THE CHINESE MONOID

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Abstract

This paper presents a combinatorial study of the Chinese monoid, a ternary monoid related to the plactic monoid and based on the rewritings $cba \equiv bca \equiv cab$. An algorithm similar to Schensted's algorithm yields a characterisation of the equivalence classes and a cross-section theorem. For this work, we had to develop some new combinatorial tools. Among other things we discovered an embedding of every equivalence class in the greatest one.

1 Definition and first properties

1.1 Definition

Definition 1.1 (Duchamp, Krob, [3]) *Let $(A, <)$ be a totally ordered alphabet over n letters. The* Chinese congruence *is the congruence defined by the relation*

$$cba \equiv cab \equiv bca \qquad \text{for every } a \leq b \leq c. \tag{1}$$

The Chinese monoid $CH(A,<)$ *is the quotient monoid of A^* by the Chinese congruence.*

For instance, Figure 1 shows the congruence class of $dcba$.

1.2 Schützenberger's Involution

Denote by $CH(A,>)$ the Chinese monoid built over the alphabet A supplied with the opposite order of $<$, denote by $(A^*)^\circ$ the opposite monoid associated with A^* and consider the natural morphism $\natural$ from A^* into $(A^*)^\circ$ that maps every word w to its mirror image. This morphism is compatible with the Chinese monoid structure, namely $u \equiv_< v$ if and only if $\natural(u) \equiv_> \natural(v)$, i.e., $\natural$ defines an isomorphism between $CH(A,<)$ and $(CH(A,>))^\circ$.

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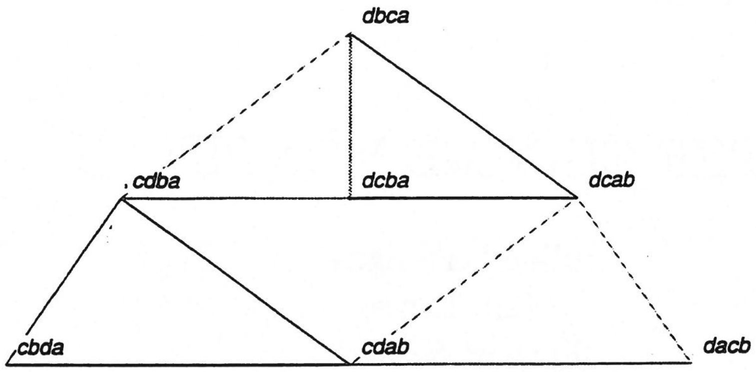

This is the congruence graph of *dcba*. Each edge stands for an elementary rewriting. The thick edge between *dcba* and *dbca* means that these words are equivalent thanks to *two* elementary congruences, namely $dcb \equiv dbc$ and $cba \equiv bca$.

Figure 1: The class of *dcba* where $a < b < c < d$

If $A = a, b, \ldots, z$, let $\mathcal{I}\colon (A^*, <) \rightarrow (A^*, >)$ be defined by $\mathcal{I}(a) = z, \mathcal{I}(b) = y, \ldots, \mathcal{I}(z) = a$. Then $\# = \mathcal{I} \circ \natural$ is an isomorphism between $CH(A,<)$ and $(CH(A,<))^{\circ}$; considerered as an involution of the set $CH(A,<)$, it is called *Schützenberger's involution* of $CH(A,<)$.

For instance, Figure 2 shows a simple example of classes that are equivalent under Schützenberger's involution. Note also the symmetry of Figure 1, since the class of *dcba* is invariant under Schützenberger's involution.

