Permutations avoiding certain patterns
The case of length 4 and some generalizations

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Abstract

Proving and disproving some earlier conjectures, we give a characterization of the numbers of permutations avoiding each pattern of length 4. Implications for longer patterns are included.

1 Introduction

Let $q = (q_1, q_2, \ldots, q_k) \in S_k$ be a permutation, and let $k \leq n$. We say that the permutation $p = (p_1, p_2, \ldots, p_n) \in S_n$ is q–avoiding if and only if there is no $1 \leq i_{q_1} < i_{q_2} < \ldots < i_{q_k} \leq n$ such that $p(i_1) < p(i_2) < \ldots < p(i_k)$. For example, a permutation is 12345-avoiding if it does not contain any increasing subsequence of length 5 in the above one-line notation. Likewise, a permutation is 132-avoiding if it doesn't contain three elements among which the leftmost is the smallest and the middle one is the largest.

It is a long-studied and hard problem to determine the number $S_n(q)$ of permutations in S_n which avoid a certain pattern q. The general conjecture, made by Herb Wilf and Richard Stanley [7] claims that only very few of them, namely less than K^n, where K is some constant depending on q. However, efforts to prove this have been unsuccessful for most patterns. (To illustrate the complexity of the problem, we note that for two permutations a and b, it is an NP-complete problem to decide whether a is a pattern of b; and the problem of counting permutations in S_n avoiding q is known to be $\#P$-complete [2]. Thus in the general case all we can expect is an upper bound or an asymptotical formula for this number, not an exact formula). If q is of length three, then it is known that the number of n-permutations avoiding q is exactly the n-th Catalan-number, $C_n = \binom{2n}{n}/(n+1)$ *no matter what pattern q is.* (See [4]). If q is longer, then the most important result is the following lemma of Amitaj Regev, which solves the problem of monotonic patterns of any length.

Lemma 1 *([3]) For all n, $S_n(1234...k)$ asymptotically equals*

$$c\frac{(k-1)^{2n}}{n^{(k^2-2k)/2}},$$

where c is a constant given by a multiple integral. □

A nice property of $123..k$-avoiding permutations is that they can be decomposed into the union of $(k-1)$ decreasing sequences.

There are some other scattered results giving the answer for some particular permutations by bijectively proving that the number of n-permutations avoiding them equals the number of those which avoid the monotonic pattern. For these kind of results, see [8].

In this paper we examine patterns of length 4. Earlier results show that we can restrict ourselves to the patterns 1234, 1324 and 1423 since any other pattern behaves identically to one of these. (See [8], [1] and [6]). We are going to prove that $S_n(1324) < 36^n$ and that $S_n(1423) \leq S_n(1234) < 9^n$, thus *the general conjecture on the exponential upper bound is true for all patterns of length 4.*

Another natural question is to ask how the different patterns relate to each other? A general (and probably hard) conjecture states that if $S_n(q_1) < S_n(q_2)$ for some n, then this inequality holds for all $N > n$. We are going to prove this conjecture in all cases when the patterns are of length 4. In other words, we show that $S_n(1423) < S_n(1234)$ if $n \geq 6$ and $S_n(1234) < S_n(1324)$ if $n \geq 7$. These are the first results we know of which prove that one pattern is more likely to occur in a random permutation than an other one. These results were suggested by numerical evidence found in [8] and [10] showing the values of $S_n(q)$ for $n \leq 8$. These values are:

- for $S_n(1423)$: 1, 2, 6, 23, 103, 512, 2740, 15485
- for $S_n(1234)$: 1, 2, 6, 23, 103, 513, 2761, 15767
- for $S_n(1324)$: 1, 2, 6, 23, 103, 513, 2762, 15793.

We will actually see what causes the split between these classes.

Another conjecture has been that for all q of length k, the numbers $S_n(q)$ are asymptotically equal. We are going to *disprove* this conjecture by showing that both inequalities mentioned in the previous paragraph hold in the asymptotical sense.

