Algebraic Shifting and Sequentially Cohen-Macaulay Simplicial Complexes

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Summary.

Björner and Wachs recently generalized the definition of shellability by dropping the assumption of purity; they also introduced the *h-triangle*, a doubly-indexed generalization of the h-vector which is combinatorially significant for shellable (nonpure) complexes. Stanley subsequently defined a (nonpure) simplicial complex to be *sequentially Cohen-Macaulay* if it satisfies algebraic conditions that generalize the (pure) Cohen-Macaulay conditions, so that a shellable (nonpure) complex is sequentially Cohen-Macaulay.

We show that algebraic shifting preserves the h-triangle of a simplicial complex K if and only if K is sequentially Cohen-Macaulay. This generalizes a result of Kalai's for pure Cohen-Macaulayness. Immediate consequences include that shellable (nonpure) complexes and sequentially Cohen-Macaulay complexes have the same set of possible h-triangles.

Pure complexes and nonpure generalizations.

A simplicial complex is pure if all of its facets (maximal faces, ordered by inclusion) have the same dimension. Cohen-Macaulayness, algebraic shifting, shellability, and the h-vector are significantly interrelated for pure simplicial complexes. We will be concerned with extending some of these relations to nonpure complexes, but first, we briefly review the pure case.

A simplicial complex is Cohen-Macaulay if its face-ring is a Cohen-Macaulay ring (an algebraic property), or, equivalently, if the complex satisfies certain topological conditions (see, *e.g.*, [St3, St6]). In particular, the complex must be pure. A pure simplicial complex is shellable if it can be constructed one facet at a time, subject to certain conditions (see, *e.g.*, [Bj1, BW1]). A shellable (pure) complex is Cohen-Macaulay, and the h-vector of a Cohen-Macaulay or shellable (pure) complex has natural combinatorial interpretations.

Algebraic shifting is a procedure that defines, for every simplicial complex K, a new complex $\Delta(K)$ with the same h-vector as K and a nice combinatorial structure ($\Delta(K)$ is shifted). Additionally, algebraic shifting preserves many algebraic and topological properties of the original complex, including Cohen-Macaulayness; a simplicial complex is Cohen-Macaulay if and only if $\Delta(K)$ is Cohen-Macaulay, which, in turn, holds if and only if $\Delta(K)$ is pure. Thus, it is easy to tell whether K is Cohen-Macaulay, if $\Delta(K)$ is known. (See, *e.g.*, [BK1, BK2].)

Now we are ready for the nonpure case.

Björner and Wachs' recent generalization of shellability to nonpure simplicial complexes, made by simply dropping the assumption of purity [BW2], generated a great deal of interest, and sparked the generalization of several other related concepts [SWa, SWe, BS, DR]. In particular, Stanley introduced sequential Cohen-Macaulayness [St6, Section III.2], a nonpure generalization of Cohen-Macaulayness, and designed the (algebraic) definition so that a shellable (nonpure) complex is sequentially Cohen-Macaulay, much as a shellable (pure) complex is (pure) Cohen-Macaulay. Meanwhile, joint work with L. Rose [DR] shows that algebraic shifting preserves the h-triangle (a non-pure generalization of the h-vector) of shellable (nonpure) complexes. These developments prompted A. Björner (private communication) to ask, "Does algebraic shifting preserve sequential Cohen-Macaulayness?" and "Does algebraic shifting preserve the h-triangle of sequentially Cohen-Macaulay simplicial complexes?"

Shifted complexes are shellable and hence sequentially Cohen-Macaulay, so $\Delta(K)$ is always sequentially Cohen-Macaulay. Thus, the "obvious" generalization, "K is sequentially Cohen-Macaulay if and only if $\Delta(K)$ is sequentially Cohen-Macaulay," is trivially false. Björner's first question may be restated as, "Can one use $\Delta(K)$ to tell if a simplicial complex K is sequentially Cohen-Macaulay?"

Our main result is to answer both of Björner's questions simultaneously, by showing that algebraic shifting preserves the h-triangle of a simplicial complex if and only if the complex is sequentially Cohen-Macaulay (Theorem 4). Two immediate corollaries, one involving shellability and the other a nonpure generalization of homology Betti numbers, follow.

f-triangle and h-triangle.