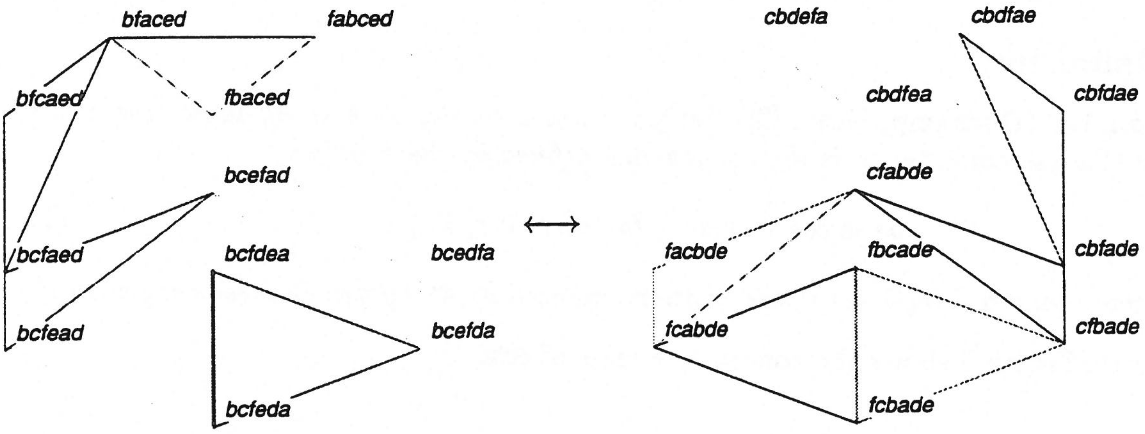

Figure 2: Schützenberger-equivalent classes: *facbde* and *bcedfa*

1.3 Standardization

Let w be a word over the alphabet A. We associate with it a standard word[1] $Std(w)$ over the alphabet $A \times \mathbb{N}$ obtained by numbering all occurrences of the same letter $1, 2, 3, \ldots$ from *right to left*. For instance, we have:

$$Std(babbaacb) = b_4 a_3 b_3 b_2 a_2 a_1 c_1 b_1 \ .$$

[1] A standard word is a word without repetition of letters.

This standardization process is compatible with the chinese congruence. Indeed:

$$u \equiv v \quad (CH(A,<)) \quad \Rightarrow \quad Std(u) \equiv Std(v) \quad (CH(A \times \mathbb{N},<)), \tag{2}$$

where the order over $A \times \mathbb{N}$ is the natural lexicographic order.

2 A representation of the Chinese monoid

2.1 Chinese Staircases

A *Chinese staircase* is a Ferrers diagram of shape $(1, 2, \ldots, n)$ filled with nonnegative integers,[2] that we draw in the following way:

$$\begin{array}{cccc|c}
 & & & \boxed{2} & a \\
 & & \boxed{1} & \boxed{} & b \\
 & \boxed{3} & \boxed{2} & \boxed{2} & c \\
\boxed{2} & \boxed{1} & \boxed{} & \boxed{1} & d \\
d & c & b & a &
\end{array}$$

We index the rows (resp. the columns) of the diagram with an initial segment of A from top to bottom (resp. from right to left.). We denote by $\sigma_{\alpha\beta}$ the cell in row α, column β, by σ_α the cell $\sigma_{\alpha\alpha}$.

A word w is said to be a *Chinese row* of type z if and only if it has the following structure

$$w = (za)^{n_a} \ldots (zy)^{n_y} (z)^{n_z}$$

where $a, b, \ldots, z$ denotes the initial segment of A ending with z, and where every n_α belongs to $\mathbb{N}$. Now let σ be a Chinese staircase. We associate to every row of σ a Chinese row in a natural way. Indeed, if the z^{th} row of σ has the form

$$\begin{array}{|c|c|c|c|}
\hline
\sigma_z & \sigma_{zy} & \cdots & \sigma_{za} \\
\hline
\multicolumn{1}{c}{z} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a}
\end{array} ,$$

then the associated Chinese row is just the word equal to

$$(za)^{\sigma_{za}} \ldots (zy)^{\sigma_{zy}} (z)^{\sigma_z}$$

A word w is a *Chinese staircase word* if and only if it can be written as $w = l_a\, l_b \ldots l_z$, where the l_α are Chinese rows of respective increasing types $a, b, \ldots, z$.

Definition 2.1 *The row-reading of a Chinese staircase σ is the word w obtained by concatenating all Chinese rows associated with the rows of σ from top to bottom.*

We can also define the column-reading of a Chinese staircase as a dual notion.

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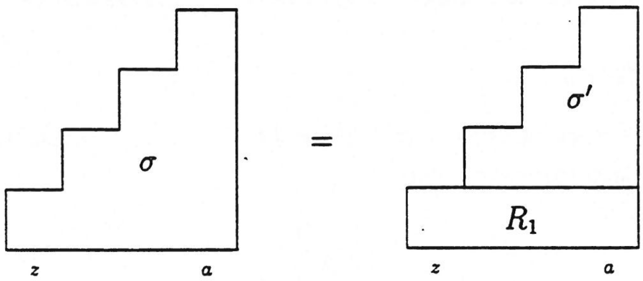
[2] We omit zeros for clarity.

