The Terwilliger Algebra of an Almost-Bipartite Graph and its Antipodal Cover

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Abstract

A distance-regular graph $\mathcal{G}=(\mathcal{X},\mathcal{E})$ with diameter $\mathcal{D}$ is said to be *almost-bipartite* if the intersection numbers satisfy $a_i(\mathcal{G})=0$ $(0\leq i\leq \mathcal{D}-1)$ and $a_{\mathcal{D}}(\mathcal{G})\neq 0$. In this case, we define a new graph, G=(X,E) by

$X = \mathcal{X}^+ \cup \mathcal{X}^-$, where $\mathcal{X}^+$ and $\mathcal{X}^-$ are two copies of $\mathcal{X}$,

$E = \{x^+y^- \mid xy \in \mathcal{E}\}$.

The graph G is a bipartite antipodal 2-cover with diameter $D=2\mathcal{D}+1$, and its quotient is $\mathcal{G}$. We investigate the relation between the Terwilliger algebras and their modules structures of two graphs related in this way.

Résumé

Un graphe $\mathcal{G}=(\mathcal{X},\mathcal{E})$ à distance-réguliere de diametre $\mathcal{D}$ est *presque bi-partie* si les nombres d'intersections est tel que $a_i(\mathcal{G})=0$ $(0\leq i\leq \mathcal{D}-1)$ et $a_{\mathcal{D}}(\mathcal{G})\neq 0$. Dans ce cas on définit un nouveau graphe G=(X,E) par

$X = \mathcal{X}^+ \cup \mathcal{X}^-$, $\mathcal{X}^+$ et $\mathcal{X}^-$ sont deux copies de $\mathcal{X}$,

$E = \{x^+y^- \mid xy \in \mathcal{E}\}$.

Le graphe G est une bi-partie antipodale graphe qui est un recouvrement de degré 2, de diametre $D=2\mathcal{D}+1$, et son quotient est $\mathcal{G}$. Pour deux tels graphes, nous êtudions la relation entre leurs algebras de Terwilliger et leurs structures modulaires.

Extended Abstract

Let G=(X,E) be a distance-regular graph with dameter D. Define the *adjacency matrix* A of G to be the |X|x|X| matrix with rows and columns indexed by X, and yz entry:

$$(A)_{yz} = \begin{cases} 1 & \text{if } \partial(y,z)=1 \\ 0 & \text{otherwise} \end{cases} \qquad (x,y \in X).$$

Fix a vertex $x \in X$. Define the *dual idempotents* $E_0^*, E_1^*, \ldots, E_D^*$ of G with respect to x to be the |X|x|X| diagonal matrices with rows and columns indexed by X, and yy entry:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y)=i \\ 0 & \text{otherwise} \end{cases} \qquad (y \in X).$$

The *Terwilliger Algebra* T= T(x) is the subalgebra of $Mat_X(\mathbb{R})$ generated by A, E_0^*, $E_1^*, \ldots, E_D^*$. The *standard module* for G is the space $V=\mathbb{R}^{|X|}$ of column vectors. For each $x \in X$, we denote by $\hat{x}$ the column vector with 1 in the x^{th} position, and 0 elsewhere.

An irreducible T-module W is said to be *thin* whenever dim $E_i^*W \leq 1$ for $0 \leq i \leq D$. The graph G is said to be *thin with respect to x* if every irreducible T-module is thin. Let W be a thin irreducible T-module. By the *endpoint* of W, we mean the integer $r=\min\{i \mid E_i^*W \neq 0\}$, and by the *diameter* of W we mean the integer $d=|\{i \mid E_i^*W \neq 0\}| - 1$. Let $a_i=a_i(W)$ denote the eigenvalue of $E_{r+i}^*AE_{r+i}^*$ on E_{r+i}^*W, and let $x_i=x_i(W)$ denote the eigenvalue of $E_{r+i-1}^*AE_{r+i}^*AE_{r+i-1}^*$ on E_{r+i-1}^*W. It is known that the isomorphism class of W as a T-module is determined by r, d and $\{a_i, x_i \mid 0 \leq i \leq d\}$.

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$X = \mathcal{X}^+ \cup \mathcal{X}^-$, where $\mathcal{X}^+$ and $\mathcal{X}^-$ are two copies of $\mathcal{X}$,

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The graph G is a bipartite antipodal 2-cover with diameter $D=2\mathcal{D}+1$, and its quotient is $\mathcal{G}$.

Let $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be an almost-bipartite distance-regular graph with diameter $\mathcal{D}$, and let G=(X,E) denote the bipartite antipodal 2-cover of $\mathcal{G}$. Let $\pi: X \to \mathcal{X}$ denote the quotient map.

Let Let ψ be the matrix with rows indexed by the elements of $\mathcal{X}$, and columns indexed by the elements of X, and

$$(\psi)_{yz} = \begin{cases} 1 & \text{if } \pi z = y \\ 0 & \text{otherwise.} \end{cases}$$

