Symplectic Analogs to L(m,n)

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The Gaussian poset L(m,n) *is the distributive lattice of all* m×n *partitions ordered by inclusion of their Ferrers diagrams. We define two families of distributive sublattices of* L(k,2n-k), *the symplectic lattices* C_{DeC}(n,k) *and* C_{KN}(n,k). *These are formed by imposing certain orders on the one-column tableaux of De Concini and of Kashiwara and Nakashima. We describe their posets of join irreducibles and relate the KN lattices to lattices of certain partitions of Andrews. We have shown that these symplectic lattices are rank symmetric, rank unimodal, and strongly Sperner (thus confirming a conjecture of Reiner and Stanton), and we remark on the representation theoretic techniques used to demonstrate this result.*

1. Introduction

Although this paper is primarily combinatorial, its motivation is to provide a nice environment for solving a problem from representation theory. It is well known that the lattices L(k,m-k) can be used to explicitly present the fundamental representations of $sl_m(\mathbb{C})$. Proctor conjectured that there should be analogous lattices for the symplectic Lie algebra $sp_{2n}(\mathbb{C})$. We have found two such families of symplectic lattices and have been able to construct fundamental representations of $sp_{2n}(\mathbb{C})$ on each family.

The symplectic lattices we construct here can be viewed as certain sets of partitions ordered in the usual way, but our construction will also emphasize circle diagrams. These were developed by Sheats to relate the symplectic tableaux of King and of De Concini [Sh]. Circle diagrams are convenient for describing "admissibility" conditions on tableaux and are crucial in the interval analysis we use to study representations on symplectic lattices. We will first describe L(k,m-k) using circle diagrams, and then we will present circle diagrams for the De Concini and KN symplectic lattices.

2. Circle Diagrams

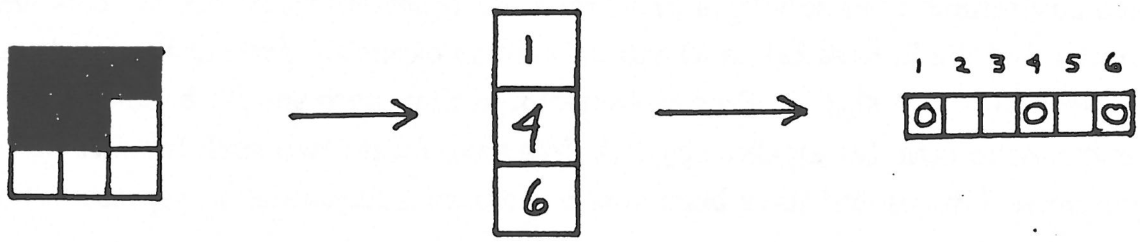
Fix $1 \leq k \leq m-k$.

We begin by associating a one-column tableau and a circle diagram to each $k \times (m-k)$ partition in $L(k,m-k)$. Recall that a column is *semistandard* if the entries strictly increase from top to bottom.

Let μ be a partition in $L(k,m-k)$. We form a semistandard column $T(\mu)$ by reversing and strictifying μ. The i^{th} entry in the column T is given by $T_i = \mu_{k+1-i} + i$. The entries for T are taken from the totally ordered set $\{1 < 2 < \ldots < m\}$. Then $L(k,m-k)$ is the set of semistandard columns of length k and with entries from $\{1, \ldots, m\}$, ordered componentwise.

We can view one of these columns as a circle diagram as follows. Construct a $1 \times m$ grid and label the squares from 1 to m, left to right. Form the circle diagram $t(\mu)$ by placing circles in the squares corresponding to the elements of T.

Example: $m = 6$, $k = 3$



The partition μ, the column $T(\mu)$, and the circle diagram $t(\mu)$ will all refer to the same object in $L(k,m-k)$.

