Intersections of several flags and a generalization of permutation matrices to higher dimensions

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Abstract

It is well-known that relative position between two complete flags in an n-dimensional vector space can be specified by an $n \times n$ permutation matrix, and that all permutation matrices arise in this way. We define a notion of "permutation matrix" of higher dimension d, such that the relative position between d complete flags is specified by such an n^d-matrix.

For fixed n and d, the n^d-permutation matrices have a natural partial order. For $d = 2$, the poset is isomorphic to the Bruhat order on the symmetric group S_n. For $n = 2$, the poset is isomorphic to the partition lattice.

Key words. flag, permutation, Bruhat order, partition lattice

1 Introduction

By a permutation matrix of size n we shall mean a square matrix with exactly one dot in each row and column, all other entries empty. We will in this paper discuss a solution to an algebraic combinatorial problem on intersections of flags, that suggests a generalization of the concept of permutation matrix to arbitrary dimensions. For each n we shall present a family of dotted arrays of dimension $d \geq 1$, such that for $d = 2$ we get the ordinary permutation matrices of size n.

1.1 On permutation matrices

By an n^d-matrix we shall mean a hypercubic array $n \times \cdots \times n$ of dimension d. Several generalizations of permutation matrices to higher dimensions can be proposed, each generalizing some aspect of the classical case.

One natural candidate is the **dense** d-dimensional permutation matrix, where we have distributed n^{d-1} dots in a n^d-matrix, such that there is exactly one dot in each one-dimensional submatrix of size n. Another candidate is the **sparse** d-dimensional permutation matrix, where n dots are distributed so that we have exactly one dot in each submatrix of size n and codimension one (that is, dimension $d-1$).

Both these concepts have been studied in the literature. For example, a dense three-dimensional permutation matrix is equivalent to a latin square. In general, it seems hard to say much about the dense matrices. On the other hand, the sparse d-dimensional permutation matrices quite simply correspond to elements in the product of $d-1$ copies of the symmetric group S_n, with $(\pi_1, \pi_2, \ldots, \pi_{d-1})$ giving the dot matrix with a dot in $(i, \pi_1(i), \pi_2(i), \ldots, \pi_{d-1}(i))$ for each $i = 1, 2, \ldots, n$. Hence, there are $(n!)^{d-1}$ different sparse permutation matrices of size n in dimension d. The sparse permutations were used by E. Pascal in 1900 to define higher-dimensional determinants [6]. An alternative definition of higher-dimensional determinants was quite recently given by Gelfand, Kapranov, and Zelevinsky [4].

1.2 A geometric property of permutation matrices

The generalization suggested in the present paper will contain the sparse permutations but also others. It is motivated by the following elegant connection between permutation matrices and geometry (possibly due to Schubert), cf. the papers by Fulton [3] and Proctor [7].

First, for any matrix P, let $P[i,j]$ denote the upper left i by j submatrix of P. If P is a dot matrix such that there is at most one dot in each row and column (like permutation matrices and their submatrices), then define the rank of P to be its number of dots.

Let V be an n-dimensional vector space, and let $E_\bullet$ and $F_\bullet$ be two complete flags in V, that is,

$$\emptyset = E_0 \subset E_1 \subset \cdots \subset E_n = V, \quad \emptyset = F_0 \subset F_1 \subset \cdots \subset F_n = V.$$

where $\dim(E_j) = \dim(F_j) = j$ for all $j = 1, \ldots, n$. Define a dot matrix $P(E_\bullet, F_\bullet)$ by putting a dot in $P_{i,j}$ if the two conditions

$$E_{i-1} \cap F_j = E_i \cap F_{j-1} \neq E_i \cap F_j$$

are satisfied. Then, elegantly, it is always true that P is a permutation matrix and $\operatorname{rank} P[i,j] = \dim(E_i \cap F_j)$ for all i, j. Conversely, every permutation matrix arises as $P(E_\bullet, F_\bullet)$ for some pair of flags.