2.2 The Insertion Algorithm

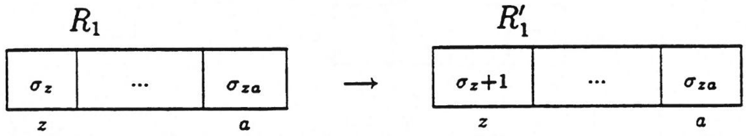
We shall now describe an algorithm that sends a word of A^* to a Chinese staircase. This is a simple adaptation of Schensted algorithm (Schensted, [12], Knuth, [5], Knuth, [6]). The basic step builds upon a Chinese staircase σ and a letter α a new Chinese staircase denoted by $\sigma.\alpha$. Hence, starting with the empty staircase ϵ, we build step by step a staircase $(\cdots((\epsilon.a_1).a_2)\cdots).a_k$ corresponding to the word $a_1a_2\ldots a_k$.

Algorithm 2.2 (The insertion algorithm)
Let σ be a staircase, α a letter to insert in σ. Start with $\sigma = (\sigma', R_1)$, where R_1 is the bottom row of σ, z the greatest letter of σ.



1. If $\alpha > z$, then $\sigma.\alpha = \sigma$.

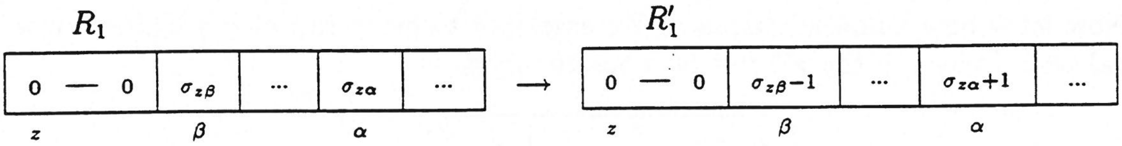
2. If $\alpha = z$, then $\sigma.\alpha = (\sigma', R_1')$ where R_1' is obtained from R_1 by adding 1 to cell σ_z:



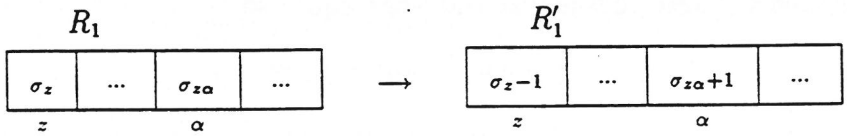
3. If $\alpha < z$, let β be the greatest letter whose cell on R_1 does not contain 0 or if such a β does not exist, set $\beta = \alpha$. Three distinct cases appear:

   3a. If $\alpha \geq \beta$, then $\sigma.\alpha = (\sigma'.\alpha, R_1)$.

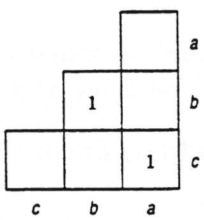
   3b. If $\alpha < \beta < z$, then $\sigma.\alpha = (\sigma'.\beta, R_1')$, where R_1' is obtained from R_1 by adding 1 to cell $\sigma_{z\alpha}$ and substracting 1 from cell $\sigma_{z\beta}$:



   3c. If $\alpha < \beta = z$, then $\sigma.\alpha = (\sigma', R_1')$, where R_1' is obtained from R_1 by adding 1 to cell $\sigma_{z\alpha}$ and substracting 1 from cell σ_z:



Example 2.3 On the alphabet a, b, c, words cba, cab, bca map to the same staircase:



Basically, inserting a lettre α into a staircase can only modify some specific cells of the staircase, based upon the set of exposed entries in the staircase.

Definition 2.4 *Let σ be a staircase. An* exposed entry *is a cell holding a non zero value such that all cells to its west and to its south-west are empty (that is: a $\sigma_{\alpha\beta}$ such that $\sigma_{\alpha\beta} > 0$, $\sigma_{\gamma\delta} = 0$ for $\gamma > \alpha$ and $\delta > \beta$, and $\sigma_{\alpha\delta} = 0$ for $\delta > \alpha$). An* exposed letter *is a letter that indexes a column corresponding to an exposed entry.*

Definition 2.5 *Let Σ be the set of Chinese staircases over A. The insertion algorithm defines an action $\mathcal{A}$ of the monoid A^* on Σ:*

$$\begin{array}{llll} \mathcal{A}: & \Sigma \times A^* & \longrightarrow & \Sigma \\ & (\sigma, w) & \longmapsto & \sigma.w = (\cdots((\sigma.a_1).a_2)\cdots).a_k, \end{array}$$

To further the analogy with Schensted algorithm, exhibiting a good Q-symbol and finding a kindred of Robinson-Schensted correspondence looks enticing. Such a correspondence exists, but this subject will not be broached here.

