Lecture Hall Partitions

MIREILLE BOUSQUET-MÉLOU *
LaBRI, Université Bordeaux 1
351 cours de la Libération
33405 Talence Cedex, FRANCE
bousquet@labri.u-bordeaux.fr

KIMMO ERIKSSON †
Dept. of Mathematics
Stockholm University
S-106 91 Stockholm, SWEDEN
kimmo@nada.kth.se

Abstract

We prove a finite version of Euler's well-known theorem that says that any integer has as many partitions into distinct parts as partitions into odd parts. Our version says that any integer has as many "lecture hall partitions of length n" as partitions into small odd parts: $1, 3, 5, \ldots, 2n-1$. We give two proofs: one via Bott's formula for the Poincaré series of the affine Coxeter group $\widetilde{C}_n$, and one direct proof. This generalizes to a whole family of identities on partitions with conditions on the quotient of consecutive parts.

1 Introduction

In 1748, Euler [7] published a pioneering result about integer partitions, saying that any integer has as many partitions into distinct parts as partitions into odd parts. In other words, the generating function for partitions $\mu = (\mu_1, \ldots, \mu_m)$, with arbitrary m, such that $\mu_{i+1} > \mu_i$ for $1 \le i < m$ is

$$\sum_{\mu} q^{|\mu|} = \prod_{i\geq 0} \frac{1}{1-q^{2i+1}} \qquad (1)$$

where $|\mu| = \mu_1 + \cdots + \mu_m$ is the *weight* of μ. The standard way to perceive the condition $\mu_{i+1} > \mu_i$ is that the *difference* between two consecutive parts must be at least one. The famous Rogers-Ramanujan identities and a score of other formulas (see Andrews's book [1]) are results of the same type, counting partitions with conditions on the difference of consecutive parts. However, the condition in Euler's theorem can equivalently be perceived as requiring that the *quotient* of consecutive parts be greater than one. We will prove that Euler's result is only one in a family of identities of this type.

Proposition 1.1 *Let k be an integer greater than or equal to 2. The generating function for integer partitions $\mu = (\mu_1, \ldots, \mu_m)$ with arbitrary m, such that $\frac{\mu_{i+1}}{\mu_i} > \frac{k+\sqrt{k^2-4}}{2}$ for $1 \le i < m$, is*

$$\sum_{\mu : \frac{\mu_{i+1}}{\mu_i} > \frac{k+\sqrt{k^2-4}}{2}} q^{|\mu|} = \prod_{i\geq 1} \frac{1}{1-q^{e_i}}$$

where $e_1 = 1$, $e_2 = k+1$ and $e_{i+1} = ke_i - e_{i-1}$ for $i \geq 2$.

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This result is a consequence of the enumeration of *lecture hall partitions.*

Suppose a lecture hall is to be built with n rows of seats of heights $\lambda_1, \lambda_2, \ldots, \lambda_n$ placed at distances $a_1, a_2, \ldots, a_n$ from the speaker. If the people in the audience are of negligible height, then the condition on the architecture for every seat to give a clear view of the speaker is that the slopes are increasing, that is, $\frac{\lambda_i}{a_i} \leq \frac{\lambda_{i+1}}{a_{i+1}}$ for $1 \leq i < n$. This justifies the following definition.

Definition 1.2 *Given a non-decreasing sequence of positive integers $a = (a_1, a_2, \ldots)$, an a-*lecture hall partition *of length n is an n-tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying*

$$0 \leq \frac{\lambda_1}{a_1} \leq \frac{\lambda_2}{a_2} \leq \cdots \leq \frac{\lambda_n}{a_n}.$$

Assuming a given sequence a, we let $\mathcal{L}_n$ denote the set of a-lecture hall partitions of length n.

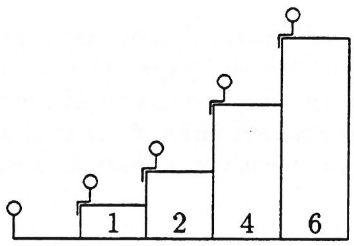

Figure 1: The design of a lecture hall of four rows at distances $1, 2, 3, 4$ from the speaker, corresponding to the lecture hall partition $(1, 2, 4, 6)$.

It turns out that for the sequence $a = (1, 2, 3, \ldots)$, counting a-lecture hall partitions gives a "finite version" of Euler's theorem. More precisely, for this sequence we will prove that a-lecture hall partitions of length n are equinumerous with partitions into small odd parts: $1, 3, \ldots, 2n-1$.

