EXTREMAL PROPERTIES OF h-VECTORS AND HILBERT FUNCTIONS

VESSELIN GASHAROV

We focus on a topic which is on the crossroad between combinatorics and commutative algebra, namely f-vectors of simplicial complexes and h-vectors of multicomplexes on the combinatorial side, and Hilbert functions on the commutative algebra side.

First we recall the definitions of the lexicographic, antilexicographic, and reverse lexicographic orders. Let n be a fixed positive integer and $S^{(d)}$ the set of all d-element subsets of $\{1,2,\ldots,n\}$. The *lexicographic order* $\mathcal{L}$ on $S^{(d)}$ is defined by:

$$A <_{\mathcal{L}} B \text{ if and only if the smallest element of } (A \cup B) \setminus (A \cap B) \text{ is in } A.$$

If we reverse the ordering of $1,2,\ldots,n$ we obtain the *antilexicographic order* $\mathcal{A}$:

$$A <_{\mathcal{A}} B \text{ if and only if the largest element of } (A \cup B) \setminus (A \cap B) \text{ is in } A.$$

The *reverse lexicographic order* (or *rev-lex order*) $\mathcal{R}$ is the reverse of the antilexicographic order, i.e.,

$$A <_{\mathcal{R}} B \text{ if and only if the largest element of } (A \cup B) \setminus (A \cap B) \text{ is in } B.$$

All definitions above generalize in a straightforward manner to multisets.

It is well known and not hard to prove that every positive integer a can be written uniquely in the form:

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta},$$

where $m_d > m_{d-1} > \cdots > m_\delta \geq \delta \geq 1$. This is called the *$d$-binomial representation* of a and $m_d, m_{d-1}, \ldots, m_\delta$ are called the *d-binomial coefficients* of a. Denote by $S_a^{(d)}$ the initial rev-lex segment of $S^{(d)}$ with cardinality a, i.e., the first a elements of $S^{(d)}$ with respect to the rev-lex order. We can describe $S_a^{(d)}$ as follows:

$S^{(d)}_{\binom{m_d}{d}}$ consists of the d-element subsets of $\{1,2,\ldots,m_d\}$,

$S^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}} \setminus S^{(d)}_{\binom{m_d}{d}}$ consists of the sets formed by adding m_d+1 to the $(d-1)$-element subsets of $\{1,2,\ldots,m_{d-1}\}$,

$S^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}+\binom{m_{d-2}}{d-2}} \setminus S^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}}$ consists of the sets formed by adding $\{m_{d-1}+1, m_d+1\}$ to the $(d-2)$-element subsets of $\{1, 2, \ldots, m_{d-2}\}$, etc.

If F is a family of d-element sets, denote by ΔF the family of $(d-1)$-element sets which are subsets of members of F. ¿From the above description of $S^{(d)}_a$ it follows that

$$\Delta S^{(d)}_a = S^{(d-1)}_b,$$

where $b = \binom{m_d}{d-1} + \binom{m_{d-1}}{d-2} + \cdots + \binom{m_\delta}{\delta-1}$.

We generalize this to multisets:

Denote by $M^{(d)}$ the set of d-element multisets on $\{1, 2, \ldots, n\}$ and by $M^{(d)}_a$ the initial rev-lex segment of $M^{(d)}$ with cardinality a. If F is a family of d-element multisets, denote by ΔF the family of $(d-1)$-element multisets which are submultisets of members of F. Then:

$M^{(d)}_{\binom{m_d}{d}}$ consists of the d-element multisets on $\{1, 2, \ldots, m_d - d + 1\}$,

$M^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}} \setminus M^{(d)}_{\binom{m_d}{d}}$ consists of the multisets formed by adding $m_d - d + 2$ to the $(d-1)$-element multisets on $\{1, 2, \ldots, m_{d-1} - d + 2\}$,

$M^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}+\binom{m_{d-2}}{d-2}} \setminus M^{(d)}_{\binom{m_d}{d}+\binom{m_{d-1}}{d-1}}$ consists of the multisets formed by adding $\{m_{d-1} - d + 3, m_d - d + 2\}$ to the $(d-2)$-element subsets of $\{1, 2, \ldots, m_{d-2} - d + 3\}$, etc.