We note that the relation "$p < q$ if and only if p is a pattern of q" is a partial ordering of the set of all finite permutations. It has recently been shown in [5] that this ordering is *not* a well-quasi ordering, that is, there is an infinite set of finite permutations in which no element is a pattern of an other one.

Terminology: Permutations in S_n will be called $n-$permutations. We will always use the one-line notation for permutations. An entry of a permutation which is smaller than all the entries it is preceeded by is called a *left-to-right minimum.* An entry which is larger than all entries it precedes is called a *right-to-left maximum.* The entries of a permutation which are not left-to-right minima or right-to-left maxima are called *remaining entries.*

2 The pattern 1324

Theorem 1 *For all $n \geq 7$, $S_n(1234) < S_n(1324)$.*

Proof: We classify all permutations of n according to the set and position of their left-to-right minima and right-to-left maxima. This definition is crucial in all this chapter, thus we announce it on its own:

Definition 1 *Two n-permutations x and y are said to be in the same* class *if and only if the left-to-right minima of x are the same as those of y and they are in the same positions and the same holds for the right-to-left maxima.*

For example, $x = 51234$ and $y = 51324$ are in the same class, but $z = 24315$ and $v = 24135$ are not as the third entry of z is not a left-to-right minimum whereas that of v is.

The outline of our proof is as follows: we show that each nonempty class contains *exactly* one 1234-avoiding permutation and *at least* one 1324-avoiding permutation. Then we exhibit some classes which contain more than one 1324-avoiding permutation.

Lemma 2 *Each nonempty class contains exactly one 1234-avoiding permutation.*

Proof: Suppose we have already picked a class, that is, we fixed the positions and values of all the left-to-right minima and right-to-left maxima. It is clear that if we write all the remaining elements into the remaining slots in decreasing order, then we get a 1234-avoiding permutation. (Indeed, the permutation obtained this way consists of 3 decreasing subsequences, that is, the left-to-right minima, the right-to-left maxima, and the remaining entries, thus, if there were a 1234-pattern, then by the pigeon-hole principle two of its elements would be in the same decreasing subsequence, which would be a contradiction). On the other hand, if two of these (remaining) elements, say a and b, were in increasing order, then together with the rightmost left-to-right minimum on the left of a and the leftmost right-to-left maximum on the right of b they would form a 1234-pattern.* Finally, if the chosen class is nonempty, then we can indeed write the remaining numbers in decreasing order

* by definition: remaining > rightmost l-to-r-min, < leftmost r-to-l-max.

without conflicting the existing constraints- otherwise the class would be empty. (In other words it is the decreasing order of the remaining elements which violates the least number of constraints). ◇

Corollary 1 *The number of nonempty classes is asymptotically $c \cdot 9^n/n^4$, where c is as in Lemma 1.*

Lemma 3 *Each nonempty class contains at least one 1324-avoiding permutation.*

Proof: First note that if a permutation contains a 1324-pattern, then we can choose such a pattern so that its first element is a left-to-right minimum and its last element is a right-to-left maximum. Indeed, we can just take any existing pattern and replace its first (last) element by its closest left (right) neighbor which is a left-to-right minimum (right-to-left maximum). Therefore, to show that a permutation avoids 1324, it is sufficient to show that it doesn't contain a 1324-pattern having a left-to-right minimum for its first element and a right-to-left maximum for its last element. (Such a pattern will be called a *good pattern*). Also note that a left-to-right minimum (right-to-left maximum) can only be the first (last) element of a 1324-pattern.

Now take any 1324-containing permutation; it has a good pattern. Interchange its second and third element. The resulting permutation is in the same class as the original because the left-to-right minima and right-to-left maxima have not been changed. Repeat this procedure as long as we can. Note that each step of the procedure decreases the number of inversions of our permutation by at least 1. Therefore, we will have to stop after at most $\binom{n}{2}$ steps. Then the resulting permutation will be in the same class as the original one, but it will have no good pattern and therefore no 1324-pattern, as we claimed. ◇

Notation (by example): in the sequel we write $a_1 * a_2 * * b_1$ for the class of permutations of length 6 which have two left-to-right minima, a_1 and a_2, which are in the first and third position, and one right-to-left maximum, b_1, which is in the last position.