Theorem 1.3 *For $a = (1, 2, 3, \ldots)$ the generating function for a-lecture hall partitions of length n is*

$$\sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} = \prod_{i=0}^{n-1} \frac{1}{1 - q^{2i+1}}.$$

In the limit (see Section 6 for the details), this theorem yields Euler's result. Our first proof of this theorem, in Section 2, will be via Bott's formula for the Poincaré series of the affine Coxeter group $\tilde{C}_n$.

We will then proceed with an alternative, more direct proof, which will actually give much more general results: we will be able to enumerate a-lecture hall partitions of length n for an infinite family of sequences a, taking into account the *even* and *odd* weights of the partitions, defined by

$$|\lambda|_e = \lambda_n + \lambda_{n-2} + \lambda_{n-4} + \ldots \quad \text{and} \quad |\lambda|_o = \lambda_{n-1} + \lambda_{n-3} + \lambda_{n-5} + \ldots,$$

where λ_i are zero for $i \leq 0$. Of course, $|\lambda| = |\lambda|_e + |\lambda|_o$.

In particular, we will prove the following refinement of Theorem 1.3:

$$\sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=0}^{n-1} \frac{1}{1 - x^{i+1} y^i}. \tag{2}$$

Setting $x = tq$ and $y = t^{-1}q$ we obtain the following alternative formulation

$$\sum_{\lambda \in \mathcal{L}_n} t^{|\lambda|_e - |\lambda|_o} q^{|\lambda|} = \prod_{i=0}^{n-1} \frac{1}{1 - tq^{2i+1}}.$$

If we introduce the new statistic $s(\lambda) = |\lambda|_e - |\lambda|_o$ and take the limit when n tends to infinity, we obtain a refined version of Euler's identity:

$$\sum_{\mu \in \mathcal{D}} t^{s(\lambda)} q^{|\mu|} = \sum_{\mu \in \mathcal{O}} t^{\ell(\mu)} q^{|\mu|},$$

where $\mathcal{D}$ and $\mathcal{O}$ denote the sets of partitions with distinct parts and odd parts respectively, and $\ell(\mu)$ stands for the *length* (the number of parts) of the partition μ.

More generally, we will prove the following theorem.

Theorem 1.4 *Let a be the sequence defined by $a_0 = 0$, $a_1 = 1$ and for $n \geq 1$,*

$$\begin{cases} a_{2n} & = \ell a_{2n-1} - a_{2n-2}, \\ a_{2n+1} & = k a_{2n} - a_{2n-1} \end{cases} \tag{3}$$

where ℓ and k are two integers ≥ 2. Then the generating functions for a-lecture hall partitions of even and odd length are respectively given by

$$\sum_{\lambda \in \mathcal{L}_{2n}} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=1}^{2n} \frac{1}{1 - x^{a_i} y^{a_i^*}} \quad \textit{and} \quad \sum_{\lambda \in \mathcal{L}_{2n-1}} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=1}^{2n-1} \frac{1}{1 - x^{a_{i+1}^*} y^{a_{i-1}}} \tag{4}$$

where the sequence a^* is defined by the initial conditions $a_1^* = 0$, $a_2^* = 1$ and the recurrence relations (3).

Our proof works as follows. In Section 3, we show that the enumeration of lecture hall partitions boils down, via a certain reduction procedure, to the enumeration of *reduced* lecture hall partitions. In Section 4 we then describe a useful involution on the set of these reduced lecture hall partitions. The main result is proved in Section 5, which is devoted to the sequences satisfying the relations (3) above. Finally, in Section 6 we derive some limit results containing Proposition 1.1.

All proofs are omitted in this version. Details can be found in [3] and [4].

2 Bott's formula for $\widetilde{C}_n$ and lecture hall partitions

We will now presume some familiarity with Coxeter group theory (see Humphreys's book [8]), and in particular with the finite group C_n: $\circ - \cdots - \circ \overset{4}{-} \circ$ and the affine Coxeter group $\widetilde{C}_n$: $\circ \overset{4}{-} \circ - \cdots - \circ \overset{4}{-} \circ$. By the *Poincaré series* of these groups one means the length generating functions,

$$C_n(q) = \sum_{\pi \in C_n} q^{\ell(\pi)} \quad \text{and} \quad \widetilde{C}_n(q) = \sum_{\pi \in \widetilde{C}_n} q^{\ell(\pi)}.$$

Bott [2] gave, as an application of Morse theory to the topology of Lie groups, a general formula for the Poincaré series of the affine groups in terms of the Poincaré series of the finite groups and their "exponents". For $\widetilde{C}_n$ this takes the following form.

