SPECTRA OF BIPARTITE P- AND Q-POLYNOMIAL ASSOCIATION SCHEMES

JOHN S. CAUGHMAN, IV [1]

Extended Abstract. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a P- and Q-polynomial association scheme, with eigenvalues $\theta_0, \theta_1, ..., \theta_D$ and dual eigenvalues $\theta_0^*, \theta_1^*, ..., \theta_D^*$. We want to find the permutations σ, τ of $0, 1, ..., D$ for which

$$\theta_{\sigma 0} > \theta_{\sigma 1} > \ \dots \ > \theta_{\sigma D},$$
$$\theta^*_{\tau 0} > \theta^*_{\tau 1} > \ \dots \ > \theta^*_{\tau D}.$$

We focus on the case where Y is bipartite, and prove the following theorem.

0.1 Theorem. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme with $D \geq 3$, and assume Y is not an ordinary cycle. Suppose Y is bipartite P-polynomial with respect to the given ordering $A_0, A_1, ..., A_D$ of the associate matrices, and Q-polynomial with respect to the ordering $E_0, E_1, ..., E_D$ of the primitive idempotents. Then the eigenvalues and dual eigenvalues satisfy exactly one of (i) - (iv).*

(i)

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \ \dots \ > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta_i^* = \theta_i \qquad (0 \leq i \leq D).$$

(ii) D is even, and

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \ \dots \ > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta_i^* = \theta_i \qquad (0 \leq i \leq D).$$

(iii) $\theta_0^* > \theta_0$, and

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \ \dots \ > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta_0^* > \theta_1^* > \theta_2^* > \theta_3^* > \ \dots \ > \theta_{D-3}^* > \theta_{D-2}^* > \theta_{D-1}^* > \theta_D^*.$$

[1]Dept. of Mathematics, University of Wisconsin, 480 Lincoln Dr., Madison, WI 53706. Email: caughman@math.wisc.edu. AMS 1991 Subject Classification: Primary 05E30.

(iv) $\theta_0^* > \theta_0$, *D is odd, and*

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \ \dots \ > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta_0^* > \theta_D^* > \theta_2^* > \theta_{D-2}^* > \ \dots \ > \theta_{D-3}^* > \theta_3^* > \theta_{D-1}^* > \theta_1^*.$$

For the remainder of this abstract, we review the standard definitions relevant to this theorem.

Association Schemes.

A D-class *symmetric association scheme* is a pair $Y = (X, \{R_i\}_{0 \le i \le D})$, where X is a non-empty finite set, and where:

(i) $\{R_i\}_{0 \le i \le D}$ is a partition of $X \times X$;

(ii) $R_0 = \{xx \mid x \in X\}$;

(iii) $R_i^t = R_i$ for $0 \le i \le D$, where $R_i^t = \{yx \mid xy \in R_i\}$;

(iv) there exist integers p_{ij}^h such that for all $x, y \in X$ with $xy \in R_h$, the number of $z \in X$ with $xz \in R_i$ and $zy \in R_j$ is p_{ij}^h.

The constants p_{ij}^h are called the *intersection numbers* of Y. By a *scheme*, we mean a symmetric association scheme with $D \ge 3$.

The Bose-Mesner Algebra M.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be any scheme, and let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X. Pick an integer i $(0 \le i \le D)$. By the i^{th} *associate matrix of* Y we mean the matrix $A_i \in \mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X). \tag{1}$$

From (1) we obtain the following relations:

$$A_0 = I, \tag{2}$$
$$A_i^t = A_i \qquad (0 \le i \le D), \tag{3}$$
$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \qquad (0 \le i, j \le D), \tag{4}$$
$$A_0 + A_1 + \cdots + A_D = J, \tag{5}$$

where I is the identity matrix, and J is the all 1's matrix.

By (2)-(4), $A_0, ..., A_D$ is a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. M is known as the *Bose-Mesner algebra* for Y.

By [2, p.45], the algebra M has a second basis $E_0, ..., E_D$ such that

$$\begin{aligned} E_0 &= |X|^{-1}J, && (6)\\ E_i^t &= E_i \qquad (0 \le i \le D), && (7)\\ E_iE_j &= \delta_{ij}E_i \qquad (0 \le i,j \le D), && (8)\\ E_0 + E_1 + &\cdots + E_D = I. && (9) \end{aligned}$$

We refer to $E_0, ..., E_D$ as the *primitive idempotents* of Y.

By the *Krein parameters* of Y, we mean the real scalars $\{q_{ij}^h \mid 0 \le h,i,j \le D\}$ satisfying

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \le i,j \le D), \tag{10}$$

where $\circ$ denotes the entry-wise matrix product [1, p.64].

Eigenvalues and Dual Eigenvalues.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be any scheme. By [7, pp.377,379], there exist real scalars $p_i(j)$, $q_i(j)$ $(0 \le i,j \le D)$ which satisfy

$$A_i = \sum_{j=0}^{D} p_i(j)E_j \qquad (0 \le i \le D), \tag{11}$$

$$E_i = |X|^{-1} \sum_{j=0}^{D} q_i(j)A_j \qquad (0 \le i \le D). \tag{12}$$

We refer to $p_i(j)$ (resp. $q_i(j)$) as the j^{th} *eigenvalue* (resp. j^{th} *dual eigenvalue*) associated with A_i (resp. A_i^*).

The P-polynomial Property.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be any scheme. We say that Y is *P-polynomial* (with respect to the given ordering $A_0, ..., A_D$ of the associate matrices) whenever for all h,i,j $(0 \le h,i,j \le D)$,

$$\begin{gathered} p_{ij}^h = 0 \textit{ if one of } h,i,j \textit{ is greater than the sum of the other two,}\\ \textit{and}\\ p_{ij}^h \ne 0 \textit{ if one of } h,i,j \textit{ equals the sum of the other two.} \end{gathered} \tag{13}$$

If a scheme Y is P-polynomial, we set

$$\theta_i \;:=\; p_1(i) \qquad (0 \leq i \leq D). \tag{14}$$

The Q-polynomial Property.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be any scheme. We say that Y is *Q-polynomial* (with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents) whenever for all h, i, j $(0 \leq h, i, j \leq D)$,

$$\begin{gathered} q_{ij}^h = 0 \text{ if one of } h,i,j \text{ is greater than the sum of the other two,} \\ \text{and} \\ q_{ij}^h \neq 0 \text{ if one of } h,i,j \text{ equals the sum of the other two.} \end{gathered} \tag{15}$$

If a scheme Y is Q-polynomial, we set

$$\theta_i^* \;:=\; q_1(i) \qquad (0 \leq i \leq D). \tag{16}$$

References

[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1984.

[2] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin, 1989.

[3] B. Curtin. Bipartite distance-regular graphs. *Graphs and Combin.* Submitted.

[4] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, New York, 1993.

[5] K. Nomura. Spin models and almost bipartite 2-homogeneous graphs. Preprint.

[6] P. Terwilliger. Eigenvalue multiplicities of highly symmetric graphs. *Discrete Math.*, 61:295–302, 1982.

[7] P. Terwilliger. The subconstituent algebra of an association scheme. I. *J. Algebraic Combin.*, 1(4):363–388, 1992.

[8] N. Yamazaki. Bipartite distance-regular graphs with an eigenvalue of multiplicity k. Preprint.