in S_n and B_n. The Stanley functions F_w can easily be computed using Proposition 1.1 below which is stated in terms of special elements in B_n. There are two types of "transpositions" in the hyperoctahedral group. These transpositions correspond with reflections in the Weyl group of the root system B_n. Let t_{ij} be a transposition of the usual type i.e.

$[\ldots, w_i, \ldots, w_j, \ldots]t_{ij} = [\ldots, w_j, \ldots, w_i, \ldots]$. Let s_{ij} be a transposition of two elements that also switches sign $[\ldots, w_i, \ldots, w_j, \ldots]s_{ij} = [\ldots, \overline{w_j}, \ldots, \overline{w_i}, \ldots]$. Let τ_{ij} be a transposition of either type. A signed permutation w is said to have a *descent* at r if $w_r > w_{r+1}$.

Proposition 1.1. [2] *The Stanley symmetric functions of type C have the following recursive formulas:*

$$F_w = \sum_{\substack{0<i<r \\ l(wt_{rs}t_{ir})=l(w)}} F_{wt_{rs}t_{ir}} + \sum_{\substack{0<i \\ l(wt_{rs}s_{ir})=l(w)}} F_{wt_{rs}s_{ir}} \tag{1.4}$$

where r is the last descent of w, and s is the largest position such that $w_s < w_r$. The recursion terminates when w is strictly increasing in which case $F_w = Q_\lambda$ where λ is the partition obtained from arranging $\{|w_i| : w_i < 0\}$ in decreasing order.

For example, let $w = [\bar{4}, 1, \bar{2}, 3]$. Then $r = 2$ since $w_2 > w_3$ is a descent and $w_3 < w_4$, and $s = 3$ since $w_3 < w_2 < w_4$. This implies $wt_{rs} = [\bar{4}, \bar{2}, 1, 3]$ and (1.4) we have

$$F_{[\bar{4},1,\bar{2},3]} = F_{[\bar{4},1,\bar{2},3]t_{23}t_{12}} + F_{[\bar{4},1,\bar{2},3]t_{23}s_{32}} = F_{[\bar{2},\bar{4},1,3]} + F_{[\bar{4},\bar{3},1,2]}. \tag{1.5}$$

Continuing to expand the right hand side we see $[\bar{4}, \bar{3}, 1, 2]$ is strictly increasing so $F_{[\bar{4},\bar{3},1,2]} = Q_{(4,3)}$ and $F_{[\bar{2},\bar{4},1,3]} = F_{[\bar{2},\bar{4},1,3]t_{12}s_{15}} = F_{[\bar{5},\bar{2},1,3,4]} = Q_{(5,2)}$. Hence, $F_{[\bar{4},1,\bar{2},3]} = Q_{(4,3)} + Q_{(5,2)}$.

Note that $l(wt_{rs})$ always equal $l(w) - 1$ in Proposition 1.1 because of the choice for r and s. If $l(wt_{rs}\tau_{ir}) = l(w)$, then $l(wt_{rs}\tau_{ir}) = l(wt_{rs})+1$. The reflections τ_{ir} which increase the length of wt_{rs} by exactly 1 are characterized by the following two propositions.

Proposition 1.2 ([16]). *If $w \in S_\infty$ or B_∞ and $i < j$, then $l(wt_{ij}) = l(w)+1$ if and only if*

- $w_i < w_j$

and no k exists such that

- $i < k < j$ *and* $w_i < w_k < w_j$.

Proposition 1.3. [2] *If $w \in B_\infty$, and $i \leq j$, then $l(ws_{ij}) = l(w) + 1$ if and only if*

- $-w_i < w_j$ *and* $-w_j < w_i$
- *if* $i \neq j$, *either* $w_i < 0$ *or* $w_j < 0$,

and no k exists such that either of the following are true:

- $k < i$ *and* $-w_j < w_k < w_i$
- $k < j$ *and* $-w_i < w_k < w_j$.

2. Main Results

In this section we give the definition of the vexillary elements in S_n and B_n. Then we present the main theorem. The proof follows after several lemmas.