Theorem 2.1 (Bott's Formula, 1956) *The Poincaré series of $\widetilde{C}_n$ is*

$$\widetilde{C}_n(q) = \frac{C_n(q)}{(1-q)(1-q^3)\dots(1-q^{2n-1})}.$$

Observe the similarity between the denominator and the generating function in Theorem 1.3. We will here give a combinatorial argument for the equivalence between these two theorems.

The affine Coxeter group $\tilde{C}_n$ can be represented by infinite permutations on $\mathbb{Z}$ as follows (see H. Eriksson's thesis [5], or H. Eriksson and K. Eriksson [6]). Start with the real line and erect mirrors at positions $x = 0$ and $x = n+1$. Let s_0 be the infinite set of transpositions generated by mirror images of the transposition $(-1, 1)$. Similarly, let s_n be the set of all mirror images of the transposition $(n, n+2)$, and for i between 1 and $n-1$, let s_i be the set of all mirror images of $(i, i+1)$. Now $\tilde{C}_n$ is the group of infinite permutations generated by $S = \{s_0, s_1, \ldots, s_n\}$. Such an infinite permutation π will be thought of as a rearrangement of $\mathbb{Z}$ written from left to right, $\ldots\pi(-2)\pi(-1)\pi(0)\pi(1)\pi(2)\ldots$, satisfying the two mirror conditions $\pi(-m) = -\pi(m)$ and $\pi(n+1-m) = -\pi(n+1+m)$ for all integers m. Note that the mirror conditions together imply the translative property $\pi(i+2n+2) = \pi(i) + 2n + 2$ for all i.

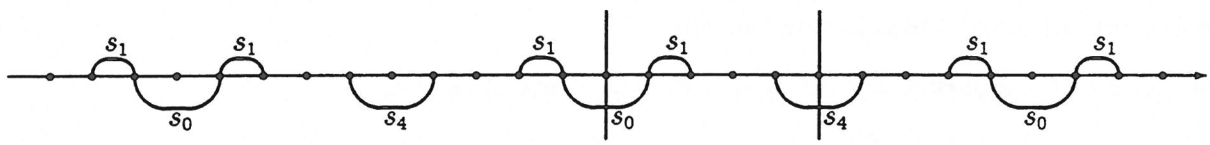

Figure 2: The actions of $s_0, s_1, s_4 \in \tilde{C}_4$ as transpositions on $\mathbb{Z}$.

We will henceforth regard as mirrors also all mirror images of mirrors, that is, the positions $k(n+1)$ for integers k. A *window* is the set of positions between two adjacent mirrors. We will frequently refer to the *class*

$$\langle i \rangle = \{i + k(2n+2), -i + k(2n+2) : k \in \mathbb{Z}\}$$

of all mirror images of a non-mirror number i. An *inversion* in a permutation π is an unordered pair $\{i, j\}$ of non-mirror numbers such that $i < j$ but i is to the right of j in π (that is, $\pi^{-1}(i) > \pi^{-1}(j)$). A *class inversion* is the set of all mirror images of an inversion $\{i, j\}$,

$$\{\{i + k(2n+2), j + k(2n+2)\}, \{-i + k(2n+2), -j + k(2n+2)\} : k \in \mathbb{Z}\},$$

and all these mirror images are also inversions. It is proved in [6] that the length of an element of $\tilde{C}_n$ is the number of its class inversions. Every class inversion can in a unique way be seen as the set of mirror images of an inversion $\{i, j'\}$ where $1 \le j \le i \le n$ and j' is a member of $\langle j \rangle$, and we refer to it as an $(i, \langle j \rangle)$-class inversion. Thus, the length of an infinite permutation $\pi \in \tilde{C}_n$ is

$$\ell(\pi) = \sum_{i=1}^{n} \sum_{j=1}^{i} I_{i,j}(\pi),$$

where $I_{i,j}(\pi)$ is the number of $(i, \langle j \rangle)$-class inversions.