EXAMPLE Let V be a three-dimensonal space spanned by e_1, e_2, e_3. Given the flags $E_\bullet : \emptyset \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset V$ and $F_\bullet : \emptyset \subset \langle e_1 + e_2 \rangle \subset$

$\langle e_1 + e_2, e_3 \rangle \subset V$, we get the dimensions of intersections and permutation matrix in the figure.

0	0	1
1	1	2
1	2	3

		•
•		
	•	

1.3 Higher dimensional matrices

Let $\mathbb{Z}_+^d$ be partially ordered by componentwise comparison:

$$X = (x_1, \ldots, x_d) \leq (y_1, \ldots, y_d) = Y \text{ if } x_i \leq y_i \text{ for all } i = 1, \ldots, d.$$

This is a lattice, with $X \vee Y$ and $X \wedge Y$ given by componentwise maximum and minimum respectively.

By a d-dimensional matrix of shape $n_1 \times n_2 \times \cdots \times n_d$ we shall mean the lower interval $[(1, \ldots, 1), (n_1, \ldots, n_d)]$ of $\mathbb{Z}_+^d$. The positions may be filled with anything. If the positions are all either empty or dotted, we speak of a **dot matrix**. We will keep the intuitive name **upper left submatrix** for a lower interval of a d-dimensional matrix (and its contents). Let $P[j_1, \ldots, j_d]$ denote the upper left $j_1 \times \cdots \times j_d$ submatrix of a d-dimensional matrix P.

1.4 Two characterization problems

While attempting [2] to generalize Fulton's essential set [3] to higher dimensions, the present authors became interested in intersecting several flags, $E_\bullet^1, \ldots, E_\bullet^d$. In particular, let $I(E_\bullet^1, \ldots, E_\bullet^d)$ and $M(E_\bullet^1, \ldots, E_\bullet^d)$ be defined by

$$I_{j_1,\ldots,j_d} = E_{j_1}^1 \cap \cdots \cap E_{j_d}^d, \quad \text{and} \quad M_{j_1,\ldots,j_d} = \dim(I_{j_1,\ldots,j_d}).$$

Problem 1 *Which n^d-matrices M can arise from intersection of several flags?*

In the two-dimensional case, we know what the answer is: the rank-matrices of permutation matrices. To be able to get an analogous answer for other dimensions, we must define a general meaning of rank. Fulton makes the trivial but useful observation that the number of dots in a submatrix of a permutation matrix is equal to the number of its rows that contain a dot, and is also equal to the number of its columns that contain a dot. We shall generalize this concept of rank.

For an arbitrary d-dimensional dot matrix P, say that the **coordinate-i-rank** of P is the number of indices j such that there exists at least one dot in P in some position whose ith coordinate is j. Say that P is **rankable** with $\operatorname{rank} P = r$ if the coordinate-i-rank is r for all $i = 1, \ldots, d$. The intuitive picture is that in whichever direction we traverse P, the number of layers containing at least one dot will be the same. Let us say that P is **totally rankable** if every upper left submatrix of P is rankable.

Answer 1 *Every n^d-matrix M that arises from intersection of several flags is the rank matrix of a totally rankable n^d-matrix.*

The converse holds for $d = 1, 2, 3$ and we conjecture that it in fact holds in general.

Problem 2 *Which are the totally rankable dot matrices?*

For any dot matrix we shall define its **redundant** positions and its **covered** positions. It will be obvious that every covered position is redundant. The key question is whether the converse holds.

Answer 2 *A dot matrix is totally rankable if every redundant position is covered.*

Among all totally rankable matrices of given ranks in all positions, there is a unique minimal one. For $d = 2$ the minimal totally rankable dot matrices of full rank are the permutation matrices.

