ON SUBSPACE ARRANGEMENTS OF TYPE $\mathcal{D}$

EVA MARIA FEICHTNER AND DMITRY N. KOZLOV

ABSTRACT. Let $\mathcal{D}_{n,k}$ denote the subspace arrangement formed by all linear subspaces in $\mathbb{R}^n$ given by equations of the form

$$\epsilon_1 x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k}$$

where $1 \leq i_1 < \cdots < i_k \leq n$ and $(\epsilon_1, \ldots, \epsilon_k) \in \{+1, -1\}^k$.

Some important topological properties of such a subspace arrangement depend on the topology of its intersection lattice. In previous work on a larger class of subspace arrangements by Björner & Sagan [BS] the topology of the intersection lattice $\mathcal{L}(\mathcal{D}_{n,k}) = \Pi^{\pm}_{n,k}$ turned out to be a particularly interesting and difficult case.

We prove in this paper that $\mathrm{Pure}(\Pi^{\pm}_{n,k})$ is shellable, hence that $\Pi^{\pm}_{n,k}$ is shellable for $k > \frac{n}{2}$. Moreover we prove that $\widetilde{H}_i(\Pi^{\pm}_{n,k}) = 0$ unless $i \equiv n-2 \pmod{k-2}$ or $i \equiv n-3 \pmod{k-2}$, and that $\widetilde{H}_i(\Pi^{\pm}_{n,k})$ is free abelian for $i \equiv n-2 \pmod{k-2}$. In the special case of $\Pi^{\pm}_{2k,k}$ we determine homology completely. Our tools are EC-shellability introduced in [Koz1] and a spectral sequence method for the computation of poset homology first used in [Han].

We state implications of our results on the cohomology of the complement of the considered arrangements.

1. INTRODUCTION

The study of arrangements of linear subspaces $\mathcal{A} = \{A_1, \ldots, A_t\}$ in real n-space has attracted considerable interest in recent years [Bj2, Bj3, Zie]. Topological invariants of a subspace arrangement $\mathcal{A}$, such as the cohomology of its *complement* $\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n \setminus \bigcup_{i=1}^t A_i$ and the homotopy type of its *link* $\mathcal{V}^0_{\mathcal{A}} = S^{n-1} \cap (\bigcup_{i=1}^t A_i)$, have been shown to be completely determined by purely combinatorial data of the arrangement [GM, ZŽ]. Here the crucial role is played by the set of intersections of subfamilies of $\mathcal{A}$ ordered by reversed inclusion, the *intersection lattice* $\mathcal{L}(\mathcal{A})$. Notably the topology of this lattice (homology, resp. homotopy types of its intervals) turned out to provide the building blocks for the above mentioned topological invariants of the arrangement.

In this paper we investigate topological properties of the intersection lattice for the arrangement $\mathcal{D}_{n,k}$, where $\mathcal{D}_{n,k}$ denotes the subspace arrangement consisting of linear subspaces of $\mathbb{R}^n$ given by equations of the form

$$\epsilon_1 x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k}$$

for all k-subsets $1 \leq i_1 < \cdots < i_k \leq n$ of $\{1, \ldots, n\}$ and all sign patterns $(\epsilon_1, \ldots, \epsilon_k) \in \{+1, -1\}^k$.

A related subspace arrangement $\mathcal{B}_{n,k,h}$ is obtained from $\mathcal{D}_{n,k}$ by adding the coordinate subspaces of codimension h

$$x_{j_1} = x_{j_2} = \cdots = x_{j_h} = 0$$

for any h-subset $1 \leq j_1 < \cdots < j_h \leq n$ of $\{1, \ldots, n\}$.

The first author's stay at KTH Stockholm was partially supported by EC grant CHRX-CT93-0400. Moreover the first author acknowledges support from the graduate school "Algorithmic Discrete Mathematics" supported by the Deutsche Forschungsgemeinschaft, grant We 1265/2 – 1.

In [BS] the intersection lattice of $\mathcal{B}_{n,k,h}$ was denoted by $\Pi_{n,k,h}$. Clearly $\mathcal{D}_{n,k}$ and $\mathcal{B}_{n,k,h}$ coincide for $h = k$, hence $\mathcal{L}(\mathcal{D}_{n,k}) = \Pi_{n,k,k}$. The special cases $\mathcal{B}_{n,2,1}$ and $\mathcal{D}_{n,2}$ are well known as the reflection hyperplane arrangements corresponding to the Coxeter groups of type B_n, resp. D_n. Subspace arrangements of type $\mathcal{B}_{n,k,h}$ and $\mathcal{D}_{n,k}$ have been studied as generalizations of those, continuing and extending the work that had been done on the *"k-equal" arrangements* $\mathcal{A}_{n,k}$ (resp. on their intersection lattices $\mathcal{L}(\mathcal{A}_{n,k}) = \Pi_{n,k}$) [BL, BLY, BWe] as natural generalizations of the reflecting hyperplane arrangements for Coxeter groups of type A_{n-1}. The arrangements $\mathcal{D}_{n,k}$ can be viewed as a signed version of the "k-equal" arrangements. Emphasizing this point of view we will write $\Pi^{\pm}_{n,k}$ instead of $\Pi_{n,k,k}$.

In Björner & Sagan [BS] an extensive study of the lattices $\Pi_{n,k,h}$ is done. Expressions for their Möbius functions and characteristic polynomials are given, for $h < k$ the lattices are shown to be lexicographically shellable and implications on the topological type of $\mathcal{B}_{n,k,h}$-arrangements are derived. In particular it is shown that the cohomology of the complement is torsion-free and $(k-2)$-periodic. However, the arguments given in [BS] do not apply for $h = k$, but a similar "nice" behaviour for arrangements of type $\mathcal{D}$ was conjectured [Bj3, Sect. 4].

