# An Alternative Method for q-Counting Directed Column-Convex Polyominoes

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**Abstract.** *The area+perimeter generating function of directed column-convex polyominoes will be written as a quotient of two expressions, each of which involves powers of q of all kinds: positive, zero and negative. The method used in the proof applies to some other classes of column-convex polyominoes as well.*

## 1. Definitions, conventions and notations

### 1.1 Directed column-convex polyominoes

Let $x=(1,0)$, $\bar{x}=(-1,0)$, $y=(0,1)$, and $\bar{y}=(0,-1)$. Suppose we have paths $\pi_1$ and $\pi_2$ such that:

*i)* $\pi_1$ lies in $\{x,y\}^*$, starts with an x-step, and ends with a y-step;

*ii)* $\pi_2$ lies in $\{x,y,\bar{y}\}^*$, has no factors $y\bar{y}$ or $\bar{y}y$, starts with a y-step, and ends with an x-step,

*iii)* $\pi_1$ and $\pi_2$ have the same origin and the same terminus, but are internally disjoint.

Let P be the plane figure bounded by $\pi_1$ and $\pi_2$. The figure P is called a *directed column-convex polyomino* (*dcc-polyomino*, Fig.1, left). The paths $\pi_1$ and $\pi_2$ are the *lower border* and the *upper border* of P, respectively. The $i^{th}$ *column* of P is the part of P that lies between the vertical lines passing through the ends of the $i^{th}$ x-step of $\pi_1$. We denote the minimal and the maximal ordinate of the $i^{th}$ column of P by $y_i(P)$ and $Y_i(P)$, respectively. If no ambiguity need be feared, we suppress the "(P)" and simply write $y_i$ and $Y_i$.

Let P be a dcc-polyomino. If the boundary of P consists of $j$ horizontal steps and $k$ vertical steps, we say that the *horizontal* and *vertical perimeters* of P are $j$ and $k$ respectively, and we write $h(P)=j$, $v(P)=k$. If the upper border of P has $m$ nonempty downward segments, P then has $m$ *descents*, and this is written $d(P)=m$. If the *area* of P is $n$, we write $a(P)=n$.

For $\Omega$ a family of dcc-polyominoes, we define the *generating function (gf)* of $\Omega$ to be the formal sum

$$gf(\Omega)= \sum_{P\in\Omega} x^{h(P)}y^{v(P)}q^{a(P)} .$$

The case y=1 of $gf(\Omega)$ is the *area* gf of $\Omega$, denoted by $agf(\Omega)$.

We denote the set of all dcc-polyominoes by $\mathcal{V}$, and we put $V=gf(\mathcal{V})$.

## 1.2 Lattice paths

Let $\mathcal{W}$ be the set of the paths on the step-set $\{x,y,\bar{y}\}$ which begin on the x-axis and have no factors $y\bar{y}$ or $\bar{y}y$.

Let $w\in\mathcal{W}$. If $|w|_x=n$, then $w$ has a unique factorization

$$w= u_1\cdot x\cdot u_2\cdot x\cdots u_n\cdot x\cdot u_{n+1} ,$$

where $u_i\in \{y\}^*\cup\{\bar{y}\}^*$ , for every $i$. We call the paths $u_i$ *nests* of $w$. Clearly enough, by the *odd nests* of $w$ we mean the nests $u_1$, $u_3$, $u_5,\ldots$ , while by the *even nests* of $w$ we mean the nests $u_2$, $u_4$, $u_6,\ldots$ . An x-step of $w$ is *odd* (resp. *even*) when it comes after an odd (resp. even) nest of $w$. The *rank* of $w$ (denoted $r(w)$) is defined to be the ordinate of the terminus of $w$. We write $a_1(w)$ (resp. $a_2(w)$) for the sum of the ordinates of the odd (resp. even) x-steps of $w$. Finally, with $\mathcal{S}\subseteq\mathcal{W}$ we associate two generating functions, $gf_1(\mathcal{S})$ and $gf_2(\mathcal{S})$, defined by

$$\langle x^iy^jq^nt^z\rangle\ gf_1(\mathcal{S})= |\{w\in\mathcal{S}:\ |w|_x=i,\ |w|_y+|w|_{\bar{y}}=j,\ a_1(w)=n,\ r(w)=z\}|,$$

$$\langle x^iy^jq^nt^z\rangle\ gf_2(\mathcal{S})= |\{w\in\mathcal{S}:\ |w|_x=i,\ |w|_y+|w|_{\bar{y}}=j,\ a_2(w)=n,\ r(w)=z\}|.$$

(The symbol $\langle u^k\rangle\ f(u)$ means the coefficient of $u^k$ in $f(u)$.)