Definition 2.6 *We denote by $C(\sigma)$ the set of words w of A^* such that $\epsilon.w = \sigma$. We also denote by $C(w)$ the Chinese classe of w. Theorem 2.7 fully justifies the use of a similar notation.*

2.3 The Cross-Section Theorem

Theorem 2.7 *The Chinese staircase words form a cross-section of the Chinese monoid. More precisely:*

- *Property 1: For any words v and w for which the insertion algorithm yields the same staircase σ, v and w are equivalent under the Chinese congruence.*
- *Property 2: For any words v and w equivalent under the Chinese congruence, for any staircase σ, $\sigma.v = \sigma.w$.*
- *Property 3: For any t and t' staircase words, $t \equiv t'$ implies $t = t'$.*

Let σ be a staircase, v, w two words. By Property 2, if $v \equiv w$, $\sigma.v = \sigma.w$. So action $\mathcal{A}$ is compatible with the Chinese congruence, and the quotient $\mathcal{A}/\equiv$ is well defined. Consider σ_1 another staircase. By Theorem 2.7, $C(\sigma_1)$ is a Chinese class, so the staircase $\sigma.\sigma_1 = \sigma.w$ is constant for $w \in C(\sigma_1)$.This law defines an action of Σ on itself which is isomorphic to $\mathcal{A}/\equiv$ thanks to the isomorphism $\sigma \to C(\sigma)$. Finally, $(\Sigma, .)$ is isomorphic to the Chinese monoid $Ch(A, <)$.

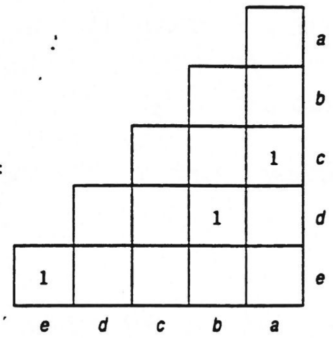
3 Backtracking the Insertion Algorithm

We now consider the standard case, where all letters used in a given word are distinct.

3.1 Link Representation

Definition 3.1 *Let σ be a standard staircase. Use σ to define a partial involution ρ of A: for every non-empty $\sigma_{\alpha\beta}$, let ρ exchange α and β.*

The link representation $\lambda(\sigma)$ *of σ is a representation of ρ that we obtain as follows: dispose all letters involved in σ in lexicographic order and link two letters whenever they appear together in σ. Link a letter on the diagonal with itself. For instance,*

if $\sigma =$ 

, *then* $\lambda(\sigma) =$ abcde.

Obviously, the link representation $\lambda(\sigma)$, the corresponding standard staircase σ and the corresponding involution ρ are equivalent representations, the link representation being more compact. For instance, recalling Figure 1 and Figure 2:

$$\lambda(\mathsf{cbda}) = \mathsf{abcd}, \quad \lambda(\mathsf{bcedfa}) = \mathsf{abcdef}, \quad \lambda(\mathsf{cbdefa}) = \mathsf{abcdef}.$$

Definition 3.2 *Let σ be a standard staircase, ρ the corresponding involution.*
A great letter *α verifies $\rho(\alpha) < \alpha$, a* small letter *α verifies $\rho(\alpha) > \alpha$, and a* neutral letter *α verifies $\rho(\alpha) = \alpha$.*
In the staircase representation, the great letters index rows that contain a 1, the small letters index columns that contain a 1, and the neutral letters occur on the diagonal.

Example 3.3 Take the class of abdc, of link representation abcd. That is, a and b are neutral, c is small, d is great.

3.2 A Converse of the Insertion Algorithm

In order to find all the words of $C(\sigma)$, we need to find back all the staircases that can occur in the sequence $\epsilon.a_1.\ldots.a_k$.