A **simplicial complex** K is a collection of finite sets (called faces) such that $F \in K$ and $G \subseteq F$ together imply that $G \in K$. We allow K to be the empty simplicial complex $\emptyset$ consisting of no faces, or the simplicial complex $\{\emptyset\}$ consisting of just the empty face, but we do distinguish between these two cases. A **subcomplex** of K is a subset of faces $L \subseteq K$ such that $F \in L$ and $G \subseteq F$ imply $G \in L$. A subcomplex is a simplicial complex in its own right. An **order filter** of K is a subset of faces $J \subseteq K$ such that $F \in J$ and $F \subseteq G \in K$ imply $G \in J$.

The **dimension** of a face $F \in K$ is $\dim F = |F| - 1$, and the **dimension** of K is $\dim K = \max\{\dim F\colon F \in K\}$. The maximal faces of K (under the set inclusion partial order) are called **facets**, and K is **pure** if all the facets have the same dimension.

Following [BW2], we define the **degree** of a face $F \in K$ to be $\deg_K F = \max\{|G|\colon F \subseteq$

$G \in K\}$. We further define the **degree** of K to be $\deg K = \min\{\deg_K F \colon F \in K\}$. Note that K is pure if and only if all the faces have the same degree.

Björner and Wachs [BW2, Definition 2.8] define the subcomplex

$$K^{(r,s)} = \{F \in K \colon \dim F \leq s,\ \deg_K F \geq r+1\}$$

for $-1 \leq r, s \leq \dim K$. We may extend this by defining $K^{(r,s)}$ to be the empty simplicial complex when $r > \dim K$.

We will frequently make use of the following subcomplexes: $K^{(s)} = K^{(-1,s)}$, the s-**skeleton** of K; $K^{<r>} = K^{(r,\dim K)}$, the subcomplex of all faces of K whose degree is at least $r+1$ (equivalently, the subcomplex generated by all facets whose dimension is at least r); and $K^{(i,i)}$, the **pure** i**-skeleton**, the pure subcomplex generated by all i-dimensional faces. Another interpretation of $K^{(r,s)}$, then, is $K^{(r,s)} = (K^{<r>})^{(s)}$.

Let K_j denote the set of j-dimensional faces of K. Recall that the f**-vector** of K is the sequence $f(K) = (f_{-1}, \ldots, f_{d-1})$, where $f_j = f_j(K) = \#K_j$ and $d - 1 = \dim K$, and that the h**-vector** of K is the sequence $h(K) = (h_0, \ldots, h_d)$ where

$$h_j = \sum_{s=0}^{j} (-1)^{j-s} \binom{d-s}{j-s} f_{s-1} \qquad (0 \leq j \leq d). \tag{1}$$

Inverting equation (1) gives

$$f_j = \sum_{s=0}^{d} \binom{d-s}{j+1-s} h_s,$$

so knowing the h-vector of a simplicial complex is equivalent to knowing its f-vector.

Definition (Björner-Wachs [BW2, Definition 3.1]): Let K be a $(d-1)$-dimensional simplicial complex. Define

$$f_{i,j}(K) = \#\{F \in K \colon \deg_K F = i,\ \dim F = j-1\}.$$

The triangular integer array $(f_{i,j})_{0 \leq j \leq i \leq d}$ is the f**-triangle** of K. Further define

$$h_{i,j}(K) = \sum_{s=0}^{j} (-1)^{j-s} \binom{i-s}{j-s} f_{i,s}(K). \tag{2}$$

The triangular array $\mathbf{h} = (h_{i,j})_{0 \leq j \leq i \leq d}$ is the h**-triangle** of K. □

Inverting equation (2) gives

$$f_{i,j} = \sum_{s=0}^{i} \binom{i-s}{j+1-s} h_{i,s}, \tag{3}$$

so knowing the h-triangle of a simplicial complex is equivalent to knowing its f-triangle.

If K is a pure $(d-1)$-dimensional simplicial complex, then every face has degree d, so

$$f_{i,j}(K) = \begin{cases} f_{j-1}(K), & if\ i = d \\ 0, & if\ i \neq d \end{cases},$$

and similarly for the h's. Thus, when K is pure, the f-triangle and the h-triangle are zero except for the last row ($f_{d,\bullet}(K)$ or $h_{d,\bullet}(K)$), which consists of the f-vector or h-vector, respectively.

Clearly,

$$f_{j-1}(K^{<i-1>}) = \sum_{p=i}^{d} f_{p,j}(K) \tag{4}$$

for all $0 \leq j, i \leq d$. Inverting equation (4), we get

$$f_{i,j}(K) = f_{j-1}(K^{<i-1>}) - f_{j-1}(K^{<i>}) \tag{5}$$

for all $0 \leq j \leq i \leq d$; this is essentially the same idea as [BW2, equation (3.2)]. In the case $i = d$, equation (5) relies upon the tail condition $f_{j-1}(K^{<d>}) = f_{j-1}(\emptyset) = 0$.