Definition . If $w \in S_n$ then w is *vexillary* if $G_w = s_\lambda$ for some shape $\lambda \vdash l(w)$. Similarly, if $w \in B_n$ then w is *vexillary* if $F_w = Q_\lambda$ for some shape $\lambda \vdash l(w)$ with distinct parts.

It follows from the definition of the Stanley symmetric functions (1.3) that if w is vexillary then the number of reduced words for w is the number of standard tableaux of a single shape (unshifted for $w \in S_n$ or shifted for $w \in B_n$).

For S_n, this definition is equivalent to the original definition of vexillary given by Lascoux and Schützenberger in [13]. They showed that vexillary permutations w are characterized by the condition that no subsequence $a < b < c < d$ exists such that $w_b < w_a < w_c < w_d$. This property is usually referred to as *2143-avoiding*. Lascoux and Schützenberger also showed that the Schubert polynomial of type A_n indexed by w is a flagged Schur function if and only if w is a vexillary permutation. One might ask if the Schubert polynomials of type B, C or D indexed by a vexillary element could be written in terms of a "flagged Schur Q-function."

Many other properties of permutations can be given in terms of pattern avoidance. For example, the reduced words of 321-avoiding [1] permutations all have the same content, and a Schubert variety in SL_n/B is smooth if and only if it is indexed by a permutation which avoids the patterns 3412 and 4231 [9]. Also, Julian West [21] and Simion and Schmidt [18] have studied pattern avoidance more generally and given formulas for computing the number of permutations which avoid combinations of patterns. Recently, Stembridge [20] has described several properties of signed permutations in terms of pattern avoidance as well.

We will define pattern avoidance in terms of the following function which *flattens* any subsequence into a signed permutation.

Definition . Given any sequence $a_1 a_2 \ldots a_k$ of distinct non-zero real numbers, define $\mathrm{fl}(a_1 a_2 \ldots a_k)$ to be the unique element $b = [b_1, \ldots, b_k]$ in B_k such that

- both a_j and b_j have the same sign.
- for all i, j, we have $|b_i| < |b_j|$ if and only if $|a_i| < |a_j|$.

For example, $\mathrm{fl}(\bar{6}, 3, \bar{7}, 0.5) = [\bar{3}, 2, \bar{4}, 1]$. Any word containing the subsequence $\bar{6}, 3, \bar{7}, 0.5$ does not avoid the pattern $\bar{3}2\bar{4}1$.

Theorem 1. *An element $w \in B_\infty$ is vexillary if and only if every subsequence of length 4 in w flattens to a vexillary element in B_4. In particular, w is vexillary if and only if it avoids the following patterns:*

$$\begin{array}{ccccc} \bar{3}2\bar{1} & \bar{3}21 & 32\bar{1} & 321 & 3\bar{1}2 \\ \bar{2}31 & \bar{1}32 & \bar{4}\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 & \bar{3}\bar{4}\bar{1}\bar{2} \\ \bar{3}\bar{4}1\bar{2} & 3\bar{4}\bar{1}\bar{2} & 3\bar{4}1\bar{2} & 3142 & \bar{2}\bar{3}4\bar{1} \\ 2413 & 2\bar{3}4\bar{1} & 2143 & & \end{array} \tag{2.1}$$

This list of patterns was conjectured in [11]. Due to the large number of non-vexillary patterns in (2.1) we have chosen to prove the theorem in two steps. First, we have verified that the theorem holds for B_6, see Lemma 2.1. Second, we show that any counter example in B_∞ would imply a counter example in B_6.

Lemma 2.1. *Let $w \in B_6$, then w is vexillary if and only if it does not contain any subsequence of length 3 or 4 which flattens to a pattern in* (2.1).

See the appendix for an outline of the code used to verify Lemma 2.1.