1.5 A partial ordering

Let $\mathcal{P}_{n,d}$ be the set of minimal totally rankable matrices of full rank of size n and dimension d. The set $\mathcal{P}_{n,d}$ can be partially ordered by entrywise comparison of ranks, that is, if $P, P' \in \mathcal{P}_{n,d}$, then

$$P \geq P' \quad \text{if} \quad \operatorname{rank} P[j_1, \ldots, j_d] \geq \operatorname{rank} P'[j_1, \ldots, j_d] \text{ for all } j_1, \ldots, j_d.$$

For $d = 2$ the dot matrices in $\mathcal{P}_{n,2}$ are the $n \times n$ permutation matrices, and the poset is in fact the Bruhat order on S_n. On the other hand, for $n = 2$ the dot matrices in $\mathcal{P}_{2,d}$ encode partitions of a d-set and the poset is in fact the partition lattice. Hence, what we have got is a common generalization of the Bruhat order and the partition lattice. These two posets have very different features. For example, the Bruhat order is not a lattice, but it is self-dual; the partition lattice is of course a lattice, but it is not self-dual. However, they are both graded, and have unique minimal and maximal elements. We can show that for all n, d, the poset $\mathcal{P}_{n,d}$ has unique minimal and maximal elements. We prove that $\mathcal{P}_{n,3}$ is graded and conjecture the grading for $d \geq 4$.

In Fig. 1 we show the first non-trivial case, $\mathcal{P}_{3,3}$, with 70 elements.

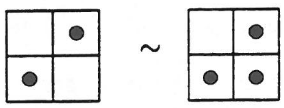
2 Totally rankable dot matrices

Given a fixed shape of matrices, a dot matrix P can be identified with the set of positions in which it has dots. Several totally rankable dot matrices may yield the same rank for each upper left submatrix. Say that such dot matrices are **rank equivalent.** Last in this section we shall see that every rank equivalence class of totally rankable dot matrices has a simple boolean lattice structure, so that in particular every rank equivalence class has a unique minimal member.

For a position X, let $P - X$ and $P + X$ denote the two dot matrices obtainable from P by removing the dot in position X (if there is one, otherwise do nothing), and adding a dot in position X (if there is none) respectively.

Lemma 2.1 *For a dot matrix P and a position $\check{X}$, the dot matrices $P - \check{X}$ and $P + \check{X}$ are rank equivalent if and only if $\check{X} = \bigvee \mathcal{X}$ for some subset of dots $\mathcal{X} \subseteq P$ not containing $\check{X}$.*

In the classical case, this means for example that adding a dot in a position that is later in the row than some other dot and later in the column than yet another dot does not alter either the column rank or the row rank.



Let $(x_1, \ldots, x_d) \prec (\check{x}_1, \ldots, \check{x}_d)$ mean that $x_i < \check{x}_i$ for all $i = 1, \ldots, j$. For a set $\mathcal{X} \subset \mathbb{Z}_+^d$, define a statistic $\sigma(\mathcal{X})$ as the number of members in $\mathcal{X}$ that have at least one coordinate in common with the join, that is, $\sigma(\mathcal{X}) = |\{X \in \mathcal{X} : X \not\prec \bigvee \mathcal{X}\}|$. For a dot matrix P, let $R(P)$ be the set of **redundant positions** defined by

$$R(P) = \{\check{X} = \bigvee \mathcal{X} : \mathcal{X} \subseteq P, \sigma(\mathcal{X}) \geq 2\}.$$

Furthermore, say that a position $\check{X} = (\check{x}_1, \ldots, \check{x}_d)$ is **covered** by a subset $\mathcal{X}$ of P if it can be written $\check{X} = \bigvee \mathcal{X}$ where

1. $\check{X} \notin \mathcal{X} \subseteq P$;
2. $X \not\prec \check{X}$ for every X in $\mathcal{X}$; and
3. for every j there is an $X = (x_1, \ldots, x_d)$ in $\mathcal{X}$ such that $x_j < \check{x}_j$.

It is evident that if $\check{X}$ is covered in P, then $\check{X}$ is redundant in P. We shall see that P is totally rankable if and only if the converse also holds.

Theorem 2.2 *A dot matrix P is totally rankable if and only if every redundant position is covered by a subset of P.*

For $d = 2$, this statement boils down to the simple fact that a two-dimensional dot matrix is totally rankable if and only if the first dot in any row is also the first dot in its column, and vice versa.