Cohen-Macaulayness.

Cohen-Macaulayness is an important algebraic concept, but we will use the equivalent algebraic topological characterizations as our definitions. For all undefined topological terms, see [Mu]; for further details on Cohen-Macaulayness, see [St6].

The **pair** (K, L) will denote a pair of simplicial complexes $L \subseteq K$. Let k denote a field, fixed throughout the rest of the paper. Recall that $\widetilde{H}_p(K)$ refers to **reduced homology** of K (over k), and $\widetilde{H}_p(K, L)$ denotes **reduced relative homology** of the pair (K, L) (over k). For K a simplicial complex, $\widetilde{H}_p(K, \emptyset) = \widetilde{H}_p(K)$; for a pair (K, L) with L non-empty, $\widetilde{H}_p(K, L) = H_p(K, L)$.

The **link** of a face F in a simplicial complex K is defined to be the subcomplex

$$lk_K F = \{G \in K : F \cup G \in K,\ F \cap G = \emptyset\}.$$

If $L \subseteq K$ are a pair of subcomplexes and $F \in K$, then define the **relative link** of F in L to be

$$lk_L F = \{G \in L : F \cup G \in L,\ F \cap G = \emptyset\}$$

(see Stanley [St4, Section 5]). If $F \in L$, this matches the usual definition of $lk_L F$, but we now allow the possibility that $F \notin L$, in which case $lk_L F = \emptyset$.

By [Re], a simplicial complex K is **pure Cohen-Macaulay** (over k) if K is pure and, for every $F \in K$ (including $F = \emptyset$), $\widetilde{H}_p(lk_K F) = 0$ for all $p < \dim lk_K F$. By [St4, Theorem 5.3], a pair of simplicial complexes (K, L) is **relative Cohen-Macaulay** (over k) if and only if, for every $F \in K$ (including $F = \emptyset$), $\widetilde{H}_p(lk_K F,\ lk_L F) = 0$ for all $p < \dim lk_K F$.

Definition (Stanley [St6, III.2.9]): Let K be a $(d-1)$-dimensional simplicial complex. Then K is **sequentially Cohen-Macaulay** if the pairs

$$\Omega_i(K) = (K^{(i,i)},\ K^{(i+1,i)})$$

are relative Cohen-Macaulay for $-1 \leq i \leq d-1$. In particular, when $i = d-1$, we require $\Omega_{d-1}(K) = (K^{(d-1,d-1)},\ \emptyset)$ to be relative Cohen-Macaulay, which is equivalent to $K^{<d-1>} = K^{(d-1,d-1)}$ being pure Cohen-Macaulay. $\square$

Remark: This definition is stated slightly differently from the one given by Stanley [St6], but it is easy to show that the two definitions are entirely equivalent. $\square$

We will use the following new characterization of sequential Cohen-Macaulayness, whose proof is omitted.

Theorem 1 *Let K be a $(d-1)$-dimensional simplicial complex. Then K is sequentially Cohen-Macaulay if and only if $K^{(i,i)}$ is pure Cohen-Macaulay for all $-1 \leq i \leq d-1$.* $\square$

Algebraic shifting.

Define the partial order $\leq_P$ on k-subsets of integers as usual: If $S = \{i_1 < \cdots < i_k\}$ and $T = \{j_1 < \cdots < j_k\}$ are two k-subsets of integers, then $S \leq_P T$ if $i_p \leq j_p$ for all p. A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex K is **shifted** if K_j is shifted for every j.

Given a simplicial complex K, **algebraic shifting** is a way to define a new complex $\Delta(K)$ that is shifted, has the same f-vector, and has many of the same algebraic and topological properties of the original complex (Kalai [Ka1]; see also [BK1, BK2]). The following result is the central property of algebraic shifting for our purposes.

Proposition 2 (Kalai [Ka2, Theorem 5.3]) *Let K be a simplicial complex. Then K is pure Cohen-Macaulay if and only if $\Delta(K)$ is pure.* $\square$

Thus, it is easy to detect whether K is pure Cohen-Macaulay, if $\Delta(K)$ is known. We extend Proposition 2 to the nonpure case as follows (the proof is omitted).