The finite group C_n is embedded in $\tilde{C}_n$ as the subgroup of permutations such that there is no inversion between any member of $\{-n, \ldots, 0, \ldots, n\}$ and any member of the complement of this set. The parabolic quotient $\tilde{C}_n/C_n$ can be viewed as the subset of permutations in $\tilde{C}_n$ such that the numbers $-n, \ldots, 0, \ldots, n$ appear in that order in π from left to right. Then every permutation π in $\tilde{C}_n$ has a unique factorization as $\pi_1 \circ \pi_2$ where $\pi_1 \in \tilde{C}_n/C_n$ and $\pi_2 \in C_n$, and its length is simply $\ell(\pi) = \ell(\pi_1) + \ell(\pi_2)$; cf. Humphreys [8, p. 123]. Thus, Bott's Formula is equivalent to

$$\sum_{\pi \in \tilde{C}_n/C_n} q^{\ell(\pi)} = \frac{1}{(1-q)(1-q^3)\ldots(1-q^{2n-1})}. \tag{5}$$

In order to show that Bott's Formula is equivalent to Theorem 1.3, it is therefore sufficient to find a bijection λ from $\tilde{C}_n/C_n$ to the set $\mathcal{L}_n$ of lecture hall partitions of length n, such that $\ell(\pi) = |\lambda(\pi)|$. Our candidate will be the n-tuple $\lambda(\pi) = (\lambda_1, \ldots, \lambda_n)$ defined by

$$\lambda_i = \sum_{j=1}^{i} I_{i,j}(\pi).$$

Clearly we have $\ell(\pi) = \lambda_1 + \ldots + \lambda_n = |\lambda(\pi)|$.

Proposition 2.2 *The correspondence λ is a bijection from $\tilde{C}_n/C_n$ to the set $\mathcal{L}_n$ of lecture hall partitions of length n.*

As observed above, this establishes equivalence between Bott's Formula and Theorem 1.3.

3 Reduction of lecture hall partitions

In this section and the following one, fix a non-decreasing sequence $a = (a_i)_{i\geq 1}$ of positive integers, and fix a positive integer n. We let $\mathcal{L}_n$ be the set of a-lecture hall partitions of length n, which will be called, for short, lecture hall partitions. We will describe a way of splitting any $\lambda \in \mathcal{L}_n$ as $\lambda = \mu + \sum_{i=1}^{n} k_i \lambda^{(i)}$ (with partitions considered as members of an n-dimensional vector space), where μ belongs to a set $\mathcal{R}_n$ of *reduced* lecture hall partitions, the $\lambda^{(i)}$ constitute a natural basis of $\mathcal{L}_n$, and the k_i are nonnegative integers. This implies that we can write the generating function for lecture hall partitions as

$$\sum_{\mu\in\mathcal{R}_n} x^{|\mu|_e} y^{|\mu|_o} \prod_{i=1}^{n} \frac{1}{1 - x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o}}.$$

We will compute in section 5 the two factors of this expression for some particular sequences a.

Given an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, we define its *D-sequence* to be the n-tuple $D(\lambda) = (d_1, \ldots, d_n)$ given by

$$d_1 = \lambda_1 \quad \text{and} \quad d_i = \lambda_i - \left\lceil \frac{a_i \lambda_{i-1}}{a_{i-1}} \right\rceil \quad \text{for } 2 \leq i \leq n.$$

The sequence $D(\lambda)$ completely defines λ. Moreover, $\lambda \in \mathbb{N}^n$ is a lecture hall partition if and only if $d_i \geq 0$ for all i.

For $1 \leq i \leq n$, let

$$\lambda^{(i)} = (0, \ldots, 0, a_i, a_{i+1}, \ldots, a_n) \in \mathbb{N}^n.$$

Then $D(\lambda^{(i)}) = (0, \ldots, 0, a_i, 0, \ldots, 0)$ and thus $\lambda^{(i)}$ is a lecture hall partition of length n, which will be called *standard*. More generally, if λ belongs to $\mathcal{L}_n$, then the sum $\lambda + \lambda^{(i)}$ also belongs to $\mathcal{L}_n$, and its D-sequence is $(d_1, \ldots, d_{i-1}, a_i + d_i, d_{i+1}, \ldots, d_n)$ where $(d_1, \ldots, d_n) = D(\lambda)$. The following lemma describes for which λ also the difference $\lambda - \lambda^{(i)}$ lies in $\mathcal{L}_n$.

Lemma 3.1 *Let $\lambda \in \mathcal{L}_n$, and let $(d_1, \ldots, d_n)$ be its D-sequence. Let $1 \leq i \leq n$. Then $\lambda - \lambda^{(i)}$ belongs to $\mathcal{L}_n$ if and only if $d_i \geq a_i$.*

Definition 3.2 *A lecture hall partition of length n is said to be* reduced *if its D-sequence $(d_1, \ldots, d_n)$ satisfies $0 \leq d_i < a_i$ for $1 \leq i \leq n$.*

Since the D-sequence completely defines the partition, there are exactly $a_1 a_2 \cdots a_n$ reduced lecture hall partitions of length n. The set of reduced partitions of $\mathcal{L}_n$ will be denoted by $\mathcal{R}_n$. Iterating Lemma 3.1 leads to the following reduction result.