Motivated by this, we investigate topological properties of the intersection lattice $\Pi^{\pm}_{n,k}$ for $\mathcal{D}_{n,k}$ and derive results on its homology which together with implications on the cohomology of the complement support the above mentioned conjecture. It is both from combinatorial and topological contexts that we gather our major tools.

Using the notion of *general lexicographic shellability* (EC-shellability) introduced in [Koz1] for the study of orbit arrangements we prove that $\mathrm{Pure}(\Pi^{\pm}_{n,k})$ (the poset obtained by taking the union of all longest unrefinable chains in $\Pi^{\pm}_{n,k}$) is shellable. In particular this implies shellability of $\Pi^{\pm}_{n,k}$ for $k > \frac{n}{2}$ and complete results on homotopy type and homology in that case.

Our second tool is a spectral sequence converging to the homology of a poset which was first used by Hanlon [Han] to compute the homology of generalized Dowling lattices. We extend his presentation to higher differentials and terms of the spectral sequence and apply it to the lattice $\Pi^{\pm}_{n,k}$. We derive completely the E^1-tableau which involves a study of topological properties for certain subposets of $\Pi^{\pm}_{n,k}$ including shellability arguments. From the tableau thus obtained we derive that for $k \geq 5$ the homology of $\Pi^{\pm}_{n,k}$ is trivial except for the dimensions of two $(k-2)$-periodic series. Moreover we conclude that the homology is torsion-free in one of these series. In the first nonpure case which is not covered by our EC-shellability argument mentioned above, namely for $\Pi^{\pm}_{2k,k}$, we carry out the spectral sequence to its very end and derive complete information on the homology of $\Pi^{\pm}_{2k,k}$.

Using the result of Goresky & MacPherson mentioned above, we deduce implications on the cohomology of the complement of $\mathcal{D}_{n,k}$-arrangements from our results on the homology of their intersection lattices. For the special cases of parameters $k > \frac{n}{2}$ and $n = 2k$ we obtain complete descriptions of the cohomology groups.

2. Basic notions and definitions

In this section we give a short summary of the notions and basic concepts used throughout the text. For additional information we refer to the textbooks by Stanley [St] for combinatorial aspects and Munkres [Mu] for topological theory. For the topology of posets see [Bj1].

All posets which we consider will be finite and bounded. We denote the order complex of a poset by $\Delta(P)$ and call P pure if $\Delta(P)$ is a pure complex. The poset obtained from P by taking the union of the longest unrefinable chains under the order induced from P we denote by $\mathrm{Pure}(P)$.

We discuss *simplicial* homology of order complexes of posets with integer coefficients and denote it $H_*(P)$ to replace the lengthy notation $H_*(\Delta(P);\mathbb{Z})$. However, we will switch to reduced homology, $\widetilde{H}_*(P)$, whenever it serves to formulate results in a convenient form.

Definition 2.1. *A simplicial complex Δ is called* **shellable** *if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ such that the subcomplex $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. Such an ordering of facets is called a* **shelling order**.

We call a poset shellable if its order complex is shellable. We state in addition the basic notions and results on lexicographic shellability in the case of nonpure bounded posets as given by Björner & Wachs [BWa] (see also [Bj3, Sect.2]). We will need this in the sequel for our discussion of subposets of $\Pi^{\pm}_{n,k}$.

Definition 2.2. *Let P denote a finite, bounded poset, $C(P)$ the set of cover relations $x \leftarrow y$ in P, Λ a totally ordered set. A map $\lambda : C(P) \longrightarrow \Lambda$ is called an* **EL-labeling** *on P if the following conditions hold for any interval (a,b) in P:*

(i) *There is a unique maximal chain $\Gamma_{(a,b)} : a < x_0 < \ldots < x_t < b$ in (a,b) with strictly increasing labels, $\lambda(a \leftarrow x_0) < \lambda(x_0 \leftarrow x_1) < \ldots < \lambda(x_t \leftarrow b)$.*

(ii) *$\Gamma_{(a,b)} \prec \Gamma'$ for any other maximal chain Γ' in (a,b), where $\prec$ denotes lexicographic order on the label sequences of maximal chains.*

The existence of an EL-labeling on a poset implies its shellability, P is called **lexicographically shellable** in that case. We state the main consequences in the following theorem.

Theorem 2.3. *Let P denote a finite, bounded poset which admits an EL-labeling $\lambda : C(P) \longrightarrow \Lambda$. Then P is shellable; it has the homotopy type of a wedge of spheres, hence is free in homology. Its t-th reduced Betti number equals the number of maximal chains $\Gamma : \hat{0} < x_0 < \ldots < x_t < \hat{1}$ with weakly decreasing labels, $\lambda(\hat{0} \leftarrow x_0) \geq \lambda(x_0 \leftarrow x_1) \geq \ldots \geq \lambda(x_t \leftarrow \hat{1})$.*

A finite collection $\mathcal{A} = \{A_1, \ldots, A_t\}$ of linear subspaces in $\mathbb{R}^n$ is called a **subspace arrangement**. The **intersection lattice** $\mathcal{L}(\mathcal{A})$ of an arrangement $\mathcal{A}$ is the collection of all intersections $A_{i_1} \cap \cdots \cap A_{i_p}$, $1 \leq i_1 < \cdots < i_p \leq t$, ordered by reverse inclusion, $x \leq y \Leftrightarrow y \subseteq x$, and extended by a unique minimal element $\hat{0}$.