Note that ψ acts as a map $V \to \mathcal{V}$, and that

$$\psi(\hat{y}) = \hat{y}, \text{ where } \pi y = y.$$

We call ψ the *quotient transformation.*

Define matrices σ_n, σ_f with rows indexed by X and columns indexed by $\mathcal{X}$ by:

$$(\sigma_n)_{yz} = \begin{cases} 1 & \text{if } \pi y = z \text{ and } \partial(x,y) \leq \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\sigma_f)_{yz} = \begin{cases} 1 & \text{if } \pi y = z \text{ and } \partial(x,y) \geq \mathcal{D}+1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that σ_n and σ_f act as maps $\mathcal{V} \to V$ as follows. Let $y \in \mathcal{X}$. Let y, y' be the antipodal vertices of X such that $\pi(y) = \pi(y') = y$. Assume that $\partial(x,y) < \partial(x,y')$. Then

$$\sigma_n(\hat{y}) = \hat{y}$$

and

$$\sigma_f(\hat{y}) = \hat{y}'.$$

We call σ_n and σ_f the *near and far transformations with respect to x.*

We have the following results:

Theorem 1 Let $G=(X,E)$ be an antipodal 2-cover with odd diameter $D=2\mathcal{D}+1$. Let $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be the quotient graph. Fix $x \in X$, and let $x = \pi(x)$. Let ψ be the quotient transformation, and let σ_n and σ_f be the near and far transformations with respect to x. Let $T=T(x)$, and let $\mathcal{T}=T(x)$.

(i) For any T-module W, $\psi(W)$ is a $\mathcal{T}$-module.

(ii) For any $\mathcal{T}$-module $\mathcal{W}$, $\sigma_n(\mathcal{W})+\sigma_f(\mathcal{W})$ is a T-module.

(iii) The maps $W \mapsto \psi(W)$ and $\mathcal{W} \mapsto \sigma_n(\mathcal{W})+\sigma_f(\mathcal{W})$ are inverse bijections between the set of T-modules and the set of $\mathcal{T}$-modules.

Theorem 2 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Let W be a T-module, and let $\mathcal{W}=\psi(W)$. The following are equivalent:

(i) W is irreducible.

(ii) $\mathcal{W}$ is irreducible.

Theorem 3 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Let W,W' be T-modules. Let $\mathcal{W}=\psi(W)$, and let $\mathcal{W}'=\psi(W')$. The following are equivalent:

(i) W and W′ are orthogonal.

(ii) $\mathcal{W}$ and $\mathcal{W}$ are orthogonal.

Theorem 4 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Let W,W' be T-modules. Let $\mathcal{W}=\psi(W)$, and let $\mathcal{W}'=\psi(W')$. The following are equivalent:

(i) W and W′ are isomorphic as T-modules.

(ii) $\mathcal{W}$ and $\mathcal{W}$ are isomorphic as $\mathcal{T}$-modules.

Theorem 5 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Then G is thin with respect to x if and only if $\mathcal{G}$ is thin with respect to χ.

Theorem 6 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Let W be a thin, irreducible T-module with diameter d. Let $\mathcal{W}=\psi(W)$. Let $\Omega = \{\theta \mid \theta$ is an eigenvalue for $A|_W\}$. Let $\Phi = \{\theta \mid \theta$ is an eigenvalue for $\mathcal{A}|_{\mathcal{W}}\}$.

Suppose the elements of Ω_W are $\theta_0>\theta_1>\ldots>\theta_d$. Then exactly one of the following holds

(i) $\Phi = \{\theta_0,\theta_2,\ldots,\theta_{d-1}\}$

(ii) $\Phi = \{\theta_1,\theta_3,\ldots,\theta_d\}$.

Corollary 7 Let $G=(X,E)$, $\mathcal{G}=(\mathcal{X},\mathcal{E})$, W, $\mathcal{W}$, Ω, and Φ be as above.

Then $\Omega = \Phi \cup -\Phi$, where $-\Phi = \{-\theta \mid \theta\in \Phi\}$.

Theorem 8 Let $G=(X,E)$ and $\mathcal{G}=(\mathcal{X},\mathcal{E})$ be as above. Let W be a thin, irreducible T-module with endpoint r and diameter d. Let $\mathcal{W}=\psi(W)$. Then $\mathcal{W}$ is a thin, irreducible $\mathcal{T}$-module with endpoint r and diameter $\mathcal{D}$-r.

Let $x_i= x_i(W)$ and $a_i=a_i(W)$, $(0\le i\le d)$. Let $\chi_i = x_i(\mathcal{W})$ and $\alpha_i=a_i(\mathcal{W})$, $(0\le i\le \mathcal{D}-r)$. Then:

(i) $\alpha_i=0$ $\qquad (0\le i\le \mathcal{D}-r-1)$

$\alpha_{\mathcal{D}-r}{}^2 = x_{\mathcal{D}-r+1}$

(ii) $\chi_i = x_i$ $\qquad (1\le i\le \mathcal{D}-r)$.

By Theorem 4, the sign of $\alpha_{\mathcal{D}-r}$ is determined by the isomorphism class of W. We will say that W has *positive type* if $\alpha_{\mathcal{D}-r}>0$, and that W has *negative type* if $\alpha_{\mathcal{D}-r}<0$. The next theorem gives criteria to determine whether a given W has positive or negative type.

Theorem 9 Let G=(X,E), $\mathcal{G}=(\mathcal{X},\mathcal{E})$, and W be as above. Let $\Omega = \{\theta \mid \theta$ is an eigenvalue for $A|_W\}$. Let $\Phi = \{\theta \mid \theta$ is an eigenvalue for $\mathcal{A}|_{\mathcal{W}}\}$.

The following are equivalent:

(i) W has positive type.

(ii) $\sum_{\theta \in \Phi} \theta > 0$.

(iii) $\Phi = \{\theta_0, \theta_2, \ldots, \theta_{d-1}\}$ (where the elements of Ω are indexed so that $\theta_0 > \theta_1 > \ldots > \theta_d$.)

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