Proposition 3.3 *Let λ be a lecture hall partition of length n with D-sequence $(d_1, \ldots, d_n)$. Then the map $\lambda \mapsto (\mu, k_1, \ldots, k_n)$ given by $k_i = \lfloor d_i/a_i \rfloor$ and*

$$\lambda = \mu + \sum_{i=1}^{n} k_i \lambda^{(i)}$$

is a bijection from $\mathcal{L}_n$ to $\mathcal{R}_n \times I\!N^n$.

Consequently, the generating function for lecture hall partitions of length n is

$$\sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \frac{P_n(x,y)}{\prod_{i=1}^{n} \left(1 - x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o}\right)} \tag{6}$$

where the polynomial $P_n(x,y)$ enumerates reduced lecture hall partitions:

$$P_n(x,y) = \sum_{\mu \in \mathcal{R}_n} x^{|\mu|_e} y^{|\mu|_o}. \tag{7}$$

Computing this polynomial is the central problem one has to solve in order to enumerate lecture hall partitions. In section 5, we compute it by induction on n for some particular sequences a, using the involution on reduced lecture hall partitions described in the following section.

4 An involution on reduced lecture hall partitions

For $\mu \in \mathcal{R}_n$, let $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ be the unique n-tuple such that

$$\begin{cases} \mu_{n-2k}^* & = \mu_{n-2k} & \text{for } n-2k \geq 1 \\ d_{n-2k-1}^* & = \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor - \mu_{n-2k-1} & \text{for } n-2k-1 \geq 1, \end{cases} \tag{8}$$

where $(d_1^*, \ldots, d_n^*)$ is the D-sequence associated with μ^*. The second equation is of course a short way to define μ_{n-2k-1}^*. Indeed,

$$\mu_{n-2k-1}^* - \left\lceil \frac{a_{n-2k-1}}{a_{n-2k-2}} \mu_{n-2k-2}^* \right\rceil = \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor - \mu_{n-2k-1}. \tag{9}$$

In this expression, and always in this section, we have set $\mu_i^* = \mu_i = 0$ if $i \leq 0$.

Proposition 4.1 *The correspondence $\mu \mapsto \mu^*$ defines an involution on the set $\mathcal{R}_n$.*

We can extend the involution $\mu \mapsto \mu^*$ into a bijection Υ from $\mathcal{R}_n \times [0, a_{n+1}-1]$ onto $\mathcal{R}_{n+1}$, by defining

$$\Upsilon(\mu, i) = \left(\mu_1^*, \ldots, \mu_n^*, \left\lceil \frac{a_{n+1}}{a_n} \mu_n^* \right\rceil + i\right).$$

It is clear that $\Upsilon(\mu, i)$ is a reduced partition. Moreover, as $\mu \mapsto \mu^*$ defines a bijection on $\mathcal{R}_n$, Υ is a bijection from $\mathcal{R}_n \times [0, a_{n+1}-1]$ onto $\mathcal{R}_{n+1}$.

For convenience, let us denote the partition $\Upsilon(\mu, i)$ by η. We want to compute $|\eta|_e$ and $|\eta|_o$. As $\mu_{n-2k}^* = \mu_{n-2k}$, it is clear that $|\eta|_o = |\mu|_e$. Moreover,

$$\begin{aligned} |\eta|_e &= i + \left\lceil \frac{a_{n+1}}{a_n} \mu_n^* \right\rceil + \sum_{n-2k-1 \in [1,n-1]} \mu_{n-2k-1}^* \\ &= i + \left\lceil \frac{a_{n+1}}{a_n} \mu_n \right\rceil + \sum_{n-2k-1 \in [1,n-1]} \left(\left\lceil \frac{a_{n-2k-1}}{a_{n-2k-2}} \mu_{n-2k-2} \right\rceil + \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor - \mu_{n-2k-1} \right) \\ &= i + \sum_{n-2k \in [1,n]} \left\lceil \frac{a_{n-2k+1}}{a_{n-2k}} \mu_{n-2k} \right\rceil + \sum_{n-2k \in [2,n]} \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor - |\mu|_o. \end{aligned} \tag{10}$$

We cannot go further in this calculation without any additional assumptions on the sequence a. We study in the next section some particular sequences for which the weight $|\eta|_e$ can be very simply expressed in terms of $|\mu|_e$ and $|\mu|_o$.