This shows that

$$\Delta M^{(d)}_a = M^{(d-1)}_c,$$

where $c = \binom{m_d - 1}{d-1} + \binom{m_{d-1}-1}{d-2} + \cdots + \binom{m_\delta - 1}{\delta-1}$.

If $a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$, then we denote

$$a^{(d)} = \binom{m_d}{d+1} + \binom{m_{d-1}}{d} + \cdots + \binom{m_\delta}{\delta+1}$$

and

$$a^{\langle d\rangle} = \binom{m_d + 1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_\delta + 1}{\delta+1},$$

so we have:

$$|S^{(d)}_a| \leq |\Delta S^{(d)}_a|^{(d-1)} \quad \text{and} \quad |M^{(d)}_a| \leq |\Delta M^{(d)}_a|^{\langle d-1\rangle}.$$

Kruskal and Katona, and Macaulay proved the remarkable results that the above inequalities generalize to arbitrary subsets of $S^{(d)}$ or $M^{(d)}$:

Theorem(Kruskal-Katona). *Let $F \subseteq S^{(d)}$. Then $|F| \leq |\Delta F|^{(d-1)}$.*

Theorem(Macaulay). *Let* $F \subseteq M^{(d)}$. *Then* $|F| \leq |\Delta F|^{\langle d-1 \rangle}$.

We can reformulate these theorems in terms of f-vectors of simplicial complexes and h-vectors of multicomplexes (see [7, p.55]) as follows:

Theorem. *A vector* $(f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}_+^d$ *is the* f*-vector of a simplicial complex if and only if*

$$f_{i+1} \leq f_i^{(i)}$$

for $0 \leq i \leq d-2$.

Theorem. *A vector* $(h_0, h_1, \ldots)$ *of nonnegative integers is the* h*-vector of a multicomplex if and only if* $h_0 = 1$ *and*

$$h_{i+1} \leq h_i^{\langle i \rangle}$$

for $i \geq 1$.

There are obvious bijections between $S^{(d)}$ and squarefree monomials in n variables, say $x_1, \ldots, x_n$, and $M^{(d)}$ and ordinary monomials in $x_1, \ldots, x_n$. Fix a field k and denote by Q_d and R_d the k-vector spaces of monomials of degree d in $x_1, \ldots, x_n$ and squarefree monomials of degree d in $x_1, \ldots, x_n$, respectively. Let V (resp. W) be a subspace of Q_d (resp. R_d) and denote by V_1 (resp. W_1) the subspaces of Q_{d+1} (resp. R_{d+1}) generated by all polynomials of degree $d+1$ (resp. squarefree polynomials of degree $d+1$) which are divisible by at least one polynomial in V (resp. W). We write $V_1 = VQ_1$ and $W_1 = WR_1$. It is well known (see [6, Theorem 2.1] for example) that there exist subspaces $\bar{V} \subseteq Q_d$, $\bar{V}_1 \subseteq Q_{d+1}$, $\bar{W} \subseteq R_d$, and $\bar{W}_1 \subseteq R_{d+1}$ generated by monomials and satisfying the following 2 conditions:

(1) $V \oplus \bar{V} = Q_d$, $V_1 \oplus \bar{V}_1 = Q_{d+1}$, $W \oplus \bar{W} = R_d$, and $W_1 \oplus \bar{W}_1 = R_{d+1}$;
(2) All monomials in Q_d (resp. R_d) which divide a monomial in $\bar{V}_1$ (resp. $\bar{W}_1$) are in $\bar{V}$ (resp. $\bar{W}$).