We define the "k-equal" arrangement $\mathcal{A}_{n,k}$. It is the subspace arrangement formed by all linear subspaces which are given by equations of the form

$$x_{i_1} = \cdots = x_{i_k}$$

for k-subsets $1 \leq i_1 < \cdots < i_k \leq n$ of $\{1, \ldots, n\}$. These arrangements have been studied in [BL, BLY, BWe].

Next we define a related arrangement $\mathcal{B}_{n,k,h}$. It is the subspace arrangement consisting of linear subspaces of $\mathbb{R}^n$ given by equations of the form

$$\epsilon_1 x_{i_1} = \epsilon_2 x_{i_2} = \cdots = \epsilon_k x_{i_k}$$

for all k-subsets $1 \leq i_1 < \cdots < i_k \leq n$ of $\{1, \ldots, n\}$ and all sign patterns $(\epsilon_1, \ldots, \epsilon_k) \in \{+1, -1\}^k$ together with coordinate subspaces of codimension h

$$x_{j_1} = x_{j_2} = \cdots = x_{j_h} = 0$$

for any h-subset $1 \leq j_1 < \cdots < j_h \leq n$ of $\{1, \ldots, n\}$.

Finally, we define $\mathcal{D}_{n,k} = \mathcal{B}_{n,k,k}$. Let $\Pi_{n,k} := \mathcal{L}(\mathcal{A}_{n,k})$, $\Pi_{n,k,h} := \mathcal{L}(\mathcal{B}_{n,k,h})$ and $\Pi^{\pm}_{n,k} := \mathcal{L}(\mathcal{D}_{n,k})$, stressing the fact that $\mathcal{D}_{n,k}$ can be viewed as a signed version of the "k-equal" arrangement $\mathcal{A}_{n,k}$.

There is a convenient way to interpret elements of the lattice $\Pi_{n,k,h}$ with the help of signed graphs. We define a **signed graph** to be a graph G on a vertex set $V(G) = \{1, \ldots, n\}$ with $E(G)$ consisting of edges of three different kinds:

1. a positive edge between vertices i and j, denoted by ij^+;
2. a negative edge between vertices i and j, denoted by ij^-;
3. a half edge with only one endpoint i, denoted by i^h.

The idea is that edges ij^+, ij^- and i^h correspond to the equations $x_i = x_j$, $x_i = -x_j$ and $x_i = 0$. We define a **complete unbalanced graph** K_V^u on the vertex set V by

$$E(K_V^u) = \{ij^+, ij^-, j^h \mid i, j \in V\}.$$

This graph corresponds to the linear subspace given by equations $x_i = 0$ for all $i \in V$. Also we define a **complete balanced graph** $K_{V,W}^b$ on the vertex set $V \dot{\cup} W$ by

$$E(K_{V,W}^b) = \{ij^+ \mid i, j \in V\} \cup \{ij^+ \mid i, j \in W\} \cup \{ij^- \mid i \in V, j \in W\}.$$

This graph corresponds in its turn to the linear subspace given by equations $x_i = x_j$ for all $i, j \in V$ and $x_i = -x_j$ for all $i \in V, j \in W$.

Every element of the lattice $\Pi_{n,k,h}$ is an intersection of some linear subspaces defined by equations of the form described above. We state a description of lattice elements in terms of signed graphs which follows from the work of Zaslavsky [Za].

Theorem 2.4. *The lattice $\Pi_{n,k,h}$ is isomorphic to the lattice consisting of the signed graphs G such that $V(G) = \{1, \ldots, n\}$ and*

1. *every connected component of G is a complete balanced or unbalanced graph;*
2. *every complete balanced component is either a single vertex or of size at least k;*
3. *there is at most one unbalanced component, and if it exists, then it is of size at least h.*

The graphs are partially ordered by inclusion of their edge sets.

To each element $a \in \Pi_{n,k,h}$ viewed as a signed graph we can associate a set partition $U/A_1/\ldots/A_m$ of $\{1, \ldots, n\}$, U being the (possibly empty) vertex set of the unbalanced component, A_i being the vertex sets of balanced and trivial components. Whenever we are only concerned with properties of the associated set partitions we will switch to the interpretation of lattice elements in terms of partitions, talking about balanced and unbalanced blocks respectively.

3. Shellability of Pure($\Pi_{n,k}^{\pm}$)

The notion of EC-shellability generalizes the notion of usual lexicographic (or EL-) shellability. The following definition first appeared in [Koz1, Def. 3.1] and was used there to study the topology of orbit arrangements.

Definition 3.1. *We say that a pure poset P has a* **edge compatible labeling,** *or for short just* **EC-labeling,** *if we can label its edges with elements of some poset Λ so that in any interval all maximal chains have different labels and the following condition is satisfied:*

Condition (EC). *For any interval (x, t), any maximal chain c in (x, t) and $y, z \in c$, such that $x < y < z < t$, if $c|_{(x,z)}$ is lexicographically least in (x, z) and $c|_{(y,t)}$ is lexicographically least in (y, t) then c is lexicographically least in (x, t).*

It was proved in [Koz1, Prop. 3.2] that condition (EC) is equivalent to the following condition.

Condition (EC′). For any interval (x, t), c a maximal chain in (x, t), and $y, z \in c$, such that $x \prec y \prec z$, if $c|_{(x,z)}$ is lexicographically least in (x, z) and $c|_{(y,t)}$ is lexicographically least in (y, t) then c is lexicographically least in (x, t).