5 The (k,ℓ)-sequences

By a (k,ℓ)-sequence we mean a sequence a defined by the initial values a_1 and a_2 and the following recurrence relations:

$$\begin{cases} a_{2n} &= \ell a_{2n-1} - a_{2n-2} \quad \text{for } n \geq 2 \\ a_{2n+1} &= k a_{2n} - a_{2n-1} \quad \text{for } n \geq 1 \end{cases} \tag{11}$$

where $k, \ell \geq 2$ are two integers.

For such a sequence, and provided that an additional condition, stated later, is satisfied by the initial value a_1, we will be able to enumerate reduced lecture hall partitions. We will then compute the weights of the standard partitions $\lambda^{(i)}$, which will finally provide the generating function for lecture hall partitions thanks to identity (6).

5.1 Reduced lecture hall partitions

Let us return to the last equation of (10), and combine it with the following simple lemma for (k,ℓ)-sequences.

Lemma 5.1 *Let $i \geq 2$ and $m \geq 0$. Then*

$$\left\lceil \frac{a_{i+1}}{a_i} m \right\rceil + \left\lfloor \frac{a_{i-1}}{a_i} m \right\rfloor = \begin{cases} km & \text{if } i \text{ is even,} \\ \ell m & \text{otherwise.} \end{cases}$$

We obtain

$$|\eta|_e = \begin{cases} i + k|\mu|_e - |\mu|_o & \text{if } n \text{ is even,} \\ i + \left\lceil \frac{a_2}{a_1}\mu_1 \right\rceil + \ell \sum_{n-2k \in [3,n]} \mu_{n-2k} - |\mu|_o & \text{otherwise,} \end{cases}$$

so that finally

$$|\eta|_e = \begin{cases} i + k|\mu|_e - |\mu|_o & \text{if } n \text{ is even,} \\ i + \ell|\mu|_e - |\mu|_o & \text{if } n \text{ is odd,} \end{cases} \tag{12}$$

as soon as one of the following three conditions holds (remember that $\mu_1 \in [0, a_1 - 1]$):

$$a_1 = 1 \quad \text{or} \quad a_2 = a_1\ell - 1 \quad \text{or} \quad a_2 = a_1\ell. \tag{13}$$

Thus, we have described a bijection $\Upsilon : \mathcal{R}_n \times [0, a_{n+1} - 1] \to \mathcal{R}_{n+1}$ such that, if $\eta = \Upsilon(\mu, i)$, then $|\eta|_o = |\mu|_e$, and $|\eta|_e$ is given by (12). This implies that the polynomials $P_n(x,y)$, defined by (7), can be computed inductively via the following recurrence relations:

$$P_{2n+1}(x,y) = \frac{1 - x^{a_{2n+1}}}{1-x} P_{2n}(x^k y, x^{-1}) \quad \text{and} \quad P_{2n}(x,y) = \frac{1 - x^{a_{2n}}}{1-x} P_{2n-1}(x^\ell y, x^{-1}),$$

with the initial condition $P_0 = 1$. We thus obtain the following result on reduced lecture hall partitions for (k,ℓ)-sequences.

Proposition 5.2 *Given a sequence a satisfying* (11) *and* (13)*, the generating functions for reduced a-lecture hall partitions of even and odd length are given by:*

$$P_{2n}(x,y) = \prod_{i=1}^{2n} \frac{1 - \left(x^{b_i} y^{b_i^*}\right)^{a_{2n-i+1}}}{1 - x^{b_i} y^{b_i^*}}$$

and

$$P_{2n-1}(x,y) = \prod_{i=1}^{2n-1} \frac{1 - \left(x^{b_{i+1}^*} y^{b_{i-1}}\right)^{a_{2n-i}}}{1 - x^{b_{i+1}^*} y^{b_{i-1}}}$$

where the sequences b and b^* are defined by $b_0 = 0$, $b_1 = 1$, $b_1^* = 0$, $b_2^* = 1$, and the recurrence relations (11).

5.2 Standard lecture hall partitions

Recall that the standard lecture hall partitions of length n are the $\lambda^{(i)} = (0, \ldots, 0, a_i, a_{i+1}, \ldots, a_n)$, for $1 \le i \le n$. We will compute their even and odd weights thanks to the following lemma.