Finally, we must show that "at least one" in the above lemma doesn't always mean exactly one. If $n = 7$, then the class $3 * 1 * 7 * 5$ contains two 1324-avoiding permutations, 3612745 and 3416725. This proves $S_7(1234) < S_7(1324)$. For larger n we can extend this example in an easy way, such as taking the class $n\,(n-1)\ldots 8\,3 * 1 * 7 * 5$. This shows that there are more 1324-avoiding permutations than 1234-avoiding ones and completes the proof of the theorem. ◇

Definition 2 *A class which contains more than one 1324-avoiding permutation is called a* large class.

All we need to do is to evaluate the number of large classes. If there is a positive constant ϵ so that the number of large classes is at least ϵ times the number of all classes for all n, then we get that $S_n(1234)$ is asymptotically smaller than $S_n(1324)$. The following theorem shows that this is indeed the case.

Theorem 2 *$S_n(1234)$ is asymptotically smaller than $S_n(1324)$.*

Proof: We exhibit a set of large classes. They will be built up from our above example, that is, the class $3*1*7*5$ for $n = 7$. Now let $n > 7$ and let us choose any class C of permutations of length $n-7$. (For example, let $n = 12$ and let C be $1**52$). Now we define the *composition* of the class C and the class $3*1*7*5$ to a class C' of length n as follows. Simply add 7 to all left-to-right minima and right-to-left maxima of C and leave the empty slots between them as they are. Then write the class $3*1*7*5$ after this modified version of C. This way our example results in the class $8 * * 12\,9\,3 * 1 * 7 * 5$. Clearly, this way we can define the composition of *permutations* of these classes as well: if $p_1 = (a_1, a_2, \ldots, a_{n-7}) \in C$ and $p_2 = (3, b_1, 1, b_2, 7, b_3, 5) \in 3*1*7*5$, then let their composition be $p = (a_1+7, a_2+7, \ldots, a_{n-7}+7, 3, b_1, 1, b_2, 7, b_3, 5) \in C'$. Now one sees that if we have the permutations $p_1 \in C$ and $p_2 \in 3*1*7*5$, and both p_1 and p_2 are 1324-avoiding, then their composition is 1324-avoiding, too. Therefore, every class obtained this way will be large.

(In our example, we get the permutations: 8 12 10 11 9 3 6 1 2 7 4 5 and 8 12 10 11 9 3 4 1 6 7 2 5).

This shows that we can build up a large class of permutations of length n from every class of permutations of length $n-7$. The number of these classes equals $S_{n-7}(1234)$ by Lemma 2 and this is larger than $(1/9^7)S_n(1234)$ by Lemma 1. This immediately implies that $S_n(1324) \geq (1+1/9^7)S_n(1234)$. $\diamond$

We have thus proven by the above theorem that $S_n(1324)$ is asymptotically larger than $S_n(1234)$, disproving the conjecture stating that all patterns of length k are equally likely to occur. We need more work to prove that $S_n(1324) < K^n$ for some constant K.

Theorem 3 *For all n, we have $S_n(1324) < 36^n$.*

Definition 3 *A map $f : S_n \to S_n$ of $n-$permutations is called* faithful *if $f(p)$ is 1324-avoiding if and only if p is.*

Proof: We show that each class of $n-$permutations contains less than 4^n 1324-avoiding permutations. We do this step by step, starting with the classes of the simplest structure and advancing towards the most sophisticated ones. In fact, we start with the classes where all left-to-right minima are in the leftmost positions and all right-to-left maxima are in the rightmost ones, then we first move the minima into any desired position, then we do the same with the maxima. When we move the minima, we will build a decreasing subsequence of length k of remaining entries. The elements of this subsequence will have an important role afterwards, when we move the maxima.