Recently, Reiner and Stanton have used certain partitions of Andrews as a generating set for certain differences of principally specialized Schur functions. Their goal is to show that these polynomials are symmetric and unimodal with non-negative coefficients [RS]. Following [RS], a partition μ in $L(k,m-k)$ is *Andrews* if $\mu_i - \mu'_i \leq m - 2k$ for $1 \leq i \leq r$, where r^2 is the size of the Durfee square of μ. Define Andrews$(k,m-k)$ to be the distributive lattice of Andrews partitions in $L(k,m-k)$. For m even, we relate these Andrews lattices to the KN symplectic lattices defined in section 4.

3. De Concini Symplectic Lattices

Fix $1 \leq k \leq n$.

The following is a set of symplectic columns developed by De Concini with entries taken from the totally ordered set $\{\bar{n} < \ldots < \bar{1} < 1 < \ldots < n\}$.

$C_{DeC}(n,k) = \{T = (T_1, \ldots, T_k) : T$ satisfies (a) and (b) below$\}$

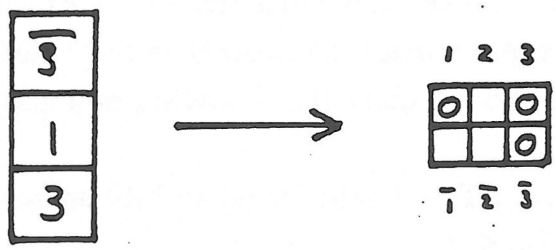
(a) $\bar{n} \leq T_1 < \ldots < T_k \leq n$

(b) If $T_p = \bar{r}$ and $T_q = r$, where $1 \leq r \leq n$, then $q - p + 1 \leq r$.

When these columns are ordered componentwise, we have shown that $C_{DeC}(n,k)$ becomes a distributive lattice.

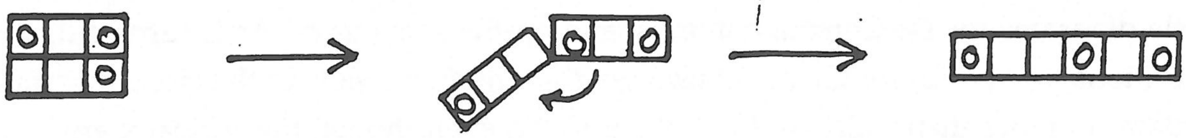
Sheats developed the following circle diagrams to compare certain symplectic tableaux [Sh]. Construct a $2\times n$ grid and label the top row of squares from 1 to n, left to right. Label the bottom row of squares from $\bar{1}$ to $\bar{n}$, left to right. The i^{th} *slot* of the $2\times n$ grid is the pair of squares labeled i and $\bar{i}$. We can view a De Concini column T as a circle diagram by placing circles in the squares corresponding to elements of T.

<u>Example:</u> $n = 3,\ k = 3$



In the language of circle diagrams, conditions (a) and (b) above say that the first r slots of the circle diagram contain no more than r circles, for $1 \leq r \leq n$.

Now if we "unfold" the circle diagrams for De Concini columns as in the following example



then we can view $C_{DeC}(n,k)$ as a distributive sublattice of L(k,2n-k). We say that $\mu \in$ L(k,2n-k) (as a partition, column, or circle diagram) is *De Concini-admissible* if μ is contained in the De Concini sublattice of L(k,2n-k).

Next we translate De Concini-admissibility into the language of partitions. For a partition $\mu \in$ L(k,2n-k), let r^2 be the size of the Durfee square of μ. Also let $\tilde{\mu}$ be the "middle" of μ, formed by removing the first n-k and the last n-k columns of the Ferrers diagram of μ. More precisely, $\tilde{\mu}$ is the $k \times k$ partition whose conjugate $\tilde{\mu}'$ is given by $\tilde{\mu}' = (\mu'_{n-k+1}, \ldots, \mu'_n)$.

<u>**Proposition 1:**</u>

(1) When k = n, then $\mu \in$ L(k,k) is DeC-admissible if and only if $\mu'_i \leq \mu_i$, for $1 \leq i \leq r$.

(2) When k < n, then $\mu \in$ L(k,2n-k) is DeC-admissible if and only if $\tilde{\mu}$ is DeC-admissible in L(k,k). □

4. KN Symplectic Lattices

Again fix $1 \leq k \leq n$.