Lemma 5.3 *For $1 \le i \le n$, let us define the sums $E(n,i)$ and $O(n,i)$ by*

$$E(n,i) = a_{2n} + a_{2n-2} + \ldots + a_{2n-2i+2}$$

and

$$O(n,i) = a_{2n-1} + a_{2n-3} + \ldots + a_{2n-2i+1}.$$

Then we have:

$$E(n,i) = b_i a_{2n-i+1} \quad \textit{and} \quad O(n,i) = b^*_{i+1} a_{2n-i}.$$

Thus the weights of standard lecture hall partitions *of length* $2n$ are given by

$$\begin{cases} |\lambda^{(2n-2i)}|_e = & a_{2n} + \cdots + a_{2n-2i} & = & E(n,i+1) & = b_{i+1} a_{2n-i} \\ |\lambda^{(2n-2i)}|_o = & a_{2n-1} + \cdots + a_{2n-2i+1} & = & O(n,i) & = b^*_{i+1} a_{2n-i} \end{cases} \quad \text{for } 0 \le i \le n-1,$$

and

$$\begin{cases} |\lambda^{(2n-2i+1)}|_e = & a_{2n} + \cdots + a_{2n-2i+2} & = & E(n,i) & = b_i a_{2n-i+1} \\ |\lambda^{(2n-2i+1)}|_o = & a_{2n-1} + \cdots + a_{2n-2i+1} & = & O(n,i) & = b^*_{i+1} a_{2n-i} \end{cases} \quad \text{for } 1 \le i \le n,$$

and the weights of standard lecture hall partitions *of length* $2n-1$ are given by

$$\begin{cases} |\lambda^{(2n-2i+1)}|_e = & a_{2n-1} + \cdots + a_{2n-2i+1} & = & O(n,i) & = b^*_{i+1} a_{2n-i} \\ |\lambda^{(2n-2i+1)}|_o = & a_{2n-2} + \cdots + a_{2n-2i+2} & = & E(n-1,i-1) & = b_{i-1} a_{2n-i} \end{cases} \quad \text{for } 1 \le i \le n,$$

and

$$\begin{cases} |\lambda^{(2n-2i)}|_e = & a_{2n-1} + \cdots + a_{2n-2i+1} & = & O(n,i) & = b^*_{i+1} a_{2n-i} \\ |\lambda^{(2n-2i)}|_o = & a_{2n-2} + \cdots + a_{2n-2i} & = & E(n-1,i) & = b_i a_{2n-i-1} \end{cases} \quad \text{for } 1 \le i \le n-1.$$

Combining these results with Proposition 5.2 and Eq. (6), we obtain the generating function for lecture hall partitions associated with these sequences.

Proposition 5.4 *The generating functions for a-lecture hall partitions of even and odd length for a sequence a satisfying* (11) *and* (13) *are given by:*

$$\sum_{\lambda \in \mathcal{L}_{2n}} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=1}^{2n} \frac{1}{1 - x^{b_i} y^{b^*_i}} \prod_{i=n+1}^{2n} \frac{1 - \left(x^{b_i} y^{b^*_i}\right)^{a_{2n-i+1}}}{1 - x^{b_{2n-i+1} a_i} y^{b^*_{2n-i+2} a_{i-1}}} \tag{14}$$

and

$$\sum_{\lambda \in \mathcal{L}_{2n-1}} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=1}^{2n-1} \frac{1}{1 - x^{b^*_{i+1}} y^{b_{i-1}}} \prod_{i=n+1}^{2n-1} \frac{1 - \left(x^{b^*_{i+1}} y^{b_{i-1}}\right)^{a_{2n-i}}}{1 - x^{b^*_{2n-i+1} a_i} y^{b_{2n-i} a_{i-1}}} \tag{15}$$

where the sequences b and b^* are defined by $b_0 = 0$, $b_1 = 1$, $b^*_1 = 0$, $b^*_2 = 1$, and the same recurrence relations as for a.

Examples

1. If $k=\ell$, then $b^*_{i+1}=b_i$ for all $i\geq 0$. The proposition above implies that the generating function for lecture hall partitions of length n is

$$\prod_{i=0}^{n-1}\frac{1}{1-x^{b_{i+1}}y^{b_i}}\prod_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}\frac{1-(x^{b_{i+1}}y^{b_i})^{a_{n-i}}}{1-(x^{a_{i+1}}y^{a_i})^{b_{n-i}}}.$$

For example, if we take $k=\ell=2$, $a_1=1$ and $a_2=3$, then property (13) is satisfied and we have $a_i=2i-1$, $b_i=i$ and $b^*_i=i-1$ for all i. We obtain that the generating function for partitions $(\lambda_1,\ldots,\lambda_n)$ such that

$$0\leq\frac{\lambda_1}{1}\leq\frac{\lambda_2}{3}\leq\ldots\leq\frac{\lambda_n}{2n-1}$$

is

$$\prod_{i=0}^{n-1}\frac{1}{1-x^{i+1}y^i}\prod_{i=\lfloor\frac{n+1}{2}\rfloor}^{n-1}\frac{1-(x^{i+1}y^i)^{2n-2i-1}}{1-(x^{2i+1}y^{2i-1})^{n-i}}.$$

