2-Homogeneous Bipartite Distance-regular Graphs

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Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 2$. For $x \in X$, write

$$\Gamma_i(x) = \{y \in X | \partial(x,y) = i\} \qquad (0 \leq i \leq D).$$

A bipartite distance-regular graph $\Gamma = (X, R)$ of diameter $D > 2$ and valency $k > 2$ is said to be *2-homogeneous* if for all integers i $(1 \leq i \leq D-1)$ and all x, y, $z \in \Gamma_i(x)$ with y, $z \in \Gamma_i(x)$ and $\partial(y,z) = 2$ the scalar

$$\gamma_i = |\Gamma_{i-1}(x) \cap \Gamma_1(z) \cap \Gamma_1(y)|$$

is independent of the choice of x, z and y.

Our main result is the following.

Theorem 1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Then the following are equivalent.*

(i) *Γ is 2-homogeneous.*

(ii) *For all integers i $(1 \leq i \leq D-1)$*

$$(b_{i-1} - 1)(c_{i+1} - 1) = p^i_{2i}.$$

(iii) *Either Γ is the k-cube, or there exists a real, nonzero scalar q such that for all i $(0 \leq i \leq D)$*

$$c_i = \frac{(q^D + q^2)(q^{2i} - 1)}{(q^D + q^{2i})(q^2 - 1)},$$
$$b_i = \frac{(q^D + q^2)(q^D - q^{2i-D})}{(q^D + q^{2i})(q^2 - 1)}.$$

(iv) *There exists a nontrivial eigenvalue θ of Γ such that*

$$(\mu - 1)\theta^2 = (k - \mu)(k - 2).$$

(v) *There exists a nontrivial primitive idempotent E of Γ and there exist $x, y \in X$ with $\partial(x, y) = 2$ such that*

$$\sum_{z \in \Gamma_1(x) \cap \Gamma_1(y)} E\hat{z} \in \text{span}\{E\hat{x}, E\hat{y}\}.$$

(vi) *There exists a nontrivial primitive idempotent E of Γ such that for all $x, y \in X$ and all i, j $(0 \leq i \leq D)$*

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \text{span}\{E\hat{x} + E\hat{y}\}.$$

(vii) *Γ has a Q-polynomial structure for which $a_i^* = 0$ $(0 \leq i \leq D - 1)$.*

We can say more about the Q-polynomial structure of the graphs described in Theorem 1.

Corollary 2 *Suppose the equivalent conditions of Theorem 1 hold. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of Γ, and let E_i denote the primitive idempotent of Γ associated with θ_i $(0 \leq i \leq D)$.*

(i) *If D is odd, then $E_0, E_1, \ldots, E_D$ is the unique Q-polynomial ordering. If D is even, then $E_0, E_1, \ldots, E_0$ and $E_{D-1}, E_2, E_{D-3}, \ldots$ are the unique Q-polynomial orderings.*

(ii) *With respect to any Q-polynomial ordering*

$$p_{ij}^h = q_{ij}^h \qquad (0 \leq h, i, j \leq D).$$

(iii) *With respect to any Q-polynomial ordering $a_i^* = 0$ $(0 \leq i \leq D)$.*