ENUMERATION OF PERFECT MATCHINGS IN GRAPHS WITH REFLECTIVE SYMMETRY

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ABSTRACT. A plane graph is called symmetric if it is invariant under the reflection across some straight line. We prove a result that expresses the number of perfect matchings of a large class of symmetric graphs in terms of the product of the number of matchings of two subgraphs. When the graph is also centrally symmetric, the two subgraphs are isomorphic and we obtain a counterpart of Jockusch's squarishness theorem. As applications of our result, we enumerate the perfect matchings of several families of graphs and we obtain new solutions for the enumeration of two of the ten symmetry classes of plane partitions (namely, transposed complementary and cyclically symmetric, transposed complementary) contained in a given box. We also consider symmetry classes of perfect matchings of the Aztec diamond and we solve the previously open problem of enumerating the matchings that are invariant under a rotation by 90 degrees.

0. Introduction

The starting point of this paper is a result [**PS**, Theorem 1] concerning domino tilings of the Aztec diamond compatible with certain barriers. This result has also been generalized and proved bijectively by Propp [**Prop**]. We present a further generalization, which allows us to prove a basic factorization theorem for the number of perfect matchings of plane bipartite graphs with a certain type of symmetry.

As a direct consequence, we obtain a counterpart of Jockusch's squarishness theorem [**J**, Theorem 1]. We then use the factorization theorem to give new solutions for the enumeration of two of the ten symmetry classes of plane partitions contained in a given box.

Motivated by the example of plane partitions, we consider the enumeration of perfect matchings of the Aztec diamond that are invariant under certain symmetries. There are a total of five enumerative problems that arise in this way. Two of them have been previously considered (one of which corresponds to matchings invariant under the trivial group). We present a solution for a previously open case and a new proof for the previously solved non-trivial case.

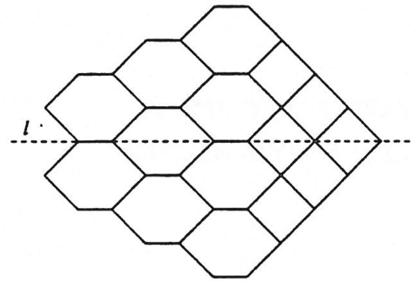

FIGURE 1.1

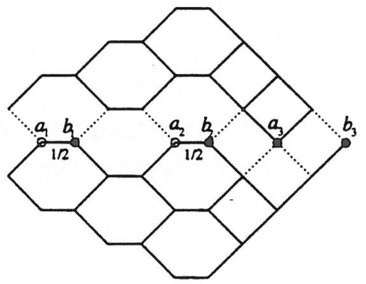

FIGURE 1.2

We conclude by presenting a simple proof of Stanley's (ex)conjecture on the number of spanning trees of the even and odd Aztec diamonds (in the case of odd orders) and by enumerating the perfect matchings of three families of graphs that either generalize or are concerned with the Aztec diamond.

1. A Factorization Theorem

A *perfect matching* of a graph is a collection of vertex-disjoint edges that are collectively incident to all vertices. We will often refer to a perfect matching simply as a matching.

Let G be a plane graph. We say that G is *symmetric* if it is invariant under the reflection across some straight line. Figure 1.1 shows an example of a symmetric graph. Clearly, a symmetric graph has no perfect matching unless the axis of symmetry contains an even number of vertices (otherwise, the total number of vertices is odd); we will assume this throughout the paper.

A *weighted* symmetric graph is a symmetric graph equipped with a weight function on the edges that is constant on the orbits of the reflection. The *width* of a symmetric graph G, denoted $w(G)$, is defined to be half the number of vertices of G lying on the symmetry axis.

For a vertex x on the symmetry axis l (considered horizontal), define the *cut above* x to be the operation of deleting all edges above l incident to x. The *cut below* x is defined similarly, as deletion of all incident edges below l.

Let G be a weighted symmetric graph that is also bipartite. Suppose that the set of vertices lying on l is a cut set (i.e., removing these vertices disconnects the graph). In such a case we say that l *separates* G. Label the vertices on l from left to right by $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$. Color the vertices in the two bipartition classes black and white. For definiteness, choose a_1 to be white.

Define two subgraphs G^+ and G^- as follows. Perform cutting operations above all white a_i's and black b_i's and below all black a_i's and white b_i's. Note that this procedure yields cuts of the same kind at the endpoints of each edge lying on l. Reduce the weight of each such edge by half; leave all other weights unchanged. Since l separates G, the graph produced by the above procedure is disconnected into one component lying above l, which we denote by G^+, and one below l, denoted by G^-. Figure 1.2 illustrates this procedure for the graph pictured in Figure 1.1 (the edges whose weight has been reduced by half are marked by 1/2).