1. The easiest case is when we have r left-to-right minima and they are in the r leftmost positions, whereas the t right-to-left maxima are in the t rightmost positions. Then all these minima are smaller than all these maxima and thus the remaining entries have to be written in increasing order to avoid 1324-patterns. Therefore, none of these classes is large. For example, $5\ 3\ 1 * * * * *\ 11\ 9\ 7$ is such a class with $n = 11$.

   Take the last left-to-right minimum (that is, the entry 1), and move it one step to the right. Thus the class of our example becomes $5\ 3 * 1 * * * *\ 11\ 9\ 7$. How does this affect the number of 1324-avoiding permutations in the class? It is clear that if the right neighbor x of 1 in the original permutation was larger than the left neighbor y of 1, then putting x in the just created empty slot does not violate the existing constraints. This map is faithful. On the other hand, if $x < y$, then we cannot put x between y and 1, as y is a left-to-right minimum. Therefore, we move x one step to the right and take the smallest remaining entry (i.e. remaining element) which is larger than y and put it in the new empty slot between y and 1. This map is again faithful. Finally, we see that we can only do this with the *smallest* remaining entry s which is larger than y, otherwise we get a 1324-pattern. Indeed, the remaining entries are written in an increasing order, thus moving any other remaining entry z between y and 1 would result in the 1324-pattern y, z, s, n. All this sets up a bijection between the 1324-avoiding permutations of the original class and those of the new class. Therefore, the new class has one such permutation. Note that the longest decreasing sequence among the remaining elements is of length 2 only.

   In our example, the only 1324-avoiding permutation of the class $5\ 3\ 1 * * * * *\ 11\ 9\ 7$ was 5 3 1 2 4 6 8 10 11 9 7; our bijection associates it with the only 1324-avoiding permutation 5 3 4 1 2 6 8 10 11 9 7 of the class $5\ 3 * 1 * * * *\ 11\ 9\ 7$.

2. Iterating this procedure, that is, moving some left-to-right minima to the right, we can build up bijections as before. By the same argument, no class obtained this way will be large.

   Therefore, if C is any class of $n-$permutations in which all the right-to-left maxima are in the rightmost positions, then C is not large.

   If there are t left-to-right minima which are preceeded by an empty slot (not necessarily directly), then the longest decreasing subsequence among the remaining elements of that 1324-avoiding permutation can have at most $t + 1$ elements. Moreover, if we choose any left-to-right minimum a, then the remaining elements which are larger than a and are on its right are written in increasing order.

Now we start moving the right-to-left maxima to the left. This time it will be harder to keep track of the number of 1324-avoiding permutations of each class. We continue our "proof by example".

3. Let us move the entry 11 one step to the left. We get the class $5 * 3 * 1 * * 11 * 9\,7$. As before, we have to separate two cases according to the entry which *previously was* in the 8th position, and has now been replaced by 11. If this entry was not 10, then we can simply put it in the 9th position. (We just interchange it with 11). This operation is again faithful. If this entry was 10 then we cannot interchange it with 11 because 9 is a right-to-left maximum. Therefore, we put the entry 8 in the new slot between 11 and 9, we put 10 at the old place of 8, and leave the other elements unchanged. It is clear that we have to pick 8 for this purpose; we cannot do it with any smaller remaining entries. This operation is faithful, therefore the only 1324-avoiding permutation of this latest class is 5 6 3 4 1 2 10 11 8 9 7.

4. Move 11 further to the left. Next we get the class $5 * 3 * 1 * 11 * * 9\,7$. Of course, for the permutations whose 7-th entry was not 10, we can again put that entry in the 8-th position and get a faithful map. However, if that entry was 10, then we can put 10 into the old place of 2, 4 or 6, then correspondingly put 2, 4 or 6 in the new slot just to the right of 11, and leave all the rest unchanged. Each of these maps is faithful. Thus there are *three* 1324-avoiding permutations in this class, namely a = 5 10 3 4 1 2 11 6 8 9 7, b = 5 6 3 10 1 2 11 4 8 9 7 and c = 5 6 3 4 1 10 11 2 8 9 7. This has become possible because we have the *decreasing* subsequence 6, 4, 2 of remaining elements and if any of them moves to the old position of 10 , it will not go to the right of remaining elements which are *larger* than itself (except 10) and will thus create no new 1324-pattern. Likewise, the entry 10 goes ahead of some small entries, but once again by the decreasing property, it will make no harm as the remaining elements on the right of 10 will be smaller than those on the left of 10. This is why decreasing subsequences help to produce more 1324-avoiding permutations.