Finally, let us prove the boolean algebra structure of the rank equivalence classes of totally rankable matrices.

Proposition 2.3 *The class of dot matrices that are rank equivalent to some totally rankable matrix P is a boolean interval under inclusion. Consequently, it has a minimal member $\hat{P}$ and a maximal member $\overline{P}$.*

3 Intersection of several flags

Given flags $E_\bullet^1, E_\bullet^2, \ldots, E_\bullet^d$, we defined in the introduction the matrix I of intersections by $I_{j_1,\ldots,j_d} = E_{j_1}^1 \cap \cdots \cap E_{j_d}^d$, and its dimension matrix M by $M_{j_1,\ldots,j_d} = \dim(I_{j_1,\ldots,j_d})$. Now define also a dot matrix $P(E_\bullet^1, \ldots, E_\bullet^d)$ as the d-dimensional dot matrix with a dot in $P_{j_1,\ldots,j_d}$ whenever

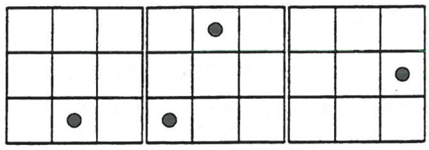
$$I_{j_1-1,j_2,\ldots,j_d} = I_{j_1,j_2-1,\ldots,j_d} = \cdots = I_{j_1,j_2,\ldots,j_d-1} \neq I_{j_1,j_2,\ldots,j_d}$$

holds. In the two-dimensional case we obtained the permutation matrices, i.e. the minimal totally rankable matrices, in this way. We shall now see that this holds also in th three-dimensional case.

Theorem 3.1 *Given flags $E_\bullet^1, E_\bullet^2, E_\bullet^3$, the dot matrix $P(E_\bullet^1, E_\bullet^2, E_\bullet^3)$ is minimal totally rankable, and $M(E_\bullet^1, E_\bullet^2, E_\bullet^3)$ is its rank matrix. Conversely, every minimal totally rankable $n \times n \times n$ matrix P has the same rank matrix as the intersection matrix for some three flags. Complete flags correspond to matrices of full rank.*

EXAMPLE Let V be a three-dimensonal space spanned by e_1, e_2, e_3. Given the flags $E_\bullet^1 : \emptyset \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset V$, $E_\bullet^2 : \emptyset \subset \langle e_3 \rangle \subset \langle e_1, e_3 \rangle \subset V$, and $E_\bullet^3 : \emptyset \subset \langle e_1 + e_3 \rangle \subset \langle e_1, e_3 \rangle \subset V$, we get the dimensions of intersections and dot matrix in the figure below. The three layers have third coordinate one, two and three respectively.

0	0	0	0	1	1	0	1	1
0	0	0	0	1	1	0	1	2
0	1	1	1	2	2	1	2	3



For $d \geq 4$ the first part of Theorem 3.1 is still true with the same proof.

Theorem 3.2 *Given flags $E_\bullet^1, \ldots, E_\bullet^d$, the dot matrix $P(E_\bullet^1, \ldots, E_\bullet^d)$ is minimal totally rankable of full rank and $M(E_\bullet^1, \ldots, E_\bullet^d)$ is the rank matrix of P.*

We conjecture that the other direction is also true in higher dimensions.

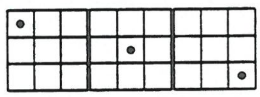
Conjecture 3.3 *Given any minimal totally rankable $n \times \ldots \times n$ dotted matrix P of full rank there are complete flags $E_\bullet^1, \ldots, E_\bullet^d$, such that the intersection rank matrix $M(E_\bullet^1, \ldots, E_\bullet^d)$ is the rank matrix of P.*

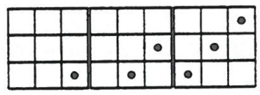
4 The rank partial order on d-dimensional permutation matrices

Let $\mathcal{P}_{n,d}$ be the set of minimal totally rankable matrices of full rank of size n and dimension d, partially ordered by entrywise comparison of rank matrices. Henceforth, in this article we will be bold enough to say **d-dimensional permutation matrices** instead of the awkward "minimal totally rankable dot matrices of full rank".