The weight of a matching μ is defined to be the product of the weights of the edges contained in μ. The matching generating function of a weighted graph G, denoted $M(G)$, is the sum of the weights of all matchings of G. The central result of this paper is the following.

THEOREM 1.1 (FACTORIZATION THEOREM). *Let G be a bipartite weighted symmetric graph separated by its symmetry axis. Then*

$$M(G) = 2^{w(G)} M(G^+) M(G^-).$$

The proof is combinatorial and consists of three steps. First, we reduce to the case when the vertices on the symmetry axis form an independent set. Second, we show that the 2^k subgraphs of G obtained by independently performing cutting operations at the a_i's have the same matching generating function. And third, we argue that for a special choice of these cuts at the a_i's, the matchings of the corresponding subgraph are the same as the matchings of $G^+ \cup G^-$.

In case G has a second symmetry axis l' perpendicular to l, and if the reflection across l' preserves bipartition classes, the graphs G^+ and G^- are isomorphic. The factorization theorem implies then that the number of matchings of such a graph is either a perfect square or twice a perfect square.

This gives a combinatorial explanation for the fact, first proved by Montroll using linear algebra (see [L, Problem 4.29] for an exposition), that the number of matchings of the $2n \times 2n$ grid graph has this "squarish" property. Moreover, we obtain a combinatorial interpretation of the square root, answering thus the last question of [J, p.114].

2. Plane Partitions

A plane partition π is a rectangular array of non-negative integers with non-increasing rows and columns and finitely many nonzero entries. We can also regard π as an order ideal of $\mathbf{N}^3$, i.e., a finite subset of $\mathbf{N}^3$ such that $(i, j, k) \in \pi$ implies $(i', j', k') \in \pi$, whenever $i \geq i'$, $j \geq j'$ and $k \geq k'$.

By permuting the coordinate axes, one obtains an action of S_3 on the set of plane partitions. Let $\pi \mapsto \pi^t$ and $\pi \mapsto \pi^r$ denote the symmetries corresponding to interchanging the x- and y-axes and to cyclically permuting the coordinate axes, respectively. For the set of plane partitions π contained in the box $B(a, b, c) := \{(i, j, k) \in \mathbf{N}^3 : i < a, j < b, k < c\}$, there is an additional symmetry

$$\pi \mapsto \pi^c := \{(i, j, k) \in \mathbf{N}^3 : (a - i - 1, b - j - 1, c - k - 1) \notin \pi\},$$

called the complement.

These three symmetries generate a group isomorphic to the dihedral group of order 12, which has 10 conjugacy classes of subgroups. These lead to 10 enumeration problems:

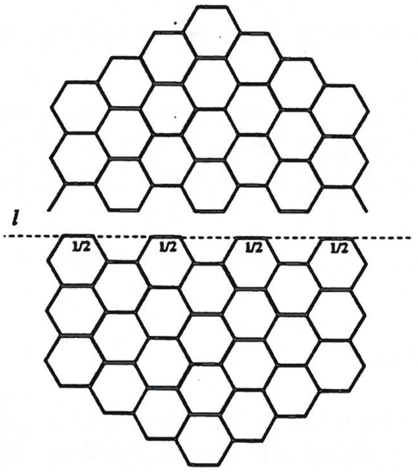

FIGURE 2.1(a)

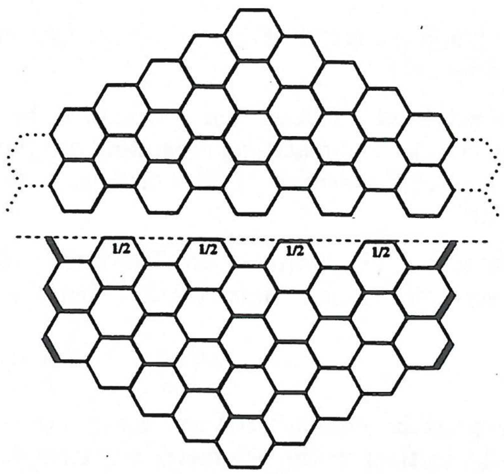

FIGURE 2.1(b)

determine the number of plane partitions contained in a given box that are invariant under the action of one of these subgroups. The program of solving these problems was formulated by Stanley [**Sta**] and has been recently completed (see [**A1**], [**K**] and [**Ste**]).