The following is a set of symplectic tableaux used by Kashiwara and Nakashima in their crystal graph constructions, with entries taken from the set $\{\bar{1}, \ldots, \bar{n}, n, \ldots, 1\}$ [KN]. The key difference from the De Concini case is that we use a different total order on these entries (equivalent to the order given in [KN]): $\bar{1} < \ldots < \bar{n} < n < \ldots < 1$. So we make the following definition:

$C_{KN}(n,k) = \{T = (T_1, \ldots, T_k) :$ T satisfies (a) and (b) below$\}$

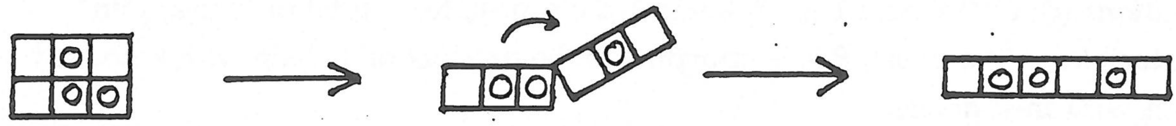
(a) $\bar{1} \leq T_1 < \ldots < T_k \leq 1$

(b) If $T_p = \bar{r}$ and $T_q = r$, where $1 \leq r \leq n$, then $p + k - q + 1 \leq r$.

When these columns are ordered componentwise, we have shown that $C_{KN}(n,k)$ becomes a distributive lattice.

The circle diagrams for KN columns are constructed in the same way as the circle diagrams for De Concini columns, and on the same grid. As it turns out, conditions (a) and (b) for $C_{KN}(n,k)$ also say that the first r slots of the circle diagram contain no more than r circles, for $1 \leq r \leq n$. So even though the tableaux are formed using different total orders, the circle diagrams are the same.

Now if we "unfold" the circle diagrams for KN columns as in the following example



then we can view $C_{KN}(n,k)$ as a distributive sublattice of $L(k,2n-k)$. This unfolding is different from the De Concini case and respects the total order $\overline{1} < \ldots < \overline{n} < n < \ldots < 1$. We say that $\mu \in L(k,2n-k)$ (as a partition, column, or circle diagram) is *KN-admissible* if μ is contained in the KN sublattice of $L(k,2n-k)$.

Next, we translate KN-admissibility into the language of partitions, recalling the Andrews partitions defined in section 2.

<u>Proposition 2:</u>
$\mu \in L(k,2n-k)$ is KN-admissible if and only if μ is Andrews. □

So $C_{KN}(n,k)$ coincides with Andrews$(k,2n-k)$ when viewed as a sublattice of $L(k,2n-k)$.

How do $C_{DeC}(n,k)$ and $C_{KN}(n,k)$ compare? For $k = 1$, notice that $C_{KN}(n,1) = C_{DeC}(n,1) = L(1,2n-1)$. Propositions 1 and 2 together with conjugation show the following:

<u>Corollary 3:</u> (Case $k = n$)
$C_{KN}(k,k) \cong C_{DeC}(k,k)$ as distributive lattices. □

However, for $1 < k < n$, $C_{KN}(n,k)$ and $C_{DeC}(n,k)$ are in fact distinct. This can be seen by comparing the posets of join irreducibles in section 5.

5. Join Irreducibles

Recall that an element of a lattice is a join irreducible if it covers precisely one element in the lattice. Our notation is $x \to y$ if x is covered by y. Also, let L be a distributive lattice and let P be the (induced) subposet of join irreducibles in L. Then L is isomorphic to the poset of order ideals of P, i.e. $L \cong J(P)$.

The join irreducibles in $L(k,m-k)$ are easy to describe. Here P is the set of partitions $(c^r, 0^{k-r})$ where $1 \le r \le k$ and $1 \le c \le m-k$, for a total of $k(m-k)$ join irreducibles. As a poset, P is isomorphic to the product of a chain with k nodes by a chain with m-k nodes.

Now fix $1 \le k \le n$. We use the partition descriptions of admissibility to locate the join irreducibles in the symplectic lattices, and there are $k(2n-k)$ of them in all.