Lemma 2.2. *Let w be any signed permutation. Suppose $w_{i_1}w_{i_2}\dots w_{i_k}$ is a subsequence of w and let $u = \mathrm{fl}(w_{i_1}w_{i_2}\dots w_{i_k})$. Then the following statements hold:*

1. *If the last decent of w appears in position $i_r \in \{i_1, \dots, i_k\}$ then the last descent of u will be in position r.*
2. *If in addition, $w_{i_s} < w_{i_r}$ and i_s is the largest index in w such that this is true then $u_s < u_r$ and s is the largest index in u such that this is true.*
3. *If $v = w\tau_{i_j i_k}$ then $\mathrm{fl}(v_{i_1}\dots v_{i_k}) = \mathrm{fl}(w_{i_1}\dots w_{i_k})\cdot\tau_{jk}$ where $\tau_{i_j i_k}$ and τ_{jk} are transpositions of the same type.*

One can check the facts above follow directly from the definition of the flatten function.

Lemma 2.3. *For any $v \in B_\infty$ and any $0 < i < r$, if $l(vt_{ir}) - l(v) > 0$ then there exists an index k such that $i \leq k < r$, $v_i \leq v_k < v_r$ and $l(vt_{kr}) - l(v) = 1$. Similarly, if $l(vs_{ir}) - l(v) > 0$ then there exists an index k such that either*

- *$k < r$, $-v_i < v_k < v_r$, and $l(vt_{kr}) - l(v) = 1$, or*
- *$k \leq i$, $-v_r < v_k \leq v_i$, and $l(vs_{kr}) - l(v) = 1$.*

Proof. If $l(vt_{ir}) - l(v) > 0$, pick k such that v_k is the largest value in $\{v_k < v_r : i \leq k < r\}$. Then no j exists such that $k < j < r$ and $v_k < v_j < v_r$, hence $l(vt_{kr}) - l(v) = 1$.

Say $l(vs_{ir}) - l(v) > 0$. If there exists $k < r$ such that $v_k < v_r$, chose k such that v_k is the largest value in $\{v_k < v_r : k < r\}$. Then no j exists such that $k < j < r$ and $v_k < v_j < v_r$, hence $l(vt_{kr}) - l(v) = 1$. On the other hand, if no such k exists, then choose k be such that v_k is the smallest value in $\{v_k > -v_r : k \leq i\}$. Then no $j < r$ exists such $-v_k < v_j < v_r$ and no $j' < k$ exists such that $-v_r < v_{j'} < v_k$, hence $l(vs_{kr}) - l(v) = 1$. □

Lemma 2.4. *Given any $w \in B_\infty$ and any subsequence of w, say $w_{i_1}w_{i_2}\dots w_{i_k}$, let $v = \mathrm{fl}(w_{i_1}w_{i_2}\dots w_{i_k}) \in B_k$. If $l(wt_{i_j,i_k}) - l(w) = 1$ then $l(vt_{jk}) - l(v) = 1$. Similarly, if $l(ws_{i_j,i_k}) - l(w) = 1$ then $l(vs_{jk}) - l(v) = 1$.*

Proof. If $l(wt_{i_j,i_k}) - l(w) \geq 1$ then $w_{i_j} < w_{i_k}$ so $v_j < v_k$ since the flatten map preserves the relative order of the elements in the subsequence and signs. Therefore, $l(vt_{jk}) - l(v) \geq 1$. If $l(wt_{i_j,i_k}) - l(w) = 1$ then no $i_j < m < i_k$ exists such that $w_{i_j} < w_m < w_{i_k}$. This in turn implies that no $j < m < k$ exists such that $v_j < v_m < v_k$, hence $l(vt_{jk}) - l(v) = 1$.