The existence of an EC-labeling on the edges of a pure poset implies its shellability [Koz1, Thm. 3.5].

Theorem 3.2. Pure($\Pi^{\pm}_{n,k}$) *is EC-shellable.*

Sketch of the proof. An edge labeling for Pure($\Pi^{\pm}_{n,k}$) is given and shown to satisfy condition (EC$'$), hence to be an EC-labeling. Details can be found in [FK]. $\square$

Corollary 3.3. (1) *$\Pi^{\pm}_{n,k}$ is shellable for $k > \frac{n}{2}$. In particular $\Pi^{\pm}_{n,k}$, $k > \frac{n}{2}$, has the homotopy type of a wedge of equal-dimensional spheres and is free in homology:*

$$\widetilde{H}_j(\Pi^{\pm}_{n,k}) = \begin{cases} \mathbb{Z}^{(2^{n-1}-1)\binom{n-1}{k-1}} & \text{for } j = n-k \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For general n, k, $\widetilde{H}_*(\Pi^{\pm}_{n,k})$ is free abelian resp. trivial in the $k-4$ highest relevant dimensions:*

$$\begin{aligned} \widetilde{H}_{n-k}(\Pi^{\pm}_{n,k}) &= \mathbb{Z}^{(2^{n-1}-1)\binom{n-1}{k-1}} \\ \widetilde{H}_j(\Pi^{\pm}_{n,k}) &= 0 \quad \text{for} \quad n-2k+3 \leq j \leq n-k-1. \end{aligned}$$

Sketch of the proof. (1) $\Pi^{\pm}_{n,k}$ is a pure lattice for $k > \frac{n}{2}$. The ranks of its free homology groups are obtained by counting so called *proper* chains under the given EC-labeling, namely those chains whose subchains are not lexicographically least in any interval of length 2.
(2) The chain groups $C_j(\Pi^{\pm}_{n,k})$ and $C_j(\text{Pure}(\Pi^{\pm}_{n,k}))$ coincide for $n-2k+3 \leq j \leq n-k$. $\square$

4. The spectral sequence

In this section we describe a spectral sequence converging to the homology of the order complex of a finite bounded poset. It will serve as our second major tool to detect homological properties of the intersection lattice of $\mathcal{D}_{n,k}$-arrangements.

A spectral sequence of this type was first used by Hanlon [Han] to calculate the homology of generalized Dowling lattices. For the general construction of spectral sequences induced by filtrations we refer to Spanier [Sp, Ch.9, Sec.1] or McCleary [McC, Ch.2]. A concise survey and elementary introduction on the usage of spectral sequences to compute poset homology can be found in [Koz2].

Let P denote a finite bounded poset, J a lower order ideal in P. Let $f : J \longrightarrow \mathbb{N}$ be an order-preserving map with the additional properties that $f(\hat{0}) = 0$ and $x < y$ implies $f(x) < f(y)$. Such mapping f might be given by a rank function on J or by a linear extension of the partial order on J. Under the conditions imposed on f the inverse image of each element in $\mathbb{N}$ forms an antichain in J.

We define an increasing filtration on the simplicial chain complex $C_*(P, \partial)$. Let $\Gamma : \hat{0} < x_0 < \ldots < x_t < \hat{1}$ be a chain in P. Define the *pivot* of Γ, piv(Γ), to be the maximal element of Γ contained in J, the *weight* of Γ, $\omega(\Gamma)$, to be its value under f, $\omega(\Gamma) := f(\text{piv}(\Gamma))$. This assignment of weights gives us the filtration of the chain complex:

$$\begin{aligned} F_s(C_t(P)) &= \big\langle \{\Gamma : \hat{0} < x_0 < \ldots < x_t < \hat{1} \,|\, \omega(\Gamma) \leq s\} \big\rangle_{\mathbb{Z}} \quad \text{for } t \geq 0,\ s \geq 0, \\ F_{-1}(C_t(P)) &= \{0\} \quad \text{for } t \geq 0, \end{aligned}$$

with $\langle \cdot \rangle_{\mathbb{Z}}$ denoting the linear span of the given chains with integer coefficients.

From the definition of the usual simplicial boundary operator,

$$\partial(\hat{0} < x_0 < \ldots < x_t < \hat{1}) = \sum_{i=0}^{t} (-1)^i (\hat{0} < x_0 < \ldots < \hat{x}_i < \ldots < x_t < \hat{1})$$

for $\Gamma : \hat{0} < x_0 < \ldots < x_t < \hat{1} \in C_t(P)$, we easily see that it respects the filtration: omitting an element other than the pivot does not alter the weight of the chain. Omitting $\mathrm{piv}(\Gamma)$ turns an element below into the pivot, the resulting chain has a strictly lower weight. Hence $\partial(F_s(C_*)) \subseteq F_s(C_*)$. Moreover the filtration is bounded from below by definition. We conclude that it induces a first quadrant spectral sequence $(E^r_{*,*}, d^r)_{r\geq 0}$ converging to $H_*(P)$ with E^0-term given by

$$\begin{aligned} E^0_{t,s} &= F_s(C_t(P)) \,/\, F_{s-1}(C_t(P)) \\ &= \langle \{\Gamma : \hat{0} < x_0 < \ldots < x_t < \hat{1} \,|\, \omega(\Gamma) = s\} \rangle_{\mathbb{Z}} \quad \text{for } t \geq 0, s \geq 0 . \end{aligned}$$