### 1.3 Notations for products

Assuming from now on any empty product to be one, we write

$$(a)_n = \prod_{i=0}^{n-1} (1-aq^i) \qquad (n\in\mathbb{N}_0) \ ; \qquad \begin{bmatrix} n \\ j \end{bmatrix} = \frac{(q^{n-j+1})_j}{(q)_j} \qquad (n,j\in\mathbb{N}_0) \ ;$$

$$h^{[n]}(x) = \prod_{i=0}^{n-1} h(q^i x) \qquad (n\in\mathbb{N}_0,\ h \text{ a formal power series}) \ .$$

The second one of the above three items are the *q-binomial coefficients* or *Gaussian polynomials.*

## 2. Introduction

The model of a *self-avoiding polygon* (*SAP*) on the step-set $\{x,\bar{x},y,\bar{y}\}$ has its origins in various physical and chemical contexts. The things that would be most interesting to know about SAP's are: what is the number of SAP's whose perimeter is $p$, or the number whose area is $n$, or the number whose perimeter is $p$ and area is $n$. But these questions are all open, and there are little chances to answer any of them in the near future.

Hoping to get insight into the above-mentioned difficult problems, scientists started studying various simplified, but still nontrivial SAP models. Such models proved to be a rich vein of appealing exact results. Here we shall recall only the necessary minimum of those results; for a comprehensive survey we refer the reader *e.g.* to Viennot [15].

So, the number of dcc-polyominoes of area $n$ is the Fibonacci number $F_{2n-1}$ (Delest and Dulucq [6]).

The number of dcc-polyominoes with $c$ columns, $2v$ vertical edges and $d$ descents is $\frac{1}{c}\begin{bmatrix} c \\ d \end{bmatrix}\binom{c+v-2}{v-d-1}\binom{c+v-d-1}{v}$ (Feretić [8]; cf.[6]). The gf for these numbers is algebraic of degree three.

The gf for dcc-polyominoes is given by

$$V = y^2 \cdot \frac{\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}(y^2-1)^{n-1} q^{n(n+1)/2}}{(q)_{n-1}(y^2q)_{n-1}(y^2q)_n}}{1-\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}(y^2-1)^{n-1} q^{n(n+1)/2}}{(q)_n(y^2q)_{n-1}(y^2q)_n}} \quad (1)$$

Formula (1) is due to Bousquet-Mélou [2].

A nice feature of the method which produced formula (1) is its wide range of applicability. Indeed, besides the dcc-polyominoes, that method can handle *e.g.* stack, parallelogram, directed and convex, convex, and column-convex polyominoes (see [2,3,9]). The same is true of the q-counting method that will be presented here. So this paper is, in fact, the first illustration of a certain fairly versatile approach. On this occasion, it seemed us appropriate to tackle the dcc-polyominoes, because they are not too complicated, and are also not too simple or over-studied.

The basic idea of our method is to combine Delest's [5] coding for column-convex polyominoes with a factorization of lattice paths used in Gessel [10]. It should be mentioned, however, that the formulas obtained in this way are somewhat different from those derived in [2,3,9]. Namely, whereas those "old" formulas involve only positive and zero powers of q, in our formulas negative powers of q are present too.

## 3. A coding for dcc-polyominoes

The first step of our method is to encode the dcc-polyominoes.

Let $\mathcal{P}_{cvn}$ be the set of dcc-polyominoes which have $c$ columns, $2v$ vertical edges and area $n$. With $P \in \mathcal{P}_{cvn}$ we associate a path $\varphi(P) \in \mathcal{W}$ which

*i)* starts and ends on the x-axis, and

*ii)* has $2c-1$ x-steps, whose ordinates are, from left to right,

$$Y_1-y_1,\ Y_1-y_2,\ Y_2-y_2,\ Y_2-y_3,\ \ldots\ ,\ Y_c-y_c. \quad (2)$$

(See Fig.1 for an example.)