Definition 3.4 *Let σ be a standard staircase. A* deletable entry *is a non empty cell such that all cells to its south-west are empty. A* deletable letter *is a letter that indexes a column corresponding to a deletable entry.*

Algorithm 3.5 (Converse of the insertion algorithm 2.2)
Let σ be a staircase, $\sigma_{\gamma\delta}$ a deletable entry of σ. The algorithm defines a set of rewriting rules $\sigma \xrightarrow[\delta]{} \tau$ as follows:

- Rule 1: if δ is on the diagonal of σ, τ is the staircase derived from σ by substracting one from the deletable entry σ_δ.
- Rule 1′: if δ is not on the diagonal of σ, τ is the staircase derived from σ by substracting one from the deletable entry $\sigma_{\gamma\delta}$ and putting a 1 on the diagonal in σ_γ.
- Rule $2_{\tilde{\delta}}$: let $\tilde{\delta}$ be a deletable letter such that $\tilde{\delta} > \delta$. This implies that $\tilde{\delta}$ is higher than δ in σ, and that the deletable entry $\sigma_{\gamma\delta}$ satisfies $\gamma > \delta$. Let $\tilde{\sigma}$ be the staircase obtained from σ by removing all the rows under row γ, inclusive. Choose $\tilde{\tau}$ recursively by

$\tilde{\sigma} \xrightarrow[\bar{\delta}]{} \tilde{\tau}$, and build τ by adding at the bottom of $\tilde{\tau}$, first the row obtained by putting a 1 in cell $\sigma_{\gamma\bar{\delta}}$ and substracting 1 from the cell $\sigma_{\gamma\delta}$, then the rows of σ of indices greater than γ.

Formally, $\tilde{\sigma}$ is defined over $A' = \{\alpha \in A \mid \quad \alpha < \gamma\}$ by $\tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta}$ for any $\alpha < \gamma$, any β. And similarly: $\tau_{\alpha\beta} = \tilde{\tau}_{\alpha\beta}$ for any $\alpha < \gamma$, any β; $\tau_{\gamma\beta} = \sigma_{\gamma\beta}$ for any β but δ, $\bar{\delta}$; $\tau_{\gamma\delta} = \sigma_{\gamma\delta} - 1$; $\tau_{\gamma\bar{\delta}} = 1$; $\tau_{\alpha\beta} = \sigma_{\alpha\beta}$ for $\alpha = \gamma$, any β but δ, $\bar{\delta}$; $\tau_{\alpha\beta} = \sigma_{\alpha\beta}$ for any $\alpha > \gamma$, any β.

Figure 3 shows a graphic explanation of what is going on.

We define the set $(\sigma \xrightarrow[\delta]{} .)$ as follows:

$$
\begin{array}{lll}
(\sigma \xrightarrow[\delta]{} .) & = \{\tau \in \Sigma \mid \quad \sigma \xrightarrow[\delta]{} \tau\} & \text{if } \delta \text{ is a deletable letter of } \sigma, \\
(\sigma \xrightarrow[\alpha]{} .) & = \emptyset & \text{if } \alpha \text{ is not a deletable letter of } \sigma.
\end{array}
$$

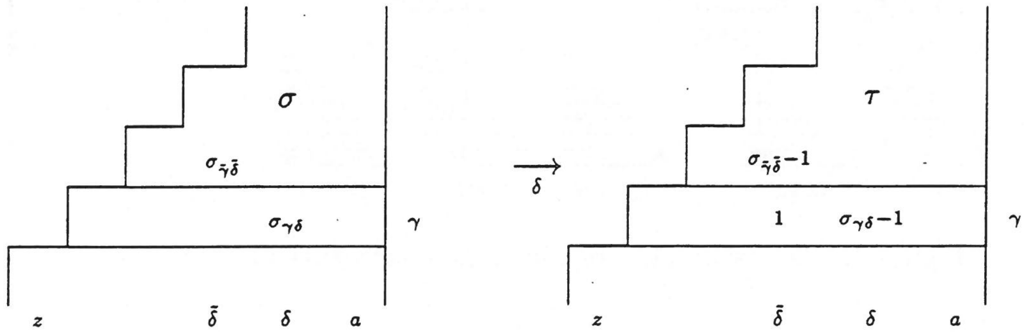


Figure 3: How Rule $2_{\bar{\delta}}$ works.

Example 3.6 Consider the Chinese staircase word *cbdfega*. We display in Figure 4 the staircases obtained from *cbdfega* by applying Algorithm 3.5.

This algorithm is a converse of the insertion algorithm. More precisely, we have the following theorem.