Proposition 4.1 *The poset $\mathcal{P}_{n,d}$ has unique maximal and minimal elements $\hat{1} = \{(i, i, \ldots, i) : i = 1, 2, \ldots, n\}$ and $\hat{0} = \{(x_1, x_2, \ldots, x_d) : x_1+x_2+\ldots+x_d = 1+n(d-1)\}$.* □

The two elements $\hat{1}$ and $\hat{0}$ for $\mathcal{P}_{3,3}$ are shown below.





4.1 $\mathcal{P}_{n,2}$ is the Bruhat order

It is clear from our previous discussion that the two-dimensional permutation matrices in our sense are precisely the classical permutation matrices.

The partial ordering on permutation matrices given by componentwise comparison of the corresponding rank matrices is known to be the Bruhat order, see Proctor [7].

4.2 $\mathcal{P}_{2,d}$ is the partition lattice

In this section we fix $n = 2$, that is, we will deal with 2^d-matrices. Given a permutation matrix $P \in \mathcal{P}_{2,d}$, associate to every dot in $(i_1, \ldots, i_d) \in P$ the set $\{a : i_a = 1\} \subseteq \{1, \ldots, d\}$. Define also $\phi(P)$ to be the family of such sets. Note that $(2, 2, \ldots, 2)$ – which will be mapped to the empty set – is a dot only in the permutation matrix that contains the dot $(1, 1, \ldots, 1)$ as well, i.e. the maximal element of $\mathcal{P}_{2,d}$. Disregard the empty set when defining ϕ.

First we claim that two dots in a d-dimensional permutation cannot both have a coordinate j equal to 1 for some j. We also claim that for every coordinate j there is a dot in the permutation having the jth coordinate equal to 1. With these claims we see that $\phi(P)$ is a partition of $\{1, \ldots, d\}$ for every permutation matrix P and that ϕ is a bijection.

Let Π_d be the lattice of partitions of $\{1, \ldots, d\}$ ordered by refinement. Regarding ϕ as a function from $\mathcal{P}_{2,d}$ to Π_d it is easy to see that it is true to the cover relations of $\mathcal{P}_{2,d}$. Hence the bijection ϕ is also an isomorphism of lattices.

Theorem 4.2 *The poset $\mathcal{P}_{2,d}$ is isomorphic to the partition lattice Π_d.*

See Fig. 2 for a picture of $\mathcal{P}_{2,4} \cong \Pi_4$.

4.3 $\mathcal{P}_{n,3}$ is graded

Both the specializations, Bruhat order and Partition lattice, are graded posets. It is an interesting question whether this common property carry over to $\mathcal{P}_{n,d}$ in general. We have not been able to prove this for arbitrary d, but at least it is true for $d = 3$.

Theorem 4.3 *$\mathcal{P}_{n,3}$ is graded, with maximal chains of length $2\binom{n}{2}$.*

Conjecture 4.4 *$\mathcal{P}_{n,d}$ is graded for all $n, d \geq 1$, and its maximal chains are of length $(d-1)\binom{n}{2}$.*

Another obviously interesting question is: "How do we generate all the n^d-permutation matrices?" Once again the proof of Theorem 3.1 gives the answer when $d = 3$.

The following algorithm will produce a dotted matrix. After the algorithm we must remove every dot that is directly under a dot in a previously layer and we will get an n^3-permutation.

Algorithm At each level k do one of the following steps:

- Choose a square (i, j, k) such that there is no dot in row $(i, \cdot, k-1)$ or in column $(\cdot, j, k-1)$. Define level k by copying level $k-1$ and adding a dot in (i, j, k).

- Choose a staircase of dots $(i_1, j_1, k-1), \ldots, (i_s, j_s, k-1)$, such that $i_1 > \ldots > i_s$ and $j_1 < \ldots < j_s$, in level $k-1$. Choose also a row $i < i_s$ and a column $j < j_1$ such that there are no dots in level $k-1$ in this row or in this column. Construct level k by copying level $k-1$, removing the dots in the chosen staircase and instead adding dots in positions $(i_1, j, k), (i_2, j_1, k), \ldots, (i_s, j_{s-1}, k), (i, j_s, k)$.