On account of the geometry of dcc-polyominoes, the numbers in (2) are all positive, which means that the internal vertices of $\varphi(P)$ all lie in the half-plane $y>0$. Next, since $y_1 \le y_2 \le y_3 \le \dots$ , each even x-step of $\varphi(P)$ stands on the same or lower level than the last x-step before it. Hence the even nests of $\varphi(P)$ all lie in $\{\bar{y}\}^*$. Further, the first differences of the sequence (2) are $y_1-y_2$, $Y_2-Y_1$, $y_2-y_3$, $\dots$ , $Y_c-Y_{c-1}$. The absolute values of these differences are the lengths of the internal nests of $\varphi(P)$. Thus

$$|\varphi(P)|_y+|\varphi(P)|_{\bar{y}} = \tag{3}$$

$$=(Y_1-y_1) + |y_1-y_2| + |Y_2-Y_1| + |y_2-y_3| + \dots + |Y_c-Y_{c-1}| + (Y_c-y_c)$$

But the sum in the second row of (3) is nothing other than the vertical perimeter of P, and thus $|\varphi(P)|_y+|\varphi(P)|_{\bar{y}} = 2v$. Further, it is obvious that the ordinates of the odd x-steps of $\varphi(P)$ sum up to the area of P, *i.e.* to $n$.

Let $\mathscr{E}_{cvn}$ be the set of those $w\in\mathscr{W}$ which meet the following conditions:

*i)* the origin and terminus of $w$ are on the x-axis, and all the internal vertices of $w$ lie in the half-plane $y>0$,

*ii)* all even nests of $w$ lie in $\{\bar{y}\}^*$ ,

*iii)* $|w|_x= 2c-1$, *iv)* $|w|_y+|w|_{\bar{y}}= 2v$,

*v)* $a_1(w)= n$.

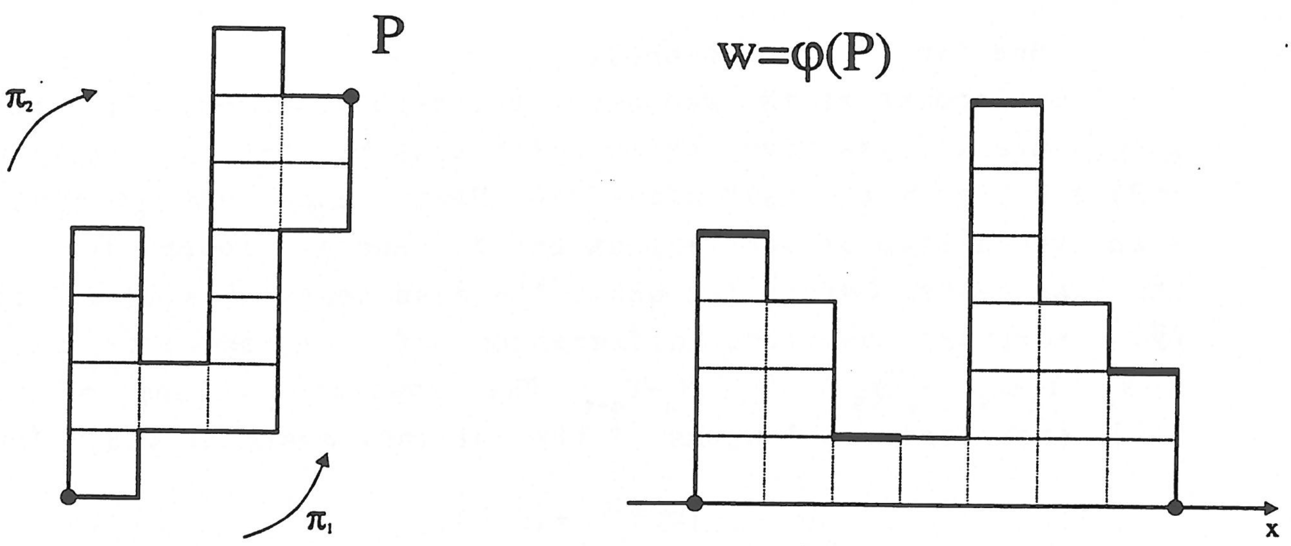
We have shown that $\varphi$ maps the set $\mathscr{P}_{cvn}$ into $\mathscr{E}_{cvn}$. What is more, this mapping is readily seen to be a bijection. Let