5. Now we move the second right-to-left maximum, the entry 9, one step to the left. We get the class $5 * 3 * 1 * 11 * 9 * 7$. Once more, if the entry in the 9-th position was not 8, then it can be simply put in the 10-th position and we get a faithful map. If it was 8, then it has to go to the left. How many different positions can it go to? We can see that it depends on how long the (longest) decreasing sequence among the remaining elements which are on its left and are smaller than 8 is. (Since as above, we can define faithful maps by interchanging 8 with any element of that sequence, provided they don't go to the left of any other remaining element they are smaller than). In other words, we get that the permutation
   - a gives rise to the 1324-avoiding permutation 5 10 3 4 1 2 11 8 9 6 7,
   - b gives rise to 5 8 3 10 1 2 11 4 9 6 7 and 5 6 3 10 1 2 11 8 9 4 7,
   - finally c gives rise to 5 8 3 4 1 10 11 2 9 6 7 and 5 6 3 8 1 10 11 2 9 4 7 and 5 6 3 4 1 10 11 8 9 2 7.

Thus this class has 6 1324-avoiding permutations.

How does this generalize to all classes? First, if there are m left-to-right minima which move away from the left end, then the longest decreasing subsequence among the remaining entries will be of length m. Therefore, when doing our procedure we reach the first large class, then that class can have at most m such permutations. When moving the next maximum, one of these m permutations will give rise to 1 new 1324-avoiding permutation, the second one will give rise to 2 new 1324-avoiding permutations, and so on, according to the position in the decreasing sequence from which we moved an element to the right of the moving right-to-left maximum. Thus after this second step we will have at most $\binom{k+1}{2}$ 1324-avoiding permutations in our class. We can go on this way: if in the previous step we moved away an entry from the i-th position of our decreasing subsequence, then in the next step we can move away an entry from i different positions, namely the first $i-1$ position and the one *to* which we have put our entry we have taken away from the i–th position last step. This shows that after $k-1$ steps the number of 1324-avoiding permutations of each class won't grow anymore. (If a remaining entry jumped backwards once, it cannot jump forward again). After all the $k-1$ steps we get a class with $\binom{2k-2}{k-1}$ such permutations. This is less than 4^{2k-2}. Moreover, $2k-2 \leq n$, therefore each class contains less than 4^n 1324-avoiding permutations. The number of classes is smaller than 9^n by Corollary 1, thus the proof is complete. $\diamond$

3 The pattern 1423

The pattern 1423 shows less similarity to the pattern 1234 than 1324 did. The important difference is that its maximal element is not the last one. This suggests that we classify the n–permutations in a slightly different way while dealing with this pattern:

Definition 4 *Two n–permutations x and y are said to be in the same* weak class *if and only if the left-to-right minima of x are the same as those of y and they are in the same positions.*

Thus here we don't require that the right-to-left maxima agree. For example, 34125 and 35124 are in the same weak class, though they are not in the same class.

The number of weak classes is easy to determine, thus we do it here even if we will need it only in the proof of Theorem 6.

Lemma 4 *The number of nonempty weak classes is $C_n = \binom{2n}{n}/(n+1)$.*

Proof: analoguous to that of lemma 2. Each weak class contains exactly one 123-avoiding permutation which is obtained by writing all the entries which are not left-to-right minima in decreasing order. The number of 123-avoiding permutations is known to be $C_n = \binom{2n}{n}/(n+1)$ (see [4]) so the proof is complete. $\diamond$

Theorem 4 *For all $n \geq 6$, $S_n(1423) < S_n(1234)$.*

Proof: We are going to show that each weak class contains at least as many 1234-avoiding permutations as 1423-avoiding ones. Then we show that there are some classes which contain less of this latter, completing the proof of the theorem.