Depending on our choice of indices the differentials d^r are of bidegree $(-1, -r)$. The homology of P can be reconstructed from the final term E^∞ by "vertical" summation:

$$H_t(P) = \bigoplus_{s\geq 0} E^\infty_{t,s}, \quad t \geq 0 .$$

We now investigate the differentials d^r, $r \geq 0$, in detail.
The differential $d^0 : E^0_{t,s} \longrightarrow E^0_{t-1,s}$ is induced by the simplicial boundary operator. Let $\Gamma : \hat{0} < x_0 < \ldots < x_{j-1} < \mathrm{piv}(\Gamma) < x_{j+1} < \ldots < x_t < \hat{1}$ be a generator of $E^0_{t,s}$, then

$$d^0(\Gamma) = [\partial(\Gamma)] = \Big[\sum_{\substack{i=0\\ i\neq j}}^{t} (-1)^i (\hat{0} < x_0 < \ldots < \hat{x}_i < \ldots < x_t < \hat{1})\Big] .$$

The weight of a chain is lowered by the omission of an element if and only if it is the pivot which is removed.

For $s > 0$ we replace the chain complexes $(E^0_{*,s}, d^0)$ (bidegree $d^0 = (-1,0)$!) by chain isomorphic complexes. The latter allow us to give an explicit description of the E^1-entries of the spectral sequence in terms of simplicial homology of certain subposets of P. There is an obvious isomorphism between the following chain complexes "dividing" each chain Γ in P with pivot $a > \hat{0}$ into two chains, namely its subchains below and above the pivot:

$$\begin{aligned} \varphi : \quad & E^0_{t,s} \longrightarrow \bigoplus_{a\in f^{-1}[\{s\}]} \Big(\widetilde{C}_*(\hat{0}, a) \otimes \widetilde{C}_*((a, \hat{1}) \cap (P \setminus J))\Big)_{t-2} \\ & (\hat{0} < \ldots < x_{j-1} < a < x_{j+1} < \ldots < \hat{1}) \\ & \qquad \longmapsto (\hat{0} < \ldots < x_{j-1} < a) \otimes (a < x_{j+1} < \ldots < \hat{1}), \end{aligned}$$

with $\widetilde{C}_*$ denoting the augmented simplicial chain complex of the respective intervals. We need to use augmented complexes including the empty chain in order to get proper counterparts for chains which lie entirely in J or except for the pivot entirely outside J. Notice however that we derive nonreduced simplicial homology from the spectral sequence. The isomorphism φ commutes with the boundary operators d^0 and $\tilde{\partial}_\otimes$, respectively. To see this, note that

$$\tilde{\partial}_\otimes = \tilde{\partial}_{(0,a)} \otimes \mathrm{id} + \mathrm{id} \otimes \tilde{\partial}_{(a,\hat{1})\cap(P\setminus J)}$$

with the usual sign conventions, namely $\tilde{\partial}_\otimes(c_p \otimes c_q) = \tilde{\partial} c_p \otimes c_q + (-1)^p c_p \otimes \tilde{\partial} c_q$ for $c_p \in \widetilde{C}_p(\hat{0}, a)$, $c_q \in \widetilde{C}_q((a, \hat{1}) \cap (P \setminus J))$.

Hence φ is actually a bijective chain map and we get

$$\begin{aligned} E^1_{t,s} &= H_t(E^0_{*,s}, d^0) \\ &\cong \bigoplus_{a\in f^{-1}[\{s\}]} H_{t-2}\Big(\widetilde{C}_*(\hat{0}, a) \otimes \widetilde{C}_*((a, \hat{1}) \cap (P \setminus J)), \tilde{\partial}_\otimes\Big) . \end{aligned}$$

In case at least one of the intervals $(\hat{0}, a)$, $(a, \hat{1}) \cap (P \setminus J)$ is free in homology we can apply the algebraic Künneth theorem and deduce

$$(4.1) \qquad E^1_{t,s} \cong \bigoplus_{a \in f^{-1}[\{s\}]} \left(\tilde{H}_*(\hat{0}, a) \otimes \tilde{H}_*((a, \hat{1}) \cap (P \setminus J)) \right)_{t-2}.$$

For $s = 0$ we directly get an explicit description of the E^1-entries in terms of simplicial homology of the poset: The E^0-entry $E^0_{t,0}$ equals $C_t(P \setminus J)$, the differential d^0 coincides with the usual simplicial boundary operator ∂, hence

$$(4.2) \qquad E^1_{t,0} = H_t(P \setminus J).$$

The differential $d^1 : E^1_{t,s} \longrightarrow E^1_{t-1,s-1}$ coincides with the connecting homomorphism of the triple $(F_s(C_*), F_{s-1}(C_*), F_{s-2}(C_*))$, hence can be described as induced by the boundary operator ∂ composed with the projection to the appropriate factor group.

An explicit description of higher differentials is obtained by a careful analysis of the construction method for spectral sequences via exact couples – an alternative approach going back to Massey [Ma]. For details we refer to [McC, §2.2.3,4]. We content ourselves here with providing the explicit descriptions for later use, omitting the lengthy reformulation which has to be done in our special case. The r-th differential d^r, $r > 1$, decomposes in two maps involving as intermediate group the image of the map $i_* : H_*(F_{s-r}(C)) \longrightarrow H_*(F_{s-1}(C))$ induced by inclusion:

$$\begin{array}{llllll} d^r : & E^r_{t,s} & \xrightarrow{d^r_1} & \operatorname{im}(H_{t-1}(F_{s-r}) \to H_{t-1}(F_{s-1})) & \xrightarrow{d^r_2} & E^r_{t-1,s-r} \\ & [c] + \operatorname{im} d^{r-1} & \longmapsto & [\partial c] = i_*[a] & \longmapsto & [a] + \operatorname{im} d^{r-1}. \end{array}$$

The first map d^r_1 is induced by the simplicial boundary operator ∂, d^r_2 projects the inverse image of the cycle obtained by d^r_1 to the factor group $E^r_{t-1,s-r}$. The problem in dealing with higher differentials actually consists in determining the inverse images of homology cycles under i_* (compare Sect. 6 in [FK]).