If $l(ws_{i_j,i_k}) - l(w) \geq 1$ then $-w_{i_j} < w_{i_k}$ and $-w_{i_k} < w_{i_j}$ so $-v_j < v_k$ and $-v_k < v_j$ since the flatten map preserves the relative order of the elements in the subsequence and signs. Also, if $i_j \neq i_k$ then either $w_{i_j} < 0$ or $w_{i_k} < 0$ so either $v_j < 0$ or $v_k < 0$. Therefore, $l(vs_{jk}) - l(v) \geq 1$. If $l(ws_{i_j,i_k}) - l(w) = 1$ then no $m < i_k$ exists such that $-w_{i_j} < w_m < w_{i_k}$, and no $m < i_j$ exists such that $-w_{i_k} < w_m < w_{i_j}$. This in turn implies that no $m < k$ exists such that $-v_j < v_m < v_k$, and no $m < j$ exists such that $-v_k < v_m < v_j$, hence $l(vs_{jk}) - l(v) = 1$. □

Lemma 2.5. *Given any $w \in B_\infty$, if w is non-vexillary then w contains a subsequence of length 4 which flattens to a non-vexillary element in B_4.*

Proof. Since w is non-vexillary then either F_w expands into multiple terms on the first step of the recurrence in (1.4) or else $F_w = F_v$ where v is again non-vexillary. Assume the first step of the recurrence gives

$$F_w = F_{wt_{rs}\tau_{ir}} + F_{wt_{rs}\tau_{jr}} + \text{ other terms}$$

Let $\alpha : \{1, 2, 3, 4\} \to \{i, j, r, s, n+1\}$ be an order preserving map onto the 4 smallest distinct numbers in the range. Let $w' = \mathrm{fl}(w_{\alpha(1)}w_{\alpha(2)}w_{\alpha(3)}w_{\alpha(4)})$. By Lemma 2.4 $l(w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(i)\alpha^{-1}(r)}]) = l(w')$ and $l(w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(j)\alpha^{-1}(r)}]) = l(w')$. Therefore, the recursion implies

$$F_{w'} = F_{w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(i)\alpha^{-1}(r)}]} + F_{w'[t_{\alpha^{-1}(r)\alpha^{-1}(s)}][\tau_{\alpha^{-1}(j)\alpha^{-1}(r)}]} + \text{ other terms.}$$

Hence, $w' \in B_4$ is not vexillary, and it follows that w contains the non-vexillary subsequence $w_{\alpha(1)}w_{\alpha(2)}w_{\alpha(3)}w_{\alpha(4)}$.

If on the other hand the first step of the recursion gives $F_w = F_v$ then $v = wt_{rs}\tau_{ir}$ and v is not vexillary. Assume, by induction on the number of steps until the recurrence branches into multiple terms that v contains a non-vexillary subsequence say $v_av_bv_cv_d$. If $i, r, s \notin \{a, b, c, d\}$ then $w_aw_bw_cw_d$ is exactly the same non-vexillary subsequence. So we can assume the order of the set $\{a, b, c, d, i, r, s\}$ is less than or equal to 6. Let

$$\phi : \{1, 2, \ldots, 6\} \to \{a, b, c, d, i, r, s\} \cup \{n+1, n+2\}$$

be an order preserving map which sends the numbers 1 through 6 to the 6 smallest distinct integers in the range. Let $w' = \mathrm{fl}(w_{\phi(1)}w_{\phi(2)} \ldots w_{\phi(6)})$ and $v' = \mathrm{fl}(v_{\phi(1)}v_{\phi(2)} \ldots v_{\phi(6)})$. By construction, $v' \in B_6$ contains a non-vexillary subsequence, hence v' is not vexillary by Lemma 2.1. We will use the recursion on $F_{w'}$ to show that w' is not vexillary in B_6. From Lemma 2.2 it follows that

$$v' = w't_{\phi^{-1}(r)\phi^{-1}(s)}\tau_{\phi^{-1}(i)\phi^{-1}(r)}.$$

By Lemma 2.3, $l(v) = l(wt_{rs}) + 1 = l(w)$ implies $l(v') = l(w')$. Therefore,

$$F_{w'} = F_{v'} + \text{possibly other terms.}$$

Irregardless of whether there are any other terms in expansion of $F_{w'}$, w' is not vexillary since v' is not vexillary. Again by Lemma 2.1, this implies w' contains a non-vexillary subsequence of length 4, say $w'_ew'_fw'_gw'_h$. Hence, w contains the non-vexillary subsequence $w_{\phi(e)}w_{\phi(f)}w_{\phi(g)}w_{\phi(h)}$. □

This proves one direction of Theorem 1.