Take a weak class W with left-to-right minima $a_1, a_2, \ldots, a_m = 1$ being in the i_1−st, i_2−nd ... i_m−th positions. (Necessarily, $i_1 = 1$). Choose any permutation p of W. For all j, let G_j be the set of entries which are larger than a_j and are on the right of a_j. Now p is 1234-avoiding if and only if all G_j are 123-avoiding. Likewise, p is 1423-avoiding if all G_j are 312-avoiding. We know by [4] that for each G_j there are as many ways to be 123-avoiding as 312-avoiding. The important difference between the two structures is in the relation among the G_j. In the sequel we show that if G_j contains 123 for some j, then this raises chances that this will happen for some other j more than it does when we replace 123 by 312. We will use a special case of the following trivial fact:

Fact 1: If A, B, C, D are finite sets so that $|A| = |B|$ and $|C| = |D|$ and $|A \bigcap C| \leq |B \bigcap D|$, then $|A \bigcup C| \geq |B \bigcup D|$. $\square$

So suppose G_j contains a 123-pattern. Then we consider three different cases.

1. If all the three elements of that pattern are on the right of some a_t, $t > j$, then of course G_t is 123-containing. We can say exactly the same for 312-patterns.

2. If there are just two elements x and y of a 123-pattern of G_j which are on the right of some a_t, $t > j$, then what we need for G_t to be 123-containing is just one element r between a_t and x which is smaller than x and larger than a_t. Thus in both cases the fact that G_j is 123-containing increases the probability of some other G_s being so. In fact, once we have such an element r, we can permute the rest of $G_t - \{x, y\}$ any way we want and G_t will remain 123-containing.

   On the other hand this will not hold for 312-patterns. For suppose that G_j is 312-containing for some j and the last two elements u and v of some of its 312-patterns are on the right of some a_t, $t > j$, but there is no 312-pattern in G_j which is entirely on the right of a_t. The point is that $u < v$, therefore we *cannot use both of them* to build up the same 312-pattern in G_t. Indeed, the only way we could possibly use both of them would be to make them play the role of 1 and

2 of that pattern, but then we would need an element $w \in G_t$ for the role of 3 and then the 312-pattern $w\ u\ v$ of G_j entirely on the right of a_t, contradicting our hypothesis. Thus, when we want to build up a permutation in which both G_j and G_t are 312-containing, we have less freedom as if we wanted them to contain 123. In other words, it is not enough to pick u and v and take any permutation of the rest of G_j as it was above.

3. Finally, if there is just one element u of any 123-pattern of G_j which is on the right of a_t, $t > j$, then what we need for G_t to be 123 containing are two elements f and h between a_t and u so that $a_t < f < h < u$. Once we have these elements f and h, we are free to permute the rest of G_t.

   Likewise, if there are just one element v of a 312-pattern of G_j which are on the left of some a_t, $t > j$, then what we need for G_t to be 312-containing are two elements d and e satisfying $a_t < d < e < v$ and being both on the right of v. One sees easily that this is equally likely to be true as the existence of f and h in the above paragraph — indeed, if $|G_t| = n_t$, then the permutations in which u is in the i-th position of G_t can be associated with the permutations in which v is in the $(n_t + 1 - i)$–th position of G_t.

Thus in cases 1 and 3 the two patterns behave identically, however, in case two we see that it is easier to have both G_j and G_t contain 123 than 312. The above argument immediately shows that any week class W *which has only two left-to-right minima* contains at most as many 1234-containing permutations than 1423-containing permutations by Fact 1. If W has $m > 2$ left-to-right minima, then we can prove the same statement by induction on m, as follows: The number of permutations in W in which G_1 is 123-containing equals the number of those in which G_1 is 312-avoiding. Restrict ourselves now to the part R of the permutations in W which is on the right of a_2 (inclusive). By induction, there are more or equal 312-containing ones among them than 123-containing ones. Moreover, we can see in a similar way to the above that once again, it is easier to have both of G_1 and R contain 123 than 312, which completes the inductive proof by Fact 1.