Theorem 3 *Let K be a simplicial complex of dimension at least i ($i \geq -1$). Then*

$$\Delta(K)^{<i>} \subseteq \Delta(K^{<i>}),$$

with equality if and only if $K^{(i,i)}$ is pure Cohen-Macaulay. $\square$

Remark: The proof of Theorem 3 relies upon Proposition 2. $\square$

Main theorem.

We now sketch the proof of our main result.

Theorem 4 *Let K be a $(d-1)$-dimensional simplicial complex. Then K is sequentially Cohen-Macaulay if and only if*

$$h_{i,j}(\Delta(K)) = h_{i,j}(K)$$

for all $0 \le j \le i \le d$.

Proof: (sketch) We show that the following statements are all equivalent:

(a) K is sequentially Cohen-Macaulay;

(b) $K^{(i,i)} = (K^{<i>})^{(i)}$ is pure Cohen-Macaulay for all $-1 \le i \le d-1$;

(c) $\Delta(K)^{<i>} = \Delta(K^{<i>})$ for all $-1 \le i \le d-1$;

(d) $f_j(\Delta(K)^{<i>}) = f_j(K^{<i>})$ for all $-1 \le j, i \le d-1$;

(e) $f_{i,j}(\Delta(K)) = f_{i,j}(K)$ for all $0 \le j \le i \le d$; and

(f) $h_{i,j}(\Delta(K)) = h_{i,j}(K)$ for all $0 \le j \le i \le d$.

(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): These equivalences are Theorem 1 and Theorem 3, respectively.

(c) $\Leftrightarrow$ (d): By Theorem 3, $\Delta(K)^{<i>} \subseteq \Delta(K^{<i>})$, so $\Delta(K)^{<i>} = \Delta(K^{<i>})$ if and only if $f_{j-1}(\Delta(K)^{<i>}) = f_{j-1}(\Delta(K^{<i>}))$ for all j. But, algebraic shifting preserves the f-vector, so $f_{j-1}(\Delta(K^{<i>})) = f_{j-1}(K^{<i>})$.

(d) $\Rightarrow$ (e): This follows immediately from equation (5) applied to $\Delta(K)$ and K, respectively. (For the $i = d$ case, we also need that $\Delta(K)^{<d>} = \emptyset = K^{<d>}$ so $f_{j-1}(\Delta(K)^{<d>}) = 0 = f_{j-1}(K^{<d>})$ for all j.)

(e) $\Rightarrow$ (d): This follows immediately from equation (4) applied to $\Delta(K)$ and K, respectively.

(e) $\Leftrightarrow$ (f): This follows immediately from equations (2) and (3). □

Shelling.

Björner and Wachs generalized the definition of shellability by dropping the assumption of purity.

Definition (Björner-Wachs [BW2, Definition 2.1]): A simplicial complex is **shellable** if it can be constructed by adding one facet at a time, so that as each facet is added, it intersects the existing complex (previous facets) in a union of codimension 1 faces. Equivalently, as each facet F is added, a *unique* new minimal face, called the **restriction face** $R(F)$, is added. (Note that the dimension of $R(F)$ is one less than the number of codimension one faces in which F intersects the existing complex when it is added.) □

The restriction faces are counted by the h-triangle [BW2, Theorem 3.4]: If K is a shellable $(d-1)$-dimensional complex, then

$$h_{i,j}(K) = \#\{facets\ F \in K\colon \dim F = i-1,\ \dim R(F) = j-1\},$$

for $0 \le j \le i \le d$. This generalizes the well-known result that the restriction faces of a shellable pure complex are counted by the h-vector.

Björner and Wachs' generalization of shellability prompted Stanley to define sequentially Cohen-Macaulay complexes, and to design the definition so that shellable complexes are sequentially Cohen-Macaulay, generalizing the well-known pure case. Our first corollary to Theorem 4 now follows easily.

Corollary 5 *Let $\boldsymbol{h} = (h_{i,j})_{0\le j\le i\le d}$ be an array of integers. Then the following are equivalent:*

(a) $\boldsymbol{h}$ is the h-triangle of a sequentially Cohen-Macaulay simplicial complex;

(b) $\boldsymbol{h}$ is the h-triangle of a shellable simplicial complex; and

(c) $\boldsymbol{h}$ is the h-triangle of a shifted simplicial complex.

Proof: (c) $\Rightarrow$ (b): A shifted complex is shellable [BW2, Theorem 11.3].

(b) $\Rightarrow$ (a): A shellable complex is sequentially Cohen-Macaulay [St6, Section III.2].