Repeating these steps in all possible ways and removing the dots that are directly beneath a dot in a previous layer, will give all possible n^3-permutaions. See Figure 1 for the 70 permutation matrices when $n = d = 3$.

One can formulate analogous algorithms for all $d \geq 3$.

5 Enumeration

Having defined d-dimensional permutations as minimal totally rankable n^d-matrices of full rank, one obvious question to ask is how many they are. Let $c(n, d)$ denote this number. This number is not easily computed in general, but when $d = 2$ or $n = 2$ we get from the characterizations in sections 4.1 and 4.2 that

$$c(n, 2) = n! \quad \text{and} \quad c(2, d) = B(d),$$

where $B(d)$ is the dth Bell number. Since the sparse permutations (see the introduction) are permutations also in our sense, we can easily conclude a lower bound.

Proposition 5.1 *A lower bound for the number of n^d-permutation matrices is given by*

$$(n!)^{d-1} \leq c(n, d)$$

We have not yet found any general nontrivial upper bound. However, for $d = 3$ we have the following.

Proposition 5.2 *For three-dimensional permutation matrices we have the following upper bound*

$$c(n,3) \leq n! \cdot 2^{\binom{n+1}{2}-1}$$

The number sequence $c(n,3)$ starts: 1,5,70,2167,...

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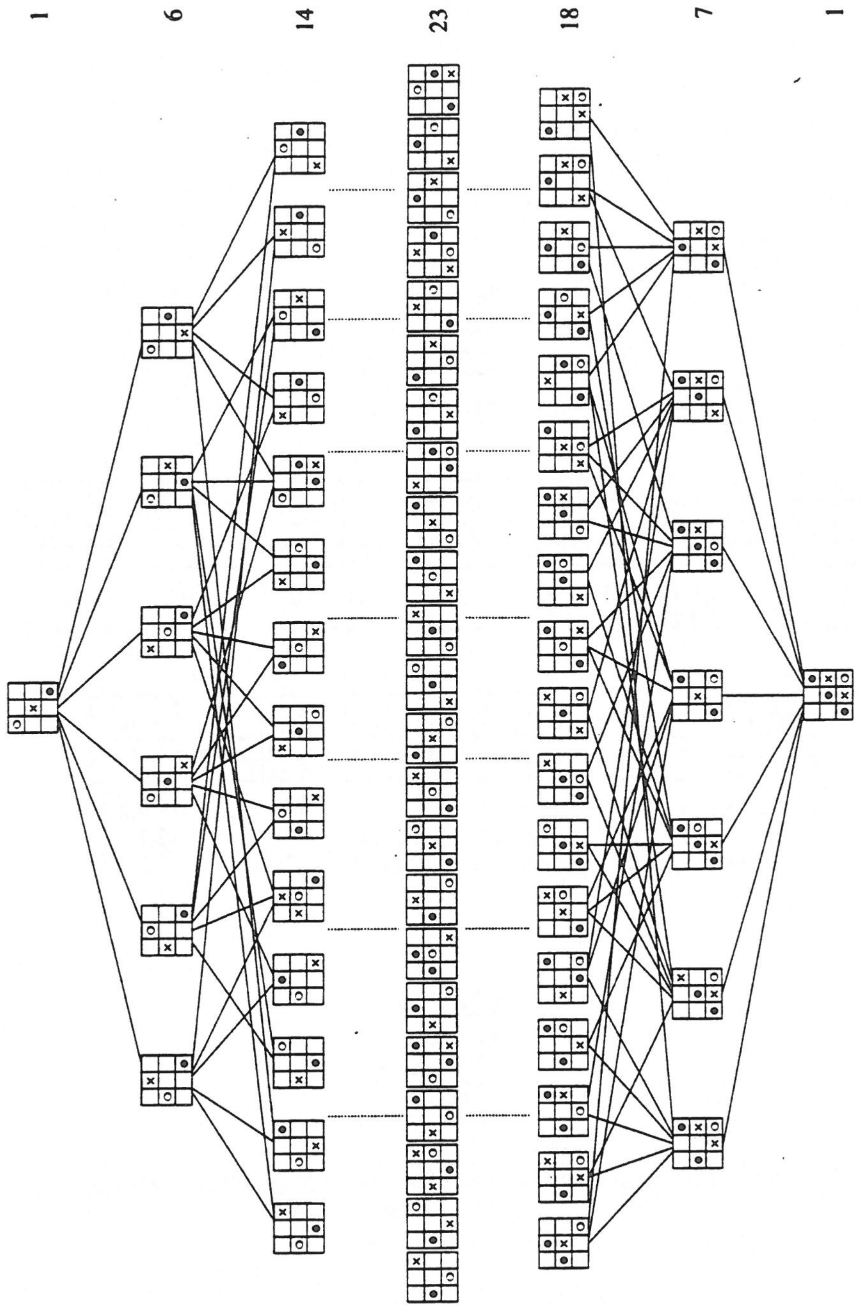