$$\mathscr{E}_0= \bigcup_{c,v,n\ge1} \mathscr{E}_{cvn} .$$

(The family $\mathscr{E}_0$ consists of those $w\in\mathscr{W}$ which possess the properties *i)*&*ii)* and have an odd number of x-steps.) For all $c,v,n\in\mathbb{N}$ we have

$$\langle x^{2c}y^{2v}q^n\rangle\ V= \langle x^{2c-1}y^{2v}q^n\rangle\ gf_1(\mathscr{E}_0) ,$$

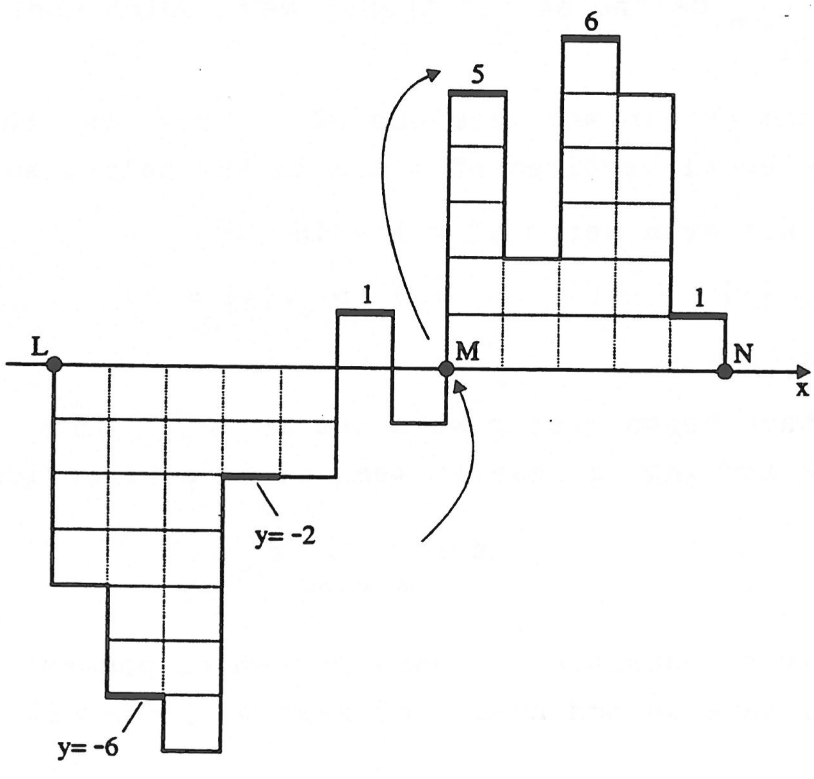
which means that $$V= x\cdot gf_1(\mathscr{E}_0) . \tag{4}$$




$h(P)=8$, $v(P)=18$,
$a(P)=13$, $d(P)=2$

$|w|_x=7$,
$|w|_y+|w|_{\bar{y}}=18$,
$a_1(w)=13$

**Figure 1.** A dcc-polyomino P and its code




from L to N: $u\in\mathcal{A}_0$, with $a_2(u)=5$

from L to M: $v\in\mathcal{B}_0$, with $a_2(v)=-7$

from M to N: $z\in\mathcal{C}_0$, with $a_1(z)=12$

**Figure 2.** A path $u\in\mathcal{A}_0$ has a unique factorization $u=vz$, where $v\in\mathcal{B}_0$ and $z\in\mathcal{C}_0$

## 4. A factorization of lattice paths

It turns out advantageous to regard the paths of $\mathscr{C}_0$ as right factors of certain other lattice paths. The relevant definitions follow.