(a) $\Rightarrow$ (c): Let K be a sequentially Cohen-Macaulay simplicial complex. Theorem 4 implies that $h_{ij}(K) = h_{ij}(\Delta(K))$ for all $0 \le i \le j \le d$. Thus $\Delta(K)$ is a shifted complex with the same h-triangle as K. □

The pure case of Corollary 5 is due to Stanley [St1, Theorem 6]. The proof of Corollary 5 is a generalization of Kalai's proof of Stanley's result [Ka2, Corollary 5.2]. It follows from Corollary 5 that characterizing the h-triangle (equivalently, characterizing the f-triangle) of sequentially Cohen-Macaulay simplicial complexes is equivalent to characterizing the h-triangle of shellable complexes or even characterizing the h-triangle of shifted complexes. (See [BW2, Theorem 3.6] and the remarks that follow it, and also Björner [Bj2].)

Iterated Betti numbers.

Another corollary to Theorem 4 involves iterated Betti numbers, a non-pure generalization of reduced homology Betti numbers ($\widetilde{\beta}_{i-1}(K) = \dim_k \widetilde{H}_{i-1}(K)$) introduced in joint work with L. Rose. Although they can be defined as the Betti numbers of a certain chain complex [DR, Section 4], we will take the following equivalent formulation as our definition of iterated Betti numbers.

Definition ([DR, Theorem 4.1]): Let K be a simplicial complex. For a set F of positive integers, let $init(F) = \max\{r\colon \{1,\dots,r\} \subseteq F\}$ (so $init(F)$ measures the largest "initial segment" in F, and is 0 if there is no initial segment, *i.e.*, if $1 \notin F$). Then define the rth **iterated Betti numbers** of K to be

$$\beta_{i-1}[r](K) = \#\{facets\ F \in \Delta(K)\colon \dim F = i-1,\ init(F) = r\}.$$

□

A special case is $r = 0$; then $\beta_i[0](K) = \tilde{\beta}_i(K)$, the (ordinary) Betti numbers of reduced homology.

Björner and Wachs [BW2, Theorem 4.1] showed that if K is shellable, then

$$\tilde{\beta}_{i-1}(K) = h_{i,i}(K), \tag{6}$$

for $0 \leq i \leq d$. Equation (6) is generalized in [DR, Theorem 1.2] to

$$\beta_{i-1}[r](K) = h_{i,i-r}(K) \tag{7}$$

for shellable K.

Theorem 4 allows us to generalize even further, by weakening the assumption on K in equation (7) from being shellable to being merely sequentially Cohen-Macaulay.

Corollary 6 *If K is sequentially Cohen-Macaulay, then $\beta_{i-1}[r](K) = h_{i,i-r}(K)$.*

Proof: By [DR, Theorem 5.4], $\beta_{i-1}[r](K) = h_{i,i-r}(\Delta(K))$, for all simplicial complexes K. Then apply Theorem 4. □

Conjecture.

Finally, we present a conjecture inspired by collapsing, which is related to shelling.

Definition (Kalai [Ka2, Section 4]): A face R of a simplicial complex K is **free** if it is included in a unique facet F. (The empty set is a free face of K if K is a simplex.) If $|R| = p$ and $|F| = q$, then we say R is of **type** (p, q). A (p, q)**-collapse step** is the deletion from K of a free face of type (p, q) and all faces containing it (*i.e.*, the deletion of the interval $[R, F]$). A **collapsing sequence** is a sequence of collapse steps that reduce K to the empty simplicial complex. □

A shelling of K gives rise to a canonical collapsing (though not conversely): If $F_1, \ldots, F_t$ is a shelling order on the facets of K, then

$$[R(F_t), F_t] \ldots [R(F_1), F_1]$$

is a collapsing sequence of K [DR, Lemma 5.5], [Ka2, Section 4]. Since $\Delta(K)$ is shifted and hence shellable, $\Delta(K)$ has a collapsing sequence whose types are given by $h(\Delta(K))$. Kalai has conjectured that K must have a decomposition into Boolean intervals of the same type as a collapse sequence of $\Delta(K)$ [Ka2, Section 9.3]. This conjecture and Theorem 4 would then imply the following conjecture.

Conjecture 7 *A sequentially Cohen-Macaulay complex K can be decomposed into a collection of Boolean intervals (indexed by the set A)*

$$K = \dot{\cup}_{a \in A}[R_a, F_a] \tag{8}$$

such that

$$h_{i,j}(K) = \#\{a \in A\colon |F_a| = j, |R_a| = i\} \tag{9}$$

and every F_a is a facet in K.