Theorem 3.7 *Let σ be a standard staircase over n letters. Define:*

$$
\begin{aligned}
\Delta(\sigma) &= \{\delta \in A \mid \quad \exists\, \sigma' \in \Sigma',\ \sigma = \sigma'.\delta\}, \\
\sigma.\alpha^{-1} &= \{\sigma' \in \Sigma \mid \quad \exists\, \alpha \in A,\ \sigma = \sigma'.\alpha\}.
\end{aligned}
$$

Then

- *$\Delta(\sigma)$ is the set of deletable letters.*
- *$\sigma.\alpha^{-1}$ is equal to $(\sigma \xrightarrow[\alpha]{} .)$ for any α.*

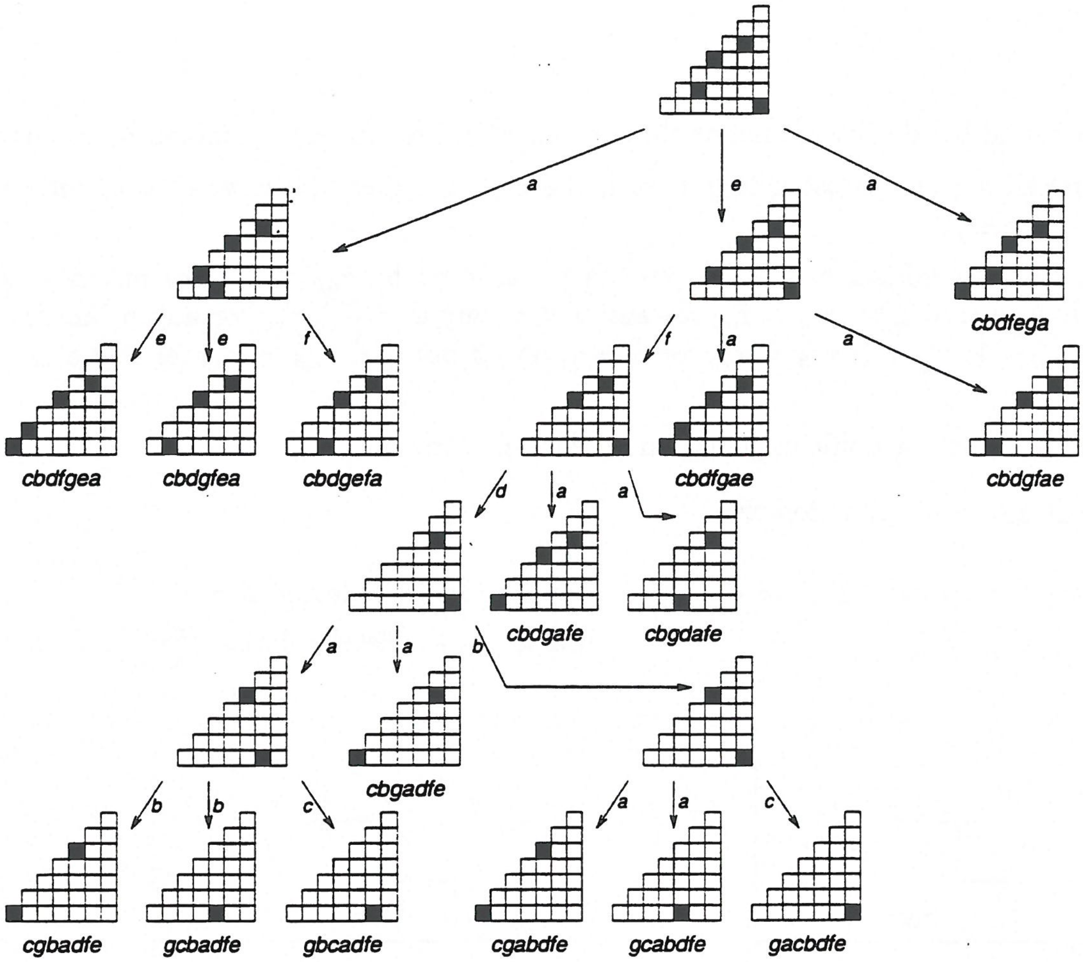


Figure 4: The cardinality of the class of *cbdfega* is 15.

3.3 Applications

Theorem 3.8 *In the standard case, every class has an odd order.*

Proposition 3.9 *The row normal form of a staircase σ is the minimal word in the class of σ for the lexicographic order.*

Define $\sigma_\perp$ to be the staircase obtained by applying Rule 1 or Rule 1′ to σ for the highest deletable element (depending whether the highest letter is on the diagonal or not).

Lemma 3.10 *$\sigma_\perp$ is the smallest staircase of $\sigma.A^{-1}$.*

4 The Great Class

We now focus on the largest standard class over a given alphabet. We first have to prove that it is effectively the largest class—in fact it encompasses all other classes in a precise sense. Trying to size it leads us to the construction of a bijection between the words of a given class and Dyck words. An integer that we call the *weight* of a Dyck word appears in a natural way.