Let $\mathscr{A}$ be the set of those $u\in\mathscr{W}$ which possess the properties:

*i)* $|u|_x$ is a nonzero even number,
*ii)* the odd nests of $u$ lie in $\{\bar{y}\}^*$,
*iii)* the last (odd) nest of $u$ is nonempty.

Let $\mathscr{A}_0=\{u\in\mathscr{A}:\ r(u)=0\}$.

Further, let $\mathscr{B}$ be the set of those $v\in\mathscr{W}$ which possess the properties:

*i)* $|v|_x$ is an odd number,
*ii)* the odd nests of $v$ lie in $\{\bar{y}\}^*$,
*iii)* the last (even) nest of $v$ lies in $\{y\}^*$.

Let $\mathscr{B}_0=\{v\in\mathscr{B}:\ r(v)=0\}$.

Now, let $u\in\mathscr{A}_0$. Consider the factorization $u=vz$, where $v$ is the longest among such left factors of $u$ which are different from $u$ and have rank zero. A little thought shows that here we have $v\in\mathscr{B}_0$ and $z\in\mathscr{C}_0$. Evidently,

$$|u|_x=|v|_x+|z|_x \qquad \text{and} \qquad |u|_y+|u|_{\bar{y}}=(|v|_y+|v|_{\bar{y}})+(|z|_y+|z|_{\bar{y}}) .$$

Since $|v|_x$ is an odd number, the odd x-steps of $z$ are even x-steps of $u$, and consequently $a_2(u)=a_2(v)+a_1(z)$. Furthermore, the factorization just described is actually a bijection between $\mathscr{A}_0$ and the cartesian product $\mathscr{B}_0\times\mathscr{C}_0$. (See Fig.2.)

Putting these remarks together, we find

$$gf_2(\mathscr{A}_0)=gf_2(\mathscr{B}_0)\cdot gf_1(\mathscr{C}_0) \quad . \tag{5}$$

From (4) and (5) it follows that

$$V=x\cdot\frac{gf_2(\mathscr{A}_0)}{gf_2(\mathscr{B}_0)} \quad . \tag{6}$$

## 5. Computations

As we see, now we need to compute the functions $gf_2(\mathscr{A}_0)$ and $gf_2(\mathscr{B}_0)$. A good way to do that is to compute $gf_2(\mathscr{A})$ and $gf_2(\mathscr{B})$ first, and then read off the coefficients of $t^0$.

In what follows, for $k$ a negative integer, we write $y^k$ to mean $\bar{y}^{(-k)}$.

Now, the family $\mathscr{A}$ consists of all paths of the form

$$u = y^{n_1}x\cdot y^{n_2}x\cdot y^{n_3}x\cdot y^{n_4}x\cdots y^{n_{2i-1}}x\cdot y^{n_{2i}}x\cdot y^{n_{2i+1}},$$

with $i\in\mathbb{N}$, the odd-indexed $n$'s up through $n_{2i-1}$ nonpositive, $n_{2i+1}$ strictly negative, and the even-indexed $n$'s arbitrary integers. It is easy to see that for such a $u$ we have

$$a_2(u) = i\cdot n_1 + i\cdot n_2 + (i-1)\cdot n_3 + (i-1)\cdot n_4 + \ldots + n_{2i-1} + n_{2i}$$

and

$$gf_2(\{u\}) = x^{2i}\cdot(y^{|n_1|}q^{in_1}t^{n_1})\cdot(y^{|n_2|}q^{in_2}t^{n_2})\cdot$$

$$\cdot(y^{|n_3|}q^{(i-1)n_3}t^{n_3})\cdot(y^{|n_4|}q^{(i-1)n_4}t^{n_4})\cdots$$

$$\cdots(y^{|n_{2i-1}|}q^{n_{2i-1}}t^{n_{2i-1}})\cdot(y^{|n_{2i}|}q^{n_{2i}}t^{n_{2i}})\cdot(y^{|n_{2i+1}|}t^{n_{2i+1}}).$$