5. The spectral sequence applied to $\Pi^{\pm}_{n,k}$

Our objective in this section is to derive the E^1-tableau of the spectral sequence for $\Pi^{\pm}_{n,k}$ and to draw conclusions on the homology of $\Pi^{\pm}_{n,k}$. According to the general description of E^1-entries given above (4.1), this involves the study of certain subposets of $\Pi^{\pm}_{n,k}$ which will turn out to have interesting topological properties on their own.

Recall the description of $\Pi^{\pm}_{n,k}$ as a lattice of signed graphs stated in Theorem 2.4. We define the ideal J in $\Pi^{\pm}_{n,k}$ to be formed by all graphs without unbalanced component. As order-preserving function f into the natural numbers we choose a linear extension of the partial order on J. From the general description of the E^1-term (4.1) we see that for each element $a \in J$ we have to determine the homology of the lower interval $(\hat{0}, a)$ as well as the homology of $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$. Moreover for the entries in the 0-th row of the E^1-tableau the homology of $(\Pi^{\pm}_{n,k} \setminus J)$ is needed.

Consider an element $a \in J$. Let $A_1 / \ldots / A_m$ be the associated set partition of $\{1, \ldots, n\}$. We can assume that $|A_1| \geq \ldots \geq |A_l| > |A_{l+1}| = \ldots = |A_m| = 1$.

<u>$(\hat{0}, a)$, $a \in J$.</u> Denote by a_i the graph obtained by splitting all nontrivial components of a except for the component on A_i into singletons. We obtain a decomposition of $(\hat{0}, a)$ as direct product of the intervals $(\hat{0}, a_i)$. These in turn are isomorphic to "k-equal" lattices on the vertex sets A_i:

$$(0, a) \cong \prod_{i=1}^{l} (\hat{0}, a_i) \cong \prod_{i=1}^{l} \Pi_{|A_i|,k}.$$

Using [BWe, Thm. 4.8(iii)] we conclude that

$$\widetilde{H}_t(\hat{0}, a) \neq 0 \quad \Leftrightarrow \quad t = n - 2 - m - r(k-2) \tag{5.1}$$

$$\text{for some integer } r,\ l \leq r \leq \sum_{i=1}^{l} \left\lfloor \frac{|A_i|}{k} \right\rfloor .$$

$(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$, $a \in J$. We propose an EL-labeling for $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$ which we actually directly adopt from [BS, Sect.4] where it is shown to be an EL-labeling for $\Pi_{n,k,h}$, $h < k$. We choose $\Lambda = [4] \times [n]$ as label set under lexicographic order and label the edges in $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$ as follows:

edge type	label
a balanced block A is converted into an unbalanced or adjoint to the existing unbalanced block	$(1, \max A)$
an unbalanced block A of size k is created	$(2, \max A)$
a singleton $\{s\}$ is inserted into the unbalanced block	$(2, s)$
two balanced blocks A_i, A_j are merged	$(3, \max(A_i \cup A_j))$
a balanced block A of size k is created	$(4, \max A)$
a singleton $\{s\}$ is inserted into a balanced block	$(4, s)$

A verification that $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$ is actually lexicographically shellable under the proposed labeling can be found in [FK].

According to Theorem 2.3 we have to count maximal chains with weakly decreasing label sequences in order to derive the homotopy type of $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$. We only state the results and refer to [FK] for details.

Case 1. a does not contain any singletons.

$$\Delta((a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)) \simeq S^{m-2} . \tag{5.2}$$

Case 2. a contains singletons, but fewer than k.

$$\Delta((a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)) \text{ is contractible.}$$

Case 3. a contains at least k singletons.

$$\Delta((a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)) \simeq \bigvee_{\binom{m-l-1}{k-1}} S^{m-k-1} . \tag{5.3}$$

We summarize the implications on the homology of $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)$ for later use:

$$\widetilde{H}_t((a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J)) \neq 0 \tag{5.4}$$

$$\Leftrightarrow \begin{cases} a \text{ has no singletons and } t = m - 2, \text{ or} \\ a \text{ has at least } k \text{ singletons and } t = m - k - 1 . \end{cases}$$

$\Pi^{\pm}_{n,k} \setminus J$. We claim that also $\Pi^{\pm}_{n,k} \setminus J$ is lexicographically shellable under the labeling proposed above. For a verification we refer to [FK]. The homotopy type is derived from a count of maximal chains with weakly decreasing label sequences:

$$\Delta(\Pi^{\pm}_{n,k} \setminus J) \quad \simeq \quad \bigvee_{\binom{n-1}{k-1}} S^{n-k-1} \text{ which implies that} \tag{5.5}$$

$$H_t(\Pi^{\pm}_{n,k} \setminus J) \neq 0 \quad \Leftrightarrow \quad t = 0 \text{ or } t = n - k - 1 . \tag{5.6}$$