This shows that for all n, $S_n(1423) \leq S_n(1234)$. All what is left to do is to show that the inequality is strict if $n \geq 6$. That is, we have exhibit weak classes in which there are strictly more 1234-avoiding permutation than 1423-avoiding one. For $n = 6$ there is one such class: $3 * 1 * **$. It has 5 1234-containing permutations: 341256, 341526, 341562, 351246 and 361245. On the other hand, it has 6 1423-containing permutations: 341625, 351624, 361452, 361425, 361245 and 361524. That is, this weak class contains 12 1423-avoiding permutations and 13 1234-avoiding permutations. If $n > 6$, then it is obvious that for example the weak class $n\,(n-1)\,(n-2)\ldots 8\ 7\ 3 * 1 * **$ will be of the desired property. Thus we have proved that for all $n \geq 6$, $S_n(1423) < S_n(1234)$. $\diamond$

We point out that this result combined with Lemma 1 immediately implies:

Theorem 5 *For all n, $S_n(1423) < 9^n$.* □

Thus we have shown that for any pattern p of length 4, we have $S_n(p) < 36^n$.

As before, the inequality holds even in the asymptotical sense. The proof is very similar to that of theorem 2 and is therefore omitted.

4 Further directions

If we know that the conjecture on the exponential upper bound holds for a pattern q, then are there any longer patterns for which it must hold as well? The following lemma gives a partial answer to this question.

Lemma 5 *Let q be a pattern with first entry 1 so that $S_n(q) < K^n$ for some absolute constant K. Then there is some absolute constant K_1 so that $S_n(1q) < K_1^n$, where $1q$ is the pattern obtained by incrementing all entries of q by 1 and writing 1 to the first position.*

Proof: Take any weak class W. One sees that if an n-permutation p avoids $1q$, then the string of its entries which are not left-to-right minima *must avoid* q. Indeed, if there were any copy of q on these entries, then one could complete it to a copy of $1q$ by joining the smallest left-to-right minimum s on the left of this copy of q and this copy of q. (By the definition of the left-to-right minima, s is smaller than the first entry of our copy of q, and therefore it is smaller than all entries of that copy because q starts with 1). Thus the number of $1q$-avoiding permutations in W is less than K^n as W has less than n entries which are not left-to-right minima. Thus the number of weak classes is smaller than 4^n by lemmas 1 and 4, therefore we get $S_n(1q) < (4K)^n = K_1^n$ as claimed. ◇

Remark: the estimate we have just used is not the best possible. It is easy to sharpen our method to get much better constants.

Dual versions of this lemma are also true, that is, we can take the reverse or complement of all permutations (that is, in which $p'_i = n + 1 - p_i$), and the lemma remains true. However, due to a lemma of Julian West, we can say more. Iterate the previous lemma to prove that if q is as in the lemma, then for all r there is a constant K_r so that $S_n(123...rq) < K_r^n$. Therefore, the following lemma applies

Lemma 6 *([1] [8] [9]). For any r we have $S_n(123...rq) = S_n(r..321q)$*

See [8] for $r = 2$, see [1] for $r = 3$ and see [9] for $r > 3$.

Corollary 2 *Let q be a pattern with first entry 1 so that $S_n(q) < K^n$ for some absolute constant K. Then for all r there is a constant K_r so that $S_n(123...rq) = S_n(r..321q) < K_r^n$.*

Theorems 1 and 4 can be generalized for longer patterns as follows:

Theorem 6 *Let q be a pattern of length k which starts with 1, ends with k and has only one inversion. Then $S_n(123..k) < S_n(q)$ if n is sufficiently large.*

If q has only one inversion, but it doesn't start with 1 or doesn't end with k, then $S_n(q) = S_n(12\ldots k)$ as it is shown in [8].

Theorem 7 *If n is large enough, then we have $S_n(1\,k\,2\,3\ldots(k-1)) < S_n(1\,2\,3\,\ldots k)$.*

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