Using the factorization theorem and the interpretation of plane partitions contained in a box as perfect matchings of suitable honeycomb graphs (see [**DT**], [**R1**] and [**K**]) we relate two pairs of these ten problems.

Namely, we express the number of transposed complementary plane partitions (i.e, plane partitions π with $\pi^t = \pi^c$) contained in a given box in terms of the total number of plane partitions (with no symmetry requirement) contained in a box. Based on MacMahon's formula [**M**] for the latter number, we obtain a new proof of the transposed complementary case, first solved by Proctor [**Proc**].

Similarly, we prove a formula that relates the number of cyclically symmetric ($\pi^r = \pi$), transposed complementary plane partitions to the number of cyclically symmetric plane partitions that fit in a given box. Then, based on the formula for the latter proved by Andrews [**A2**], we obtain a new proof of the cyclically symmetric, transposed complementary case, first solved by Mills, Robbins and Rumsey [**MRR2**].

Let $TC(a, a, 2b)$ be the number of transposed complementary plane partitions contained in $B(a, a, 2b)$. Let $P(a, b, c)$ denote the number of plane partitions contained in $B(a, b, c)$.

THEOREM 2.1.

$$TC(a, a, 2b) = 2 \cdot TC(a + 2, a + 2, 2b - 2)\frac{P(a, a, 2b)}{P(a + 1, a + 1, 2b - 1)}.$$

Sketch of proof. Let $H(a, b, c)$ denote the $a \times b \times c$ honeycomb graph. Using the above mentioned interpretation of plane partitions as matchings of honeycombs, $P(a, a, 2b)$ can be regarded as the number of matchings of $H(a, a, 2b)$, and $TC(a, a, 2b)$ as the number of such matchings that are invariant under reflection across the symmetry axis l of the honeycomb perpendicular to the sides of length $2b$ (see Figure 2.1(a)).

By applying the factorization theorem to $H(a,a,2b)$ with respect to l, we obtain

$$P(a,a,2b) = 2^a TC(a,a,2b) M(H(a,a,2b)^-). \tag{2.1}$$

On the other hand, the factorization theorem applied to $H(a+1,a+1,2b-1)$ yields (see Figure 2.2(b))

$$P(a+1,a+1,2b-1) = 2^{a+1} TC(a+2,a+2,2b-2) M(H(a,a,2b)^-). \tag{2.2}$$

Relations (2.1) and (2.2) imply the statement of the theorem. □

Let $CS(a)$ and $CSTC(a)$ be the number of cyclically symmetric and cyclically symmetric, transposed complementary plane partitions contained in $B(a,a,a)$, respectively. Note that $CSTC(a)$ is nonzero only for even a. Using ideas similar to those in the proof of Theorem 2.1 we obtain the following result.

THEOREM 2.2.

$$2 \cdot CSTC(2a+2) = CSTC(2a) \frac{CS(2a+1)}{CS(2a)}.$$

It is interesting that Theorem 2.2 is also a consequence of relations (0.2) and (0.3) of **[Ste]**, relations that may be used to solve the totally symmetric case.

3. Symmetries of matchings of the Aztec diamond

The honeycomb graphs can be described as follows. Consider the tiling of the plane by congruent regular hexagons. Let H_1 be one of these hexagonal tiles, and define H_n for $n \geq 2$ to be the union of the set of tiles sharing at least one edge with some tile contained in H_{n-1}. Then H_n is the $n \times n \times n$ honeycomb graph.

Motivated by the simple product formulas that enumerate the symmetry classes of matchings of honeycombs, it is natural to investigate symmetry classes of matchings of the graphs arising by applying the inductive construction in the previous paragraph to the tiling of the plane by squares. Moreover, these graphs are the *Aztec diamonds* AD_n, and their matchings have been the subject of a great deal of investigation. In particular, in **[EKLP]** it is proved that $M(AD_n) = 2^{n(n+1)/2}$ (see **[C1]** for an alternative proof).

The symmetry group of the Aztec diamond is isomorphic to the dihedral group of order 8. Let r and t be the symmetries corresponding to rotation by 90° and reflection across a diagonal, respectively (the diagonals of AD_n are the two symmetry axes not containing any vertex). Then r and t generate the symmetry group of the Aztec diamond. Since the elements rt and r^3t act as reflections across lines containing vertices of the Aztec diamond, there are no rt- or r^3t-invariant matchings. Up to conjugacy, there are five distinct subgroups of $\langle r,t \rangle$ not containing any of these two elements. Imposing the condition that a matching is invariant under the action of one of these subgroups G leads to five different enumeration problems.