Figure 1: The poset $\mathcal{P}_{3,3}$ of the generalized $3 \times 3 \times 3$-permutation matrices. The empty circles, crosses, and filled circles signify dots in the first, second, and third layer respectively. The edges in the middle were too many to be drawn conveniently.

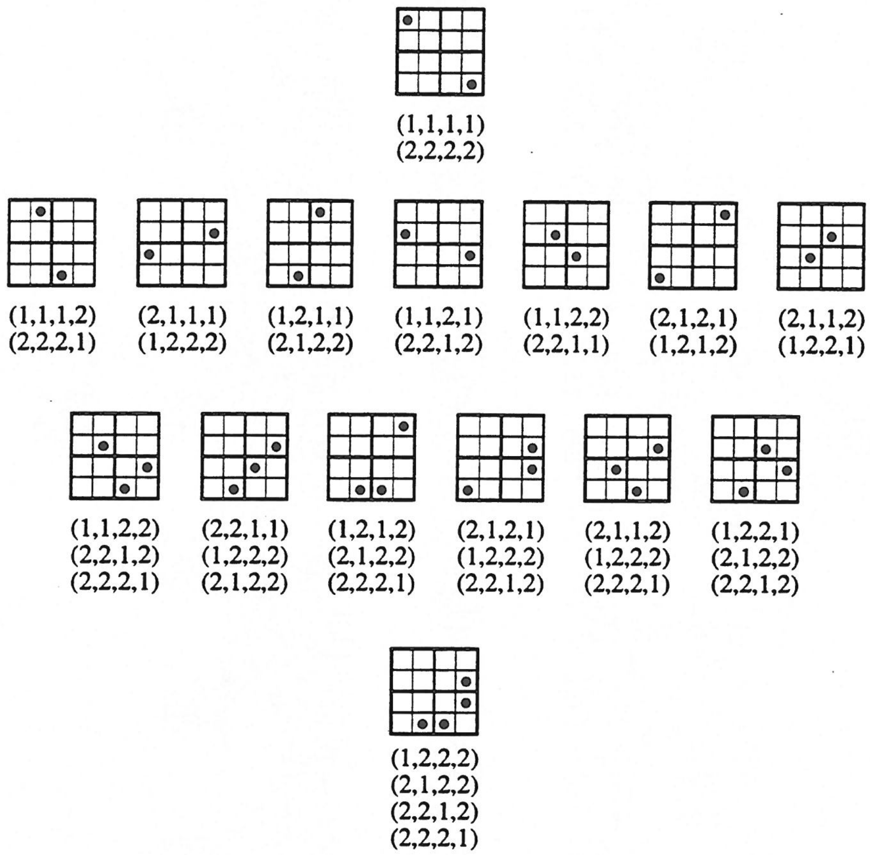

Figure 2: The partition lattice of a set of four elements: $\mathcal{P}_{2,4}$.