2. If $a_1=1$ and $a_2=\ell$, we have $a_i=b_i$ for all i. Moreover, the recursive properties of the sequences imply that if the integers i and j are equal modulo two, then

$$b^*_{i+1}b_j=b_ib^*_{j+1}.$$

This implies that all the terms occurring in the second product of (14) and (15) are equal to 1, and thus Proposition 5.4 becomes Theorem 1.4.

6 Limit theorems

In this section, we give two limit theorems corresponding to the identities (4). The sequence a is defined by $a_0=0$, $a_1=1$, and the recurrence relations (3).

First, note that removing the empty parts of lecture hall partitions puts the set $\mathcal{L}_n$ in one-to-one correspondence with the following set:

$$\mathcal{D}_n=\left\{(\mu_1,\ldots,\mu_m):m\leq n\text{ and }0<\frac{\mu_1}{a_{n-m+1}}\leq\frac{\mu_2}{a_{n-m+2}}\leq\ldots\leq\frac{\mu_m}{a_n}\right\}.$$

The conditions on the partitions of $\mathcal{D}_n$ can be restated as

$$\frac{\mu_{i+1}}{\mu_i}\geq\frac{a_{n-m+i+1}}{a_{n-m+i}}.$$

The following lemma implies that $\mathcal{D}_n\subset\mathcal{D}_{n+2}$ for all n.

Lemma 6.1 *For $i\geq 1$, let us denote $q_i=a_{i+1}/a_i$. Then the two sequences $(q_{2i})_{i\geq1}$ and $(q_{2i-1})_{i\geq1}$ are both decreasing, and converge respectively towards θ_e and θ_o where*

$$\theta_e=\frac{k\ell+\sqrt{k\ell(k\ell-4)}}{2\ell}\quad\text{and}\quad\theta_o=\frac{k\ell+\sqrt{k\ell(k\ell-4)}}{2k}.$$

Consequently, the sequence $\mathcal{D}_{2n}$ converges, when n tends to infinity, to the set $\mathcal{D}_e$ of partitions $(\mu_1,\ldots,\mu_m)$ such that $\mu_1>0$ and, for $i\geq1$,

$$\frac{\mu_{i+1}}{\mu_i}>\begin{cases}\theta_e & \text{if } m+i \text{ is even,}\\ \theta_o & \text{if } m+i \text{ is odd.}\end{cases}$$

Similarly, the sequence $\mathcal{D}_{2n+1}$ converges, when n tends to infinity, to the set $\mathcal{D}_o$ of partitions $(\mu_1, \ldots, \mu_m)$ such that $\mu_1 > 0$ and, for $i \geq 1$,

$$\frac{\mu_{i+1}}{\mu_i} > \begin{cases} \theta_o & \text{if } m+i \text{ is even,} \\ \theta_e & \text{if } m+i \text{ is odd.} \end{cases}$$

Taking the limit $n \to \infty$ in the main theorem leads to the following result.

Proposition 6.2 *The generating function for the elements of $\mathcal{D}_e$ is*

$$\sum_{\mu \in \mathcal{D}_e} x^{|\mu|_e} y^{|\mu|_o} = \prod_{i \geq 1} \frac{1}{1 - x^{a_i} y^{a_i^*}}.$$

The generating function for the elements of $\mathcal{D}_o$ is

$$\sum_{\mu \in \mathcal{D}_o} x^{|\mu|_e} y^{|\mu|_o} = \prod_{i \geq 1} \frac{1}{1 - x^{a_{i+1}} y^{a_{i-1}^*}}.$$

The sequences a and a^* are defined by $a_1 = 1$, $a_2 = \ell$, $a_1^* = 0$, $a_2^* = 1$ and the recurrence relations (11).

When $k = \ell$, the two limits θ_e and θ_o are equal. The sets $\mathcal{D}_e$ and $\mathcal{D}_o$ coincide, and are equal to the set of partitions $(\mu_1, \ldots, \mu_m)$ such that $\frac{\mu_{i+1}}{\mu_i} > \frac{k+\sqrt{k^2-4}}{2}$ for $1 \leq i < m$. We thus derive Proposition 1.1 from the proposition above.

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