Proposition 4:

(1) When $k = n$, the join irreducibles for $C_{DeC}(k,k)$ are given by

$(c^r, 0^{k-r})$ $\quad 1 \le r \le k$ and $1 \le c \le k$ and $r \le c$

$(r^c, c^{r-c}, 0^{k-r})$ $\quad 1 \le r \le k$ and $1 \le c \le k$ and $c < r$.

(2) When $k \le n$, the join irreducibles for $C_{DeC}(n,k)$ are given by

$(c^r, 0^{k-r})$ $\quad 1 \le r \le k$ and $1 \le c \le n-k$

$(c^r, 0^{k-r})$ $\quad 1 \le r \le k$ and $n-k+1 \le c \le n$ and $r \le c-n+k$

$((n-k+r)^{c-n+k}, c^{n-k+r-c}, 0^{k-r})$ $\quad 1 \le r \le k$ and $n-k+1 \le c \le n$ and $c-n+k < r$

$(c^r, 0^{k-r})$ $\quad 1 \le r \le k$ and $n+1 \le c \le 2n-k$.

(3) When $k \le n$, the join irreducibles for $C_{KN}(n,k)$ are given by

$(c^r, 0^{k-r})$ $\quad 1 \le r \le k$ and $1 \le c \le 2n-k$ and $c-r \le 2n-2k$

$(c^r, r^{c-r-(2n-2k)}, 0^{2n-k-c})$ $\quad 1 \le r \le k$ and $1 \le c \le 2n-k$ and $2n-2k < c-r$. □

Each poset of join irreducibles will include some relations in addition to the usual relations in a $k\times(2n-k)$ product of chains. To write down the covering relations in these posets, we view each poset as sitting on a $k\times(2n-k)$ arrangement of vertices. Edges are added to the $k\times(2n-k)$ product of chains to account for the additional relations, while redundant edges are erased.

Proposition 5: Poset of join irreducibles for $C_{DeC}(k,k)$

For $1 \le r \le k$ and $1 \le c \le k$, identify the pair (r,c) with the appropriate join irreducible in $C_{DeC}(k,k)$ from Proposition 4 (1).

Then in the poset of join irreducibles, $(r,c) \to (s,d)$ if and only if one of the following holds:

(1) $d = c+1$ with $s = r$
(2) $s = c$ and $d = r$ with $r < c$
(3) $s = r+1$ with $d = c$ and either $c < r$ or $r+1 < c$. □

Since $C_{DeC}(k,k) \cong C_{KN}(k,k)$, Proposition 5 accounts for the poset of join irreducibles for $C_{KN}(k,k)$ as well.

__Proposition 6:__ Poset of join irreducibles for $C_{DeC}(n,k)$

Let $k < n$, and for $1 \leq r \leq k$ and $1 \leq c \leq 2n-k$, identify the pair (r,c) with the appropriate join irreducible in $C_{DeC}(n,k)$ from Proposition 4 (2).

Let $B = \{ (r,c) : 1 \leq r \leq k \text{ and } 1 \leq c \leq n-k\}$
$M = \{ (r,c) : 1 \leq r \leq k \text{ and } n-k+1 \leq c \leq n\}$
$T = \{ (r,c) : 1 \leq r \leq k \text{ and } n+1 \leq c \leq 2n-k\}$

Then in the poset of join irreducibles, $(r,c) \rightarrow (s,d)$ if and only if one of the following holds:

(1) $(r,c) \in B$ and $(s,d) \in B$ and either $s = r+1$ and $d = c$
or $s = r$ and $d = c+1$
(2) $(r,c) \in T$ and $(s,d) \in T$ and either $s = r+1$ and $d = c$
or $s = r$ and $d = c+1$
(3) $(r,c) \in M$ and $(s,d) \in M$ and if we identify $(r,c-n+k)$ and $(s,d-n+k)$ with join irreducibles in $C_{DeC}(k,k)$, then $(r,c-n+k) \rightarrow (s,d-n+k)$ in the poset of join irreducibles for $C_{DeC}(k,k)$
(4) $(r,c) \in B$ and $(s,d) \in M$ and $c = n-k$ and $d = n-k+1$ and $r = s$
(5) $(r,c) \in M$ and $(s,d) \in T$ and $c = n$ and $d = n+1$ and $r = s$. □