We have now gathered all information which is needed to build up the E^1-tableau for $\Pi^{\pm}_{n,k}$. Recall that we have chosen f to be a linear extension of the partial order on the ideal J. Let us indicate this extension by simply numbering the elements of J, $J = (a_i)_{i\in\mathbb{N}}$. Following (4.1) we obtain E^1-entries of the form

$$E^1_{t,s} \cong \left(\widetilde{H}_*(\hat{0}, a_s) \otimes \widetilde{H}_*((a_s, \hat{1}) \cap (P \setminus J))\right)_{t-2} \quad \text{for } s > 0\,.$$

Using the homology of subposets derived above (5.1, 5.4), we conclude that

$$E^1_{t,s} \neq 0 \quad \Leftrightarrow \quad \begin{cases} a_s \text{ has no singletons and} \\ t = n-2-r(k-2) \text{ for some integer } r, \\ l \le r \le \sum_{i=1}^{l} \left\lfloor \frac{|A_i|}{k} \right\rfloor \qquad \text{or} \\ \\ a_s \text{ has at least } k \text{ singletons and} \\ t = n-3-(r+1)(k-2) \text{ for some integer } r, \\ l \le r \le \sum_{i=1}^{l} \left\lfloor \frac{|A_i|}{k} \right\rfloor\,. \end{cases}$$

For entries in the 0-th row of the E^1-tableau (compare (4.2)) we obtain from (5.6)

$$E^1_{t,0} \neq 0 \quad \Leftrightarrow \quad t = 0 \text{ or } t = n-k-1\,.$$

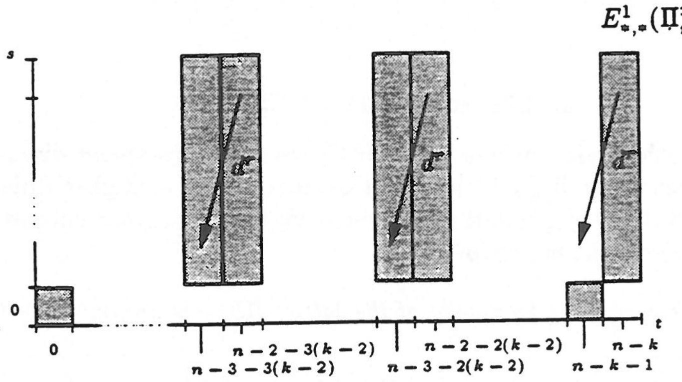
Theorem 5.1. *The reduced homology of $\Pi^{\pm}_{n,k}$ might be nontrivial only in two $(k-2)$-periodic series, namely in dimensions*

$$n-2-r(k-2) \quad \textit{for } r \ge 1 \quad \textit{and}$$
$$n-3-r(k-2) \quad \textit{for } r \ge 1\,.$$

For $k \ge 5$, $\widetilde{H}_*(\Pi^{\pm}_{n,k})$ is free abelian in dimensions

$$n-2-r(k-2) \quad \textit{for } r \ge 1\,.$$

Proof. We depict entries of sequence tableaus as squares on an integer grid. From what we derived above we easily see that the nonzero entries of the E^1-tableau for $\Pi^{\pm}_{n,k}$ concentrate in a series of column pairs $(E_{n-2-r(k-2),*}, E_{n-3-r(k-2),*})_{r\ge 1}$.

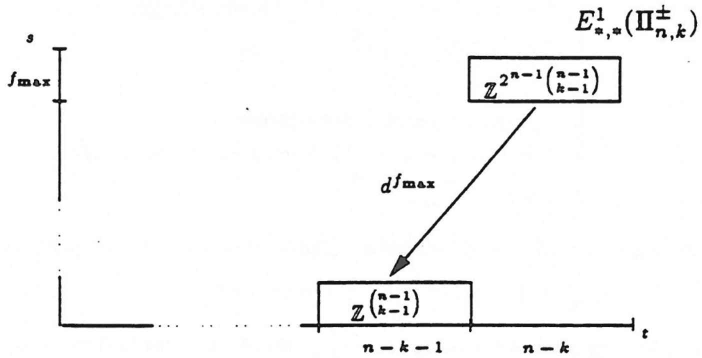


Regarding the bidegree of higher differentials and the recovering of $H_*(\Pi^{\pm}_{n,k})$ from the final stage of the sequence by vertical summation we obtain the result stated above. □

Remark 5.2. *From the spectral sequence we can reprove the results on the homology groups $\widetilde{H}_*(\Pi^{\pm}_{n,k})$ that we obtained above from the shellability of* Pure$(\Pi^{\pm}_{n,k})$ *(Cor. 3.3(2)).*

We shortly sketch the proof of Remark 5.2. Choose f to be a linear extension of the partial order on J on all elements which have more than one block and assign a single value $f_{\max}$ to all elements which consist of a balanced n-block. We get entries in the right most column of the E^1-tableau only in rows which correspond to elements of the latter type. For those $(\hat{0}, a) \cong \Pi_{n,k}$ and $(a, \hat{1}) \cap (\Pi^{\pm}_{n,k} \setminus J) = \emptyset$. In the column next to the left we only get an entry in the 0-th row; compare (5.5). Using [BWe, Cor. 4.6(i)] we conclude that

$$\begin{array}{rcl} E^1_{n-k,f_{\max}} & \cong & \bigoplus_{2^{n-1}} \widetilde{H}_{n-k-1}(\Pi_{n,k}) \cong \mathbb{Z}^{2^{n-1}\binom{n-1}{k-1}} \\ E^1_{n-k,s} & = & 0 \quad \text{for } s < f_{\max} \\ E^1_{n-k-1,0} & \cong & \mathbb{Z}^{\binom{n-1}{k-1}}. \end{array}$$