It is not hard to see that if K is sequentially Cohen-Macaulay and has the decomposition (8), then the decomposition satisfies equation (9) if and only if every F_a is a facet.

This is the nonpure generalization of a conjecture made (separately) by Garsia [Ga, Remark 5.2] and Stanley [St2, p. 149], that a pure Cohen-Macaulay complex can be decomposed into Boolean intervals whose tops are facets (see also [St5, Du]). Conjecture 7 is equivalent to being able to decompose a relative Cohen-Macaulay complex into Boolean intervals whose tops are facets.

Acknowledgements.

I am grateful to Anders Björner for informing me about sequential Cohen-Macaulayness and its possible relation to the h-triangle and shelling. Richard Stanley kindly provided details about sequential Cohen-Macaulayness. Anders Björner and Gil Kalai provided encouragement by saying that my preliminary conjectures "seemed right." Ping Zhang and the referees suggested several improvements.

References

[Bj1] A. Björner, "Shellable and Cohen-Macaulay partially ordered sets," *Trans. Amer. Math. Soc.* **260** (1980), 159–183.

[Bj2] A. Björner, "Face numbers, Betti numbers and depth," in preparation.

[BK1] A. Björner and G. Kalai, "An extended Euler-Poincaré Theorem," *Acta Math.* **161** (1988), 279–303.

[BK2] A. Björner and G. Kalai, "On f-vectors and homology," in *Combinatorial Mathematics: Proceedings of the Third International Conference* (G. Bloom, R. Graham, J. Malkevitch, eds.); *Ann. N. Y. Acad. Sci.* **555** (1989), 63–80.

[BS] A. Björner and B. Sagan, "Subspace arrangements of type B_n and D_n," *J. Alg. Comb.*, to appear.

[BW1] A. Björner and M. Wachs, "On lexicographically shellable posets," *Trans. Amer. Math. Soc.* **277** (1983), 323–341.

[BW2] A. Björner and M. Wachs, "Shellable nonpure complexes and posets, I" *Trans. Amer. Math. Soc.*, to appear.

[Du] A. Duval, "A combinatorial decomposition of simplicial complexes," *Israel J. Math.* **87** (1994), 77–87.

[DR] A. Duval and L. Rose, "Iterated homology of simplicial complexes," preprint, 1995.

[Ga] A. Garsia, "Combinatorial methods in the theory of Cohen-Macaulay rings," *Adv. in Math.* **38** (1980), 229–266.

[Ka1] G. Kalai, "Characterization of f-vectors of families of convex sets in R^d, Part I: Necessity of Eckhoff's conditions," *Israel J. Math.* **48** (1984), 175–195.

[Ka2] G. Kalai, Algebraic Shifting, unpublished manuscript (July 1993 version).

[Mu] J. Munkres, *Elements of Algebraic Topology,* Benjamin/Cummings, Menlo Park, CA, 1984.

[Re] G. Reisner, *Cohen-Macaulay quotients of polynomial rings,* thesis, University of Minnesota, 1974; *Adv. Math.* **21** (1976), 30–49.

[St1] R. Stanley, "Cohen-Macaulay complexes," in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht and Boston, 1977, 51–62.

[St2] R. Stanley, "Balanced Cohen-Macaulay complexes," *Trans. Amer. Math. Soc.* **249** (1979), 139–157.

[St3] R. Stanley, "The number of faces of simplicial polytopes and spheres," in *Discrete Geometry and Convexity* (J. Goodman, *et. al.*, eds.); *Ann. N. Y. Acad. Sci.* **440** (1985), 212–223.

[St4] R. Stanley, "Generalized h-vectors, intersection cohomology of toric varieties, and related results," in *Commutative Algebra and Combinatorics* (M. Nagata and H. Matsumura, eds.), Advanced Studies in Pure Mathematics **11**, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, 187–213.

[St5] R. Stanley, "A combinatorial decomposition of acyclic simplicial complexes," *Disc. Math.* **120** (1993), 175–182.

[St6] R. Stanley, *Combinatorics and Commutative Algebra, 2nd ed.*, Birkhäuser, Boston, 1995.

[SWa] S. Sundaram and M. Wachs, "The homology representations of the k-equal partition lattice," *Trans. Amer. Math. Soc.*, to appear.

[SWe] S. Sundaram and V. Welker, "Group actions on arrangements of linear subspaces and applications to configuration spaces," *Trans. Amer. Math. Soc.*, to appear.