B ("bottom") and T ("top") should be thought of as portions of the poset of join irreducibles for $C_{DeC}(n,k)$ that retain the usual edge relations in the $k \times 2n-k$ product of chains. M should be thought of as the "middle" of the poset of join irreducibles for $C_{DeC}(n,k)$. Proposition 6 says that M looks like the poset of join irreducibles for $C_{DeC}(k,k)$.

__Proposition 7:__ Poset of join irreducibles for $C_{KN}(n,k)$

Let $k < n$, and for $1 \leq r \leq k$ and $1 \leq c \leq 2n-k$, identify the pair (r,c) with the appropriate join irreducible in $C_{KN}(n,k)$ from Proposition 4 (3).

Then in the poset of join irreducibles, $(r,c) \rightarrow (s,d)$ if and only if one of the following holds:

(1) $s = r+1$ with $d = c$
(2) $d = c+1$ with $r = s$
(3) $s = c$ and $d = 2n-2k+r$, where $c < r$. □

In this case the poset of join irreducibles should be thought of as the $k \times 2n-k$ product of chains together with the additional edges from condition (3) of the proposition.

6. Rank Symmetry, Rank Unimodality, and the Strong Sperner Property

Tableaux are often used to index a weight basis for an irreducible representation of a semisimple Lie algebra. Often such tableaux can be organized into ranked posets in such a way that the rank symmetry and rank unimodality of the poset follow from a general result due to Dynkin. He showed that vectors of a weight basis for an irreducible representation of a semisimple Lie algebra form a symmetric and unimodal arrangement according to their weight (see [St]). For example, the one-column De Concini and KN symplectic tableaux index a weight basis for fundamental representations of $sp_{2n}(\mathbb{C})$. Within each symplectic lattice, the poset rank respects a certain Lie theoretic weight and so we can apply Dynkin's result.

A ranked poset that is rank symmetric, rank unimodal, and strongly Sperner is said to be Peck. Proctor has shown that a ranked poset is Peck if and only if it "carries" a representation of $sl_2(\mathbb{C})$ in certain sense [Pr]. Recall that $sl_2(\mathbb{C})$ is generated by X, Y and H, satisfying certain relations. Proctor's condition requires that X acts by moving up in the poset, Y acts by moving down, and H acts by staying at the same level.

Now a semisimple Lie algebra is built up out of copies of $sl_2(\mathbb{C})$, and together the generators $\{X_i, Y_i, H_i\}$ for these copies of $sl_2(\mathbb{C})$ are called *Chevalley generators* for the Lie algebra. An "explicit construction" of a representation of a semisimple Lie algebra is a rule specifying how each Chevalley generator acts on each vector of a weight basis.

In order to produce explicit constructions of fundamental representations of $sp_{2n}(\mathbb{C})$, we first "color" the edges of the symplectic lattices. Each color corresponds to a particular copy of $sl_2(\mathbb{C})$. Our convention is that X_i acts by moving up in the lattice along edges colored i, Y_i acts by moving down in the lattice along edges colored i, and H_i acts by staying at the same level. Once the lattices are colored, the challenge is to determine the coefficients for the actions of X_i, Y_i, and H_i. We accomplish this by analyzing colored connected components (or intervals) of the lattices. As a corollary we can conclude that $C_{KN}(n,k)$ and $C_{DeC}(n,k)$ are both Peck, since the principal three-dimensional embedding of $sl_2(\mathbb{C})$ in $sp_{2n}(\mathbb{C})$ will act upon the lattices in the way required by Proctor.

Reiner and Stanton conjectured that the lattice Andrews(k,2n-k) is Peck [RS], and we have confirmed this conjecture by noting that Andrews(k,2n-k) = $C_{KN}(n,k)$.

7. References

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