Lemma 2.6. *Given any $w \in B_\infty$, if w contains a subsequence of length 4 which flattens to a non-vexillary element in B_4 then w is non-vexillary.*

Proof. Assume w is vexillary then let $w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ be the sequence of signed permutations which arise in expanding $F_w = F_{w^{(1)}} = F_{w^{(2)}} = \ldots = F_{w^{(k)}}$ using the recurrence (1.4). This recurrence terminates when the signed permutation $w^{(k)}$ is strictly increasing, hence $w^{(k)}$ does not contain any of the patterns in (2.1). Replace w by the first $w^{(i)}$ such that $w^{(i)}$ contains a non-vexillary subsequence and $w^{(i+1)}$ does not, and let $v = w^{(i+1)}$.

Say $w_a w_b w_c w_d$ is a non-vexillary subsequence in w. If $i, r, s \notin \{a, b, c, d\}$ then $v_a v_b v_c v_d$ would be exactly the same non-vexillary subsequence. This contradicts our choice of v. So we can assume that the order of the set $\{a, b, c, d, i, r, s\}$ is less than or equal to 6. As in the proof of Lemma 2.5, let

$$\phi : \{1, 2, \ldots, 6\} \to \{a, b, c, d, i, r, s\} \cup \{n+1, n+2\}$$

be an order preserving map onto the smallest 6 distinct numbers in the range. Let $w' = \mathrm{fl}(w_{\phi(1)} w_{\phi(2)} \cdots w_{\phi(6)})$ and $v' = \mathrm{fl}(v_{\phi(1)} v_{\phi(2)} \cdots v_{\phi(6)})$. To simplify notation, we also let $i' = \phi^{-1}(i)$, $r' = \phi^{-1}(r)$, and $s' = \phi^{-1}(s)$. By construction, $w' \in B_6$ contains a non-vexillary subsequence hence w' is not vexillary by Lemma 2.1. As in 2.5 one can show

$$F_{w'} = F_{v'} + \text{ other terms.}$$

Since w' contains a non-vexillary subsequence and v' does not there must be another term in $F_{w'}$ indexed by a reflection $\tau_{j'r'} \neq \tau_{i'r'}$ with $l(w' t_{r's'} \tau_{j'r'}) = l(w')$. One should note that it is possible that $i' = j'$ but then $\tau_{i'r'}$ and $\tau_{j'r'}$ must be different types of transpositions. Let $j = \phi(j')$. By Proposition 1.3 and the definition of the flatten function, we have $l(wt_{rs}\tau_{jr}) - l(wt_{rs}) > 0$. By Lemma 2.3 there exists a reflection τ_{kr} such that $l(wt_{rs}\tau_{kr}) - l(wt_{rs}) = 1$.

We must have $\tau_{kr} \neq \tau_{ir}$ since $\tau_{i'r'} \neq \tau_{j'r'}$. Hence,

$$F_w = F_{wt_{rs}\tau_{ir}} + F_{wt_{rs}\tau_{kr}} + \text{possibly other terms.}$$

This proves w is not vexillary contrary to our assumption. □

This completes the proof of Theorem 1.

The definition of vexillary can be extended to Stanley symmetric functions of type B and D. These cover the remaining infinite families of root systems. For these cases, we define vexillary to be the condition that the function is exactly one Schur P-function. The signed permutations which are B and D vexillary can again be determined by avoiding certain patterns of length 4.