Now we sum this latter equation over $i\geq 1$ and over all legal values of $n_1,\ldots,n_{2i+1}$. Using the evaluation

$$\sum_{n\in\mathbb{Z}} y^{|n|}q^{kn}t^n = \frac{1-y^2}{(1-yq^{-k}t^{-1})\cdot(1-yq^k t)},$$

which is valid for every $k\in\mathbb{R}$, we find that

$$gf_2(\mathscr{A}) = \sum_{i\geq 1} \frac{x^{2i}(1-y^2)^i y t^{-1}}{(a)_i (b)_i (b)_{i+1}},$$

where $a = yqt$ and $b = yq^{-i}t^{-1}$. In order to expand $gf_2(\mathscr{A})$ in a series in powers of $t$, next we apply the familiar identity

$$\frac{1}{(c)_n} = \sum_{r\geq 0} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} \cdot c^r \qquad (n\in\mathbb{N}), \tag{7}$$

the proof of which can be found *e.g.* in [13, p.18, Ex.3]. The result is

$$gf_2(\mathscr{A}) = \sum_{\substack{i\geq 1 \\ j,k,l\geq 0}} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+l \\ i \end{bmatrix} x^{2i}(1-y^2)^i y^{j+k+l+1} q^{-i(k+l)+j} t^{j-k-l-1}$$

Finally, we take the coefficient of $t^0$ to find that

$$gf_2(\mathscr{A}_0) = \sum_{i,j\geq 1} x^{2i}(1-y^2)^i y^{2j} q^{i+j-ij} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \cdot \sum_{k=0}^{j-1} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k-1 \\ i \end{bmatrix} . \tag{8}$$

The function $gf_2(\mathscr{B}_0)$ is found in much the same way as $gf_2(\mathscr{A}_0)$, so we omit the derivation and merely state that

$$gf_2(\mathscr{B}_0) = x \cdot \left\{ \frac{1}{1-y^2} + \sum_{i\geq 1,\, j\geq 0} x^{2i}(1-y^2)^i y^{2j} q^{-ij} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot \sum_{k=0}^{j} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k \\ i \end{bmatrix} \right\} \tag{9}$$

## 6. The theorem

Combining (6), (8) and (9), we establish our main result:

**Theorem 1.** The generating function for dcc-polyominoes is given by

$$V = \frac{\displaystyle\sum_{i,j\geq 1} x^{2i}(1-y^2)^i y^{2j} q^{i+j-ij} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \cdot \sum_{k=0}^{j-1} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k-1 \\ i \end{bmatrix}}{\displaystyle\frac{1}{1-y^2} + \sum_{i\geq 1,\, j\geq 0} x^{2i}(1-y^2)^i y^{2j} q^{-ij} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot \sum_{k=0}^{j} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k \\ i \end{bmatrix}} \tag{10}$$

□

Comparison of the formulas (1) and (10) is now in order. First, we must confess that there is one nice property of formula (1) which formula (10) does not share. Namely, in (1) we can put y=1 to find that the area gf of dcc-polyominoes is given by

$$\mathrm{agf}(\mathcal{V}) = \frac{x^2 q(1-q)}{(1-q)^2 - x^2 q} \quad .$$

In contrast to that, we cannot put y=1 in (10), because the numerator, and so too the denominator, would no longer formally converge. In other respects, however, it seems that (10) can bear comparison with its rival formula (1). In fact, in view of the identity (7), it might be said that the numerator (resp. denominator) of (10) virtually involves one summation less than the upper (resp. lower) two floors of (1).

● We have not attempted to give an independent (no-polyominoes) proof that the right sides of (1) and (10) are the same. However, we think that such an attempt would have good chances of success, especially in view of the possibility to consult the now-known similar proofs for parallelogram polyominoes.

## 7. Applications to other models

As we have said, besides dcc-polyominoes, there are also some other models to which our method applies. Here are two examples.

### 7.1 Parallelogram polyominoes

Let P be the gf of parallelogram polyominoes. Our method gives the formula

$$P = \frac{A}{C} \quad , \tag{11}$$

where

$$A = \sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i} y^{2j} q^{i+j-ij}$$

and

$$C = \sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij} \quad .$$

It is not difficult to show that (11) implies the related results of Pólya [14] and Gessel [10, Proposition 11.1], and that (11) is equivalent to the result of the exercise 5.5.2.*b)* in Goulden and Jackson [11].