The only nontrivial differential $d^{f_{\max}}$ which occurs on the right most column pair can be shown to be surjective. Hence we obtain

$$\begin{array}{rcl} H_{n-k}(\Pi^{\pm}_{n,k}) & \cong & \mathbb{Z}^{(2^{n-1}-1)\binom{n-1}{k-1}} \\ H_{n-k-1}(\Pi^{\pm}_{n,k}) & = & 0. \end{array}$$

Conjecture 5.3. *The results of Section 6 on $\widetilde{H}_*(\Pi^{\pm}_{2k,k})$ as well as further investigations on higher differentials in the general case lead us to conjecture that $\widetilde{H}_t(\Pi^{\pm}_{n,k})$ is torsion-free and is nonzero if and only if $t = n-2-r(k-2)$ for some integer r, $1 \leq r \leq \lfloor \frac{n}{k} \rfloor$.*

6. The homology of $\Pi^{\pm}_{2k,k}$

In the full length version of our paper [FK] we give a complete discussion of the spectral sequence for $\Pi^{\pm}_{2k,k}$ including a detailed study of higher differentials and explicit descriptions of generating homology cycles for relevant subposets. We finish with deriving the homology of $\Pi^{\pm}_{2k,k}$.

Theorem 6.1. *The reduced homology of the lattice $\Pi^{\pm}_{2k,k}$ is nontrivial only in dimensions 2 and k:*

$$\begin{array}{rcl} \widetilde{H}_2(\Pi^{\pm}_{2k,k}) & = & \mathbb{Z}^{(2^{2k-3}-2^{k-1}+2^{2k-2})\binom{2k}{k}} \\ \widetilde{H}_k(\Pi^{\pm}_{2k,k}) & = & \mathbb{Z}^{(2^{2k-1}-1)\binom{2k-1}{k-1}} \\ \widetilde{H}_i(\Pi^{\pm}_{2k,k}) & = & 0 \ \textit{otherwise}. \end{array}$$

7. On the complement of subspace arrangements of type $\mathcal{D}$

We state implications of our results on the topology of $\mathcal{D}_{n,k}$-arrangements. In particular, we will be concerned with the cohomology of their complements in $\mathbb{R}^n$ using the following fundamental result of Goresky & MacPherson [GM].

Theorem 7.1. *Let $\mathcal{A}$ be a linear subspace arrangement, $\mathcal{L}(\mathcal{A})$ its intersection lattice. Then the cohomology of the complement of $\mathcal{A}$, $\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n \setminus \bigcup \mathcal{A}$, is given by*

$$\widetilde{H}^t(\mathcal{M}_{\mathcal{A}}) \cong \bigoplus_{a \in \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}} \widetilde{H}_{\operatorname{codim}(a)-2-t}(\hat{0}, a), \quad t \geq 0. \tag{7.1}$$

Let $\mathcal{M}^{\pm}_{n,k}$ denote the complement of $\mathcal{D}_{n,k}$ in $\mathbb{R}^n$, $(\widetilde{\beta}^{\pm}_{n,k})^t := \operatorname{rk} \widetilde{H}^t(\mathcal{M}^{\pm}_{n,k})$ its t-th reduced Betti number. We obtain the following results on the cohomology of $\mathcal{M}^{\pm}_{n,k}$:

Corollary 7.2. (1) *The cohomology of $\mathcal{M}^{\pm}_{n,k}$ is zero in any dimension which does not belong to one of the following $(k-2)$-periodic series:*

$$\begin{array}{ll} r(k-2), & \\ r(k-2)+1 & \text{for } r \geq 0. \end{array}$$

For $k \geq 5$, $\widetilde{H}^*(\mathcal{M}^{\pm}_{n,k})$ is free abelian in dimensions

$$r(k-2) \qquad \text{for } r \geq 0.$$

(2) *For $k > \frac{n}{2}$ the cohomology of $\mathcal{M}^{\pm}_{n,k}$ is free abelian. The only nontrivial reduced cohomology group occurs in dimension $k-2$ with Betti number*

$$(\widetilde{\beta}^{\pm}_{n,k})^{k-2} = \sum_{j=k}^{n} (2^j - 1) \binom{n}{j} \binom{j-1}{k-1}.$$

(3) *For $n = 2k$, $\widetilde{H}^*(\mathcal{M}^{\pm}_{n,k})$ is free abelian and nonzero cohomology groups only occur in dimensions $k-2$ and $2k-4$:*

$$\begin{aligned} (\widetilde{\beta}^{\pm}_{n,k})^{k-2} &= \sum_{j=k}^{2k} (2^j - 1) \binom{2k}{j} \binom{j-1}{k-1}, \\ (\widetilde{\beta}^{\pm}_{n,k})^{2k-4} &= (2^{2k} - 2^k) \binom{2k}{k}. \end{aligned}$$

Acknowledgements

We would like to thank Volkmar Welker for bringing Hanlon's work on spectral sequences in [Han] to our attention. We also wish to thank professor Anders Björner and professor Günter M. Ziegler for several valuable comments during our work on this paper.

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Fachbereich Mathematik, Sekr. MA 6-1, Technische Universität Berlin, D-10623 Berlin, Germany

E-mail address: `feichtne@math.tu-berlin.de`

Department of Mathematics, Royal Institute of Technology, S-100 44, Stockholm, Sweden

E-mail address: `kozlov@math.kth.se`