Definition 4.1 *Let ω denote the maximal standard word over an alphabet A on n letters (for the lexicographic order): $\omega = zy\ldots ba$. The* great class $Gr(n) = Gr(A)$ *is the Chinese* *class of ω.*

Note 4.2 Let $p = [\frac{n}{2}]$. In $Gr(n)$, the great letters are the p greatest letters of the alphabet and the small letters are the p smallest. If n is odd, the $(p+1)^{\text{th}}$ letter is neutral, if n is even, there is no neutral letter. The i^{th} letter is associated with the $(n-i)^{\text{th}}$ letter. The non-zero entries occur precisely along the second diagonal of σ and are filled with 1.

4.1 Embedding other classes into the Great Class

We will now prove that the great class is the largest class of the Chinese monoid. In fact, it contains all the other subclasses in a very precise way.

Definition 4.3 *Let u, v, w be three words. An elementary rewriting between u, v, w is an elementary congruence of the Chinese relation: there exists three letters a, b, c, two words x, y such that $\{u, v, w\} = \{x.cba.y, x.cab.y, x.bca.y\}$.* An embedding *of a Chinese class C_1 into another Chinese class C_2 is an injection i from C_1 into C_2 that preserves the congruence graph. Namely, for any elementary rewriting between words u, v, w of C_1, there exists an elementary rewriting between $i(u)$, $i(v)$, $i(w)$. The embedding i is* strict *if both elementary rewritings occur at the same position.*

Here is the outline of the proof: the following algorithm yields an embedding of every Chinese class which is not the great class into a greater class. Therefore, each class belongs to a sequence of increasing classes that ends necessarily at the great class.

Algorithm 4.4 (Class Embedding)
Let σ be a full standard staircase. Provided $C(\sigma)$ is not the Great Class, the algorithm finds an embedding of $C(\sigma)$ into another Chinese class. Find β such that $\sigma_{\beta\rho(\beta)}$ be the right-most, non exposed entry. Let γ be the successor of β. The embedding is t_β^γ, the elementary tranposition which exchanges β and γ and leave other letters invariant.

Proposition 4.5 *This algorithm is correct, namely β does exist if and only if $C(\sigma)$ is not a great class, in which case t_β^γ is a strict embedding.*

We can rephrase this proposition as follows:

Theorem 4.6 *For a given n, all classes have an order less or equal to the cardinality of $Gr(n)$. In fact, for any Chinese class, there exists a permutation of A which is a strict embedding of this class into $Gr(n)$.*

Example 4.7 Consider the class of abcdefijhkg of order 35. The algorithm embeds it successively into the classes of

bacdefijhkg (order 35),	bcadefijhkg (105),	bcdaefijhkg (175),
bcdeafijhkg (245),	bcdefaijhkg (315),	bcdegaijhkf (315),
bcdehaijgkf (315),	bcdehiajgkf (329),	bcdehigjakf (399),
bcdehigjfka (1 225),	cbdehigjfka (1 295),	cdbehigjfka (4 165),
cdebhigjfka (7 175),	cdfbhigjeka (7 175),	cdgbhifjeka (7 175),
cdghbifjeka (7 725),	cdghfibjeka (10 607),	cdghfiejbka (60 037),
dcghfiejbka (67 597),	ecghfidjbka (67 597),	fcgheidjbka (67 597),
fgcheidjbka (92 323),	fgehcidjbka (228 305),	

and finally fgehdicjbka (3 705 075) ! The embedding sends abcdefghijk to kajbicdefgh.

4.2 Dyck words

We begin by studying $Gr(n)$ where $n = 2p$ is an even number.