Only one of these problems besides the case $G = 1$ has been previously solved: the case $G = \langle r^2 \rangle$, which was solved by Yang **[Y]**. This case is also implicit in the unpublished work of Robbins **[R2]**.

n	$\langle t\rangle$ – invariant	factorization
1	2	2
2	6	$2 \cdot 3$
3	24	$2^3 \cdot 3$
4	132	$2^2 \cdot 3 \cdot 11$
5	1048	$2^3 \cdot 131$
6	11960	$2^3 \cdot 5 \cdot 13 \cdot 23$

n	$\langle r^2, t\rangle$ – invariant	factorization
1	2	2
2	4	2^2
3	10	$2 \cdot 5$
4	28	$2^2 \cdot 7$
5	96	$2^5 \cdot 3$
6	384	$2^7 \cdot 3$
7	1848	$2^3 \cdot 3 \cdot 7 \cdot 11$
8	10432	$2^6 \cdot 163$
9	70560	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2$
10	564224	$2^{10} \cdot 19 \cdot 29$
11	5386080	$2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 229$

TABLE 3.1. $\langle t\rangle$-invariant and $\langle r^2, t\rangle$-invariant matchings of the Aztec diamond.

Using the factorization theorem, we obtain a new solution for this case and we solve the case $G = \langle r\rangle$.

The two problems that remain open are the enumeration of matchings invariant under reflection across one or both diagonals (i.e., invariant under $\langle t\rangle$ or $\langle r^2, t\rangle$). For the first few orders of the Aztec diamond the corresponding numbers and their factorizations are shown in Table 3.1. Apparently these numbers do not all factor into small primes, so a simple product formula seems unlikely in this two cases.

Our proofs involve the planar regions known as quartered Aztec diamonds defined in [JP], which can be described as follows. Let us consider the planar region whose dual is the Aztec diamond of order n. This region can be divided into two congruent parts by a zig-zag lattice path that changes direction after every two steps, as shown in Figure 3.1.

By superimposing two such paths that intersect at the center of the region we divide it into four pieces that are called quartered Aztec diamonds. Up to symmetry, there are two different ways we can superimpose the two cuts. For one of them, the obtained pattern has fourfold rotational symmetry and the four pieces are congruent (see Figure 3.2(a)); denote the number of their domino tilings by $R(n)$. For the other, the resulting pattern has Klein 4-group reflection symmetry and there are two different kinds of pieces (see Figure 3.2(b)); they are called abutting and non-abutting quartered Aztec diamonds and the numbers of their domino tilings are denoted by $K_a(n)$ and $K_{na}(n)$, respectively.

Denote by $Q(n)$ and $H(n)$ the number of $\langle r\rangle$-invariant and $\langle r^2\rangle$-invariant matchings of the Aztec diamond of order n, respectively (this notation is motivated by the words "quarter-turn" and "half-turn," which describe the corresponding symmetries). Using the factorization theorem, we obtain the following results.

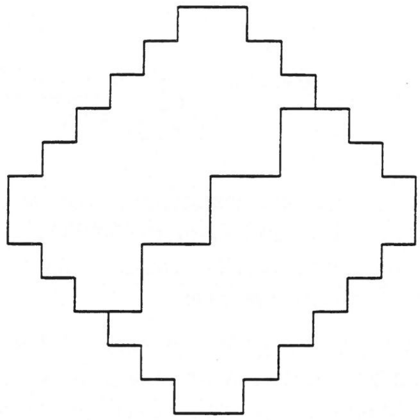

FIGURE 3.1

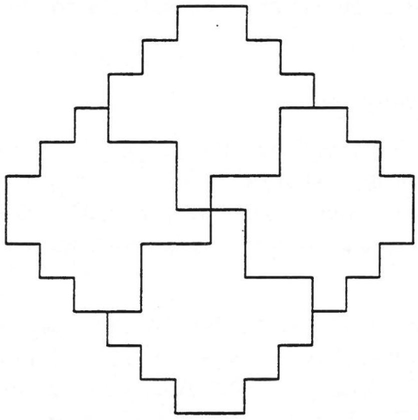

FIGURE 3.2(a)