Bousquet-Mélou and Viennot [4] obtained two formulas for P which (unlike (11)) do not involve negative powers of q. One of those formulas generalizes earlier results due to Klarner and Rivest [12], and to Delest and Fédou [7].

## 7.2 Directed convex polyominoes

Let K be the gf of directed convex polyominoes. Our method finds the formula

$$K=\frac{A+B}{C}, \qquad (12)$$

where
$$B=\sum_{i,j=2}^{\infty}\sum_{k=1}^{i-1}\sum_{l=1}^{j-1}\begin{bmatrix} i+l-1 \\ i \end{bmatrix}\begin{bmatrix} j+k-1 \\ j \end{bmatrix}\begin{bmatrix} i+j-k-l-2 \\ i-k-1 \end{bmatrix} x^{2i}y^{2j}q^{i+j-kl},$$

while A and C are as in (11). Note that the knowledge of (12) enables everybody (whether he know or not where does the function K come from) to prove it easily that K is symmetric in x and y. (Let $\overline{B}$ be the expression B with the variables x and y exchanged. We can convert B into $\overline{B}$, and so too $\overline{B}$ into B, by doing two interchanges of indices: $i\leftrightarrow j$ and $k\leftrightarrow l$.)

Once more, our formula has a different-looking precursor: it is Bousquet-Mélou and Viennot's [4] formula for K, which involves nonnegative powers of q only. Incidentally, the formula of [4] was elegantly rederived in the later-day papers [2,3].

## References


[1] E.Barcucci, R.Pinzani, E.Rodella, R.Sprugnoli, *A characterization of binary search networks*, in: L.Budach (Ed.), Proceedings FCT'91, Lecture Notes in Computer Science **529** (Springer--Verlag, Berlin, 1991), 126-135.

[2] M.Bousquet-Mélou, *A method for the enumeration of various classes of column-convex polygons*, rapport n° 578-93 (1993), LaBRI, Université Bordeaux I.

[3] M.Bousquet-Mélou, *Polyominoes and polygons*, Contemporary Math. **178** (1994), 55-70.

[4] M.Bousquet-Mélou and X.G.Viennot, *Empilements de segments et q-énumération de polyominos convexes dirigés*, J.Combin.Theory Ser.A **60** (1992), 196-224.

[5] M.P.Delest, *Generating functions for column-convex polyominoes*, J.Combin.Theory Ser.A **48** (1988), 12-31.

[6] M.P.Delest and S.Dulucq, *Enumeration of directed column-convex animals with given perimeter and area*, Croat.Chem.Acta **66** (1993), 59-80.

[7] M.P. Delest and J.M. Fédou, *Enumeration of skew Ferrers diagrams*, Discrete Math. **112** (1993), 65-79.

[8] S.Feretić, *A new coding for column-convex directed animals*, Croat.Chem.Acta **66** (1993), 81-90.

[9] S.Feretić and D.Svrtan, *On the number of column-convex polyominoes with given perimeter and number of columns*, in: A.Barlotti, M.Delest, and R.Pinzani (Eds.), $5^{th}$ FPSAC proceedings, Firenze 1993, 201-214.

[10] I.Gessel, *A noncommutative generalization and q-analog of the Lagrange inversion formula*, Trans.Amer.Math.Soc. **257** (1980), 455-482.

[11] I.P. Goulden and D.M. Jackson, *"Combinatorial Enumeration"*, Wiley, New York, 1983.

[12] D.A.Klarner and R.L.Rivest, *Asymptotic bounds for the number of convex n-ominoes*, Discrete Math. **8** (1974), 31-40.

[13] I.G.Macdonald, *"Symmetric Functions and Hall Polynomials"*, Oxford University Press, Oxford, 1979.

[14] G.Pólya, *On the number of certain lattice polygons*, J.Combin. Theory **6** (1969), 102-105.

[15] X.G.Viennot, *A survey of polyominoes enumeration*, in: P. Leroux and C.Reutenauer (Eds.), $4^{th}$ FPSAC proceedings, Montréal 1992, 399-420.