Definition 4.8 *Let $D = \{x, \bar{x}\}$ be the alphabet over two letters x and $\bar{x}$.*

- *The* height *of a word w over D is $h(w) = |w|_x - |w|_{\bar{x}}$.*
- *A word $w \in D^*$ is* a Dyck word *if every prefix word u of w verifies $h(u) \geq 0$ and if $h(w) = 0$.*
- *A word w is* a proper Dyck word *if every proper prefix word u of w verifies $h(u) > 0$ and if $h(w) = 0$.*
- *If w is a Dyck word, and u a prefix of w, we say that u is* a return to zero *of w if $h(u) = 0$ (Therefore, a proper Dyck word is a Dyck word with just one return to zero.)*

Definition 4.9 *Let A be an alphabet. We denote by π the monoid morphism defined on A by:*

$$\begin{array}{cccl} \pi : & A & \longrightarrow & D \\ & \alpha & \longmapsto & \begin{cases} x \text{ if } \alpha \text{ is a great letter of } Gr(A) \\ \bar{x} \text{ if } \alpha \text{ is a small letter of } Gr(A). \end{cases} \end{array}$$

Theorem 4.10 *Fix an alphabet A over $2p$ letters. Denote by c and c' the median letters of A; model $Gr(2p-2)$ over $A \setminus \{c, c'\}$. Then the great class $Gr(2p)$ is characterized as follows:*

- *if w belongs to $Gr(2p)$, if w' is obtained by deleting c and c' from w, then w' does belong to $Gr(2p-2)$.*
- *if w' belongs to $Gr(2p-2)$, if w is obtained by inserting c and c' in such a way that the image $\pi(w)$ is a Dyck word, then w does belong to $Gr(2p)$.*
- *if w' belongs to $Gr(2p-2)$, if w is obtained by inserting c and c' in such a way that the image $\pi(w)$ is not a Dyck word, then w does not belong to $Gr(2p)$.*

For example, faeb $\in Gr(4)$ and $\pi(\text{faeb}) = x\bar{x}x\bar{x} \in D_4$. Insert c and d to obtain fadebc whose image by π is $x\bar{x}xx\bar{x}\bar{x}$ which is a Dyck word, hence fadebc belongs to $Gr(6)$. Insert c and d to obtain facebd whose image by π is $x\bar{x}\bar{x}x\bar{x}x$ which is not a Dyck word, and correspondingly, facebd does not belong to $Gr(6)$.

We now describe a procedure to compute the order of the great class.

Definition 4.11 *A Dyck word of length $2p$, $p > 0$, can be reduced to a Dyck word of length $2p-2$ in a variety of ways by deleting one x and one $\bar{x}$. Each of these ways will be called a Dyck reduction. We define* the weight *of a Dyck word inductively:*

- *$Weight(x\bar{x}) = 1$.*
- *The weight of a Dyck word of length $2p$ is the sum of all the weights of Dyck words obtained by all possible Dyck reductions.*

Example 4.12 Since $xx\bar{x}\bar{x}$ reduces as $\cancel{x}x\cancel{\bar{x}}\bar{x}$, $\cancel{x}x\bar{x}\cancel{\bar{x}}$, $x\cancel{x}\cancel{\bar{x}}\bar{x}$, and $x\cancel{x}\bar{x}\cancel{\bar{x}}$, $Weight(xx\bar{x}\bar{x}) = 4$. On the other hand, $x\bar{x}x\bar{x}$ reduces as $\cancel{x}\cancel{\bar{x}}x\bar{x}$, $x\cancel{\bar{x}}\cancel{x}\bar{x}$, and $x\bar{x}\cancel{x}\cancel{\bar{x}}$, so $Weight(x\bar{x}x\bar{x}) = 3$. Similarly the reader can check that $Weight(xxx\bar{x}\bar{x}\bar{x}) = 36$.

Theorem 4.13 *For every Dyck word w of length n, Weight(w) verifies:*

$$Weight(w) = |\{u \in Gr(n),\ \pi(u) = w\}|.$$

Proof — By definition, $Weight(w) = |\nu^{-1}(w)|$. □

Corollary 4.14 *The sum of the weights of all Dyck words of n letters is equal to the cardinality of $Gr(n)$.*

Let us now assume that n is odd, $n = 2p+1$. A similar argument yields the following theorem.

Theorem 4.15 *Fix an alphabet A over $2p+1$ letters. Denote by c the median letter of A; model $Gr(2p)$ over $A \setminus \{c\}$. Then the great class $Gr(2p+1)$ is characterized as follows:*

- *if w belongs to $Gr(2p+1)$, if w' is obtained by deleting c from w, then w' does belong to $Gr(2p)$.*
- *if w' belongs to $Gr(2p)$, if w is obtained by inserting c in w, then w does belong to $Gr(2p+1)$.*

Corollary 4.16 *If n is an odd integer. Then $|Gr(n)| = (n)\,|Gr(n-1)|$.*

Example 4.17 For $n = 11$, we have $|Gr(11)| = 3\,705\,075 = 11 \times 336\,825 = 11\,|Gr(10)|$.

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