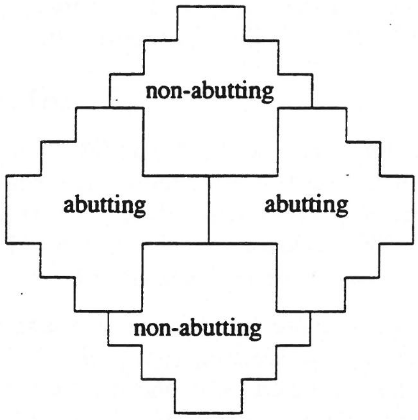

FIGURE 3.2(b)

THEOREM 3.1. *For all $n \geq 1$, we have*

$$Q(n) = 2^{\lfloor (n+1)/4 \rfloor} R(n).$$

Therefore, by [JP, Theorem 1] we obtain that $Q(n) = 0$ for n congruent to 1 or 2 (mod 4), and

$$Q(4k) = 2^k Q(4k-1) = 2^{k(3k-1)/2} \prod_{1 \leq i < j \leq k} \frac{2i+2j-1}{i+j-1}.$$

THEOREM 3.2. *For all $n \geq 1$, we have*

$$H(n) = 2^{\lfloor (n+1)/2 \rfloor} K_a(n) K_{na}(n).$$

Therefore, by [JP, Theorem 1] we obtain that

$$H(4k) = 2^{2k} H(4k-1) = 2^{k(3k-1)} \prod_{1 \leq i < j \leq k} \frac{2i+2j-3}{i+j-1} \prod_{1 \leq i \leq j \leq k} \frac{2i+2j-1}{i+j-1},$$

and

$$H(4k-2) = 2^{2k-1}H(4k-3) = 2^{3k^2-4k+2} \prod_{1\leq i<j\leq k} \frac{2i+2j-3}{i+j-1} \prod_{1\leq i\leq j\leq k-1} \frac{2i+2j-1}{i+j-1}.$$

The matchings of the Aztec diamond are strongly connected to the alternating sign matrices introduced by Mills, Robbins and Rumsey [**MRR1**] (see [**EKLP**] and [**C1**]). In fact, using the ideas of [**C1**], we can phrase the results of the two theorems above in terms of symmetry classes of alternating sign matrices.

To be specific, define the 2-count of a symmetry class of alternating sign matrices as the sum of the weights of the matrices of that symmetry class, where each matrix is weighted by 2 raised to the number of orbits of -1's. Then Theorem 3.1 provides a product formula for the 2-count of quarter-turn invariant alternating sign matrices, a result that appears to be new. On the other hand, Theorem 3.2 is then equivalent to a result stated (but not proved) by Robbins [**R2**, Theorem 3.4].

4. Further applications

Consider a $(2m+1)\times(2n+1)$ chessboard with black corners. The graph whose vertices are the white squares and whose edges connect precisely those pairs of white squares that are diagonally adjacent is called the *even Aztec rectangle* of order $m\times n$, and is denoted $ER_{m,n}$. The analogous graph constructed on the black vertices is denoted $OR_{m,n}$ and is called the *odd Aztec rectangle* of order $m\times n$.

Let $k(G)$ denote the number of spanning trees of the graph G. Stanley conjectured that $k(ER_{n,n}) = 4k(OR_{n,n})$ for all $n\geq 1$. This has been recently proved by Knuth by an algebraic method, who showed that in fact $k(ER_{m,n}) = 4k(OR_{m,n})$ for all $m,n\geq 1$. A combinatorial proof of this is given in [**C2**]. It turns out that the factorization theorem (together with a construction of Temperley that relates spanning trees to perfect matchings) can be used to obtain a surprising geometric proof in case m and n are odd. This proof also affords a weighted version of this result, with mn free parameters for the rectangles of order $(2m+1)\times(2n+1)$.

Finally, we present three families of graphs whose matchings we enumerate using the factorization theorem. First, the "holey Aztec diamond" considered by Jockusch and Propp is obtained by removing the central four vertices of the Aztec diamond. Jockusch conjectured formulas for the number of its matchings, formulas that have been recently proved [**Prop**]. The factorization theorem yields a short proof in the case when the order of the diamond is congruent to 2 or 3 modulo 4.

Second, we consider "holey Aztec rectangles," i.e., subgraphs of Aztec rectangles obtained by removing a suitable number of vertices from one of the symmetry axes of the rectangle. Using the factorization theorem and ideas from [**MRR1**], we enumerate their perfect matchings.

And third, we consider "quasi-quartered Aztec diamonds," planar regions that differ slightly from the quartered Aztec diamonds defined in the previous section, and we prove recurrences satisfied by the number of their domino tilings. We obtain that these numbers are powers of 2.

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