Theorem 2. *An element $w \in B_\infty$ is vexillary for type B if and only if every subsequence of length 4 in w flattens to a vexillary element of type B in B_4. In particular, w is vexillary if and only if it avoids the following patterns:*

$$\begin{matrix} 21 & \bar{3}2\bar{1} & 234\bar{1} \\ \bar{2}34\bar{1} & 34\bar{1}\bar{2} & \bar{3}41\bar{2} \\ \bar{3}4\bar{1}\bar{2} & 4\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 \end{matrix} \tag{2.2}$$

An element $w \in D_\infty$ is vexillary for type D if and only if every subsequence of length 4 avoids the following patterns:

$$\begin{array}{cccccc} 132 & \bar{1}32 & 321 & 32\bar{1} & \bar{3}21 & \bar{3}2\bar{1} \\ 2\bar{3}41 & 2\bar{3}4\bar{1} & \bar{2}\bar{3}41 & \bar{2}\bar{3}4\bar{1} & 3412 & 34\bar{1}2 \\ 3\bar{4}1\bar{2} & 3\bar{4}\bar{1}\bar{2} & \bar{3}412 & \bar{3}4\bar{1}2 & \bar{3}\bar{4}1\bar{2} & \bar{3}\bar{4}\bar{1}\bar{2} \\ 41\bar{2}3 & 4\bar{1}\bar{2}3 & \bar{4}1\bar{2}3 & \bar{4}\bar{1}\bar{2}3 & & \end{array} \tag{2.3}$$

Note, that the patterns that are avoided by vexillary elements of type D are not all type D signed permutations but instead include some elements with an odd number of negative signs. The proof of Theorem 2 is very similar to the proof of Theorem 1 given above. We omit the details in this abstract.

3. Open Problems

The vexillary permutations in S_n have many interesting properties. We would like to explore the possibility that these properties have analogs for the vexillary elements in B_n.

1. Is there a direct way to determine which shape λ will appear in the equation $F_w = Q_\lambda$ if w is vexillary?

   We can answer this when w is of the form where all w_i have positive signs or all have negative signs.
2. Is there a relationship between smooth Schubert varieties in $SO(2n+1)/B$ and vexillary elements? In particular, does smooth imply vexillary as in the case of S_n?
3. Is there a way to define flagged Schur Q-functions so that the Schubert polynomial indexed by w of type B or C is a flagged Schur Q-functions if and only if w is vexillary.
4. Is there an efficient method for multiplying Schur Q-functions similarly to the rule that Lascoux and Schützenberger have given for multiplying Schur functions [13]?

   This question seems to have a partial solution. If there exits a permutation $v \in S_n$ which is vexillary of type C and whose Stanley function is one of the two Schur Q-functions then the answer is yes. However, it is not true that for any shifted shape λ there exists such a $v \in S_n$. In fact the only shifted shapes with this property are the ones which are equivalent to a rectangle under jeu-de-taquin.
5. Are there other possible ways to define vexillary elements in B_n so that any of the above questions can be answered?

4. Appendix

Below is a portion of the LISP code used to verify Theorem 1 for B_6. The calculation was done on a Sparc 1 by running (grind-patterns 6 'c).

```
(setf *avoid-patterns*
   (list '(-3 2 -1) '(-3 2 1) '(3 2 -1) '(3 2 1) '(3 -1 2) '(-2 3 1) '(-1 3 2)
         '(-4 -1 -2 3)  '(-4 1 -2 3)  '(-3 -4 -1 -2)  '(-3 -4 1 -2)
         '(3 -4 -1 -2)  '(3 -4 1 -2)  '(3 1 4 2)  '(-2 -3 4 -1)  '(2 4 1 3)
         '(2 -3 4 -1)  '(2 1 4 3)))

(defun grind-patterns (n type)
  (flet ((helper (perm)
          (if (not (eq (and (avoid-subsequences perm 3)
                            (avoid-subsequences perm 4))
                       (vex-p perm type)))
              (format t "ERROR::NEW PATTERN: ~a ~%" perm)
              (format t "."))))
    (all-perm-tester n #'helper type)))

(defun avoid-subsequences (the-list size)
  (let ((results t))
    (catch 'foo
      (flet ((helper (tail)
              (when (member (flatten-seq tail) *avoid-patterns* :test #'equal)
                (setf results nil)
                (throw 'foo nil)
                )))
        (all-subsequences-tester (reverse the-list) size  #'helper nil))
      (throw 'foo t))
    results))
```

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