By the theorems of Macaulay and Kruskal-Katona we see that $|\bar{V}_1| \leq |\bar{V}|^{\langle d \rangle}$ and $|\bar{W}_1| \leq |\bar{W}|^{(d)}$. Equivalently, we can restate these inequalities as follows:

Theorem. *Let* V, V_1, W, *and* W_1 *be as above. Then*

$$\operatorname{codim}(V_1, Q_{d+1}) \leq \operatorname{codim}(V, Q_d)^{\langle d \rangle}$$

and

$$\operatorname{codim}(W_1, R_{d+1}) \leq \operatorname{codim}(W, R_d)^{(d)}.$$

A natural question to ask is what can be said about the vector spaces V which achieve Macaulay's bound, i.e.,

$$\operatorname{codim}(V_1, Q_{d+1}) = \operatorname{codim}(V, Q_d)^{\langle d \rangle}.$$

One important result in this direction was obtained by Gotzmann:

Gotzmann Persistence Theorem. *Let V and V_1 be as above and V_2 be the subspace of Q_{d+2} generated by all polynomials of degree $d+2$ which are divisible by at least one polynomial in V, i.e., $V_2 = VQ_2$. If $\operatorname{codim}(V_1, Q_{d+1}) = \operatorname{codim}(V, Q_d)^{\langle d\rangle}$, then*

$$\operatorname{codim}(V_2, Q_{d+2}) = \operatorname{codim}(V_1, Q_{d+1})^{\langle d+1\rangle}.$$

Definition. If $\operatorname{codim}(V_1, Q_{d+1}) = \operatorname{codim}(V, Q_d)^{\langle d\rangle}$, then V is called a Gotzmann vector space.

We show that a "reverse" version of Gotzmann Persistence Theorem also holds:

Theorem. *Let $V \subseteq Q_d$ be a Gotzmann vector space and let the d-binomial expansion of $c = \operatorname{codim}(V, S_d)$ be $c = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$ with $\delta > 1$. Let V_{-1} be the vector space generated by all polynomials p in Q_{d-1}, such that $px_1, px_2, \ldots, px_n$ are in V. Then $V = V_{-1}Q_1$ and V_{-1} is a Gotzmann vector space with $\operatorname{codim}(V_{-1}, Q_{d-1}) = \binom{m_d - 1}{d-1} + \binom{m_{d-1}-1}{d-2} + \cdots + \binom{m_\delta - 1}{\delta - 1}$.*

For any vector space $U \subseteq Q_d$, there exists a number $r \in \mathbb{Z}_+$ such that $UQ_s \subseteq Q_{d+s}$ is a Gotzmann vector space for all $s \geq r$. This shows that in general we cannot say much about the structure of Gotzmann vector spaces. However, in some cases we can completely determine their structure:

Theorem. *Let $V \subseteq Q_d$ be a Gotzmann vector space and let the d-binomial expansion of $c = \operatorname{codim}(V, S_d)$ be $c = \binom{a+d}{d} + \binom{a+d-1}{d-1} + \cdots + \binom{a+2}{2} + \binom{b+1}{1}$ for some $a \geq b \geq 0$. Then there exists a vector space $L \subseteq Q_1$ with $\dim L = n - a - 2$ and one of the following is satisfied:*

1. *If $a > b$, then there exists a vector space $K \subseteq Q_1$ with $L \cap K = 0$ and an element $h \in Q_{d-1} \setminus LQ_{d-2}$ such that $\dim K = a - b + 1$ and $V = LQ_{d-1} + hK$.*
2. *If $a = b$, then there exists an element $f \in Q_d \setminus LQ_{d-1}$ such that $V = LQ_{d-1} + \operatorname{span}(f)$.*

The special case $a = b$ of the previous theorem was first proved by Green [3, Theorem 4].

Let $\bar{M}_m^{(d)}$ be the set of elements of $M_m^{(d)}$ not containing 1. ¿From the description of $M_m^{(d)}$ we see that

$$|\bar{M}_m^{(d)}| = \binom{m_d - 1}{d} + \binom{m_{d-1} - 1}{d-1} + \cdots + \binom{m_\delta - 1}{\delta}.$$

This observation was generalized by Green [3, Theorem 1] as follows. We will say that some property P is true for a general element of a vector space L if there exists a dense open subset $U \subseteq L$, such that P is true for all elements of U. An element of U is called a *general element*.

Theorem (Mark Green). *Let $V \subseteq Q_d$ be a vector space of codimension c. Let x be a general element of Q_1 and $\overline{V}$ be the image of V under the projection $Q_d \to \overline{Q}_d = Q/(x)$. Let $c_x = \operatorname{codim}(\overline{V}, \overline{S}_d)$. Then $c_x \leq c_{\langle d\rangle}$.*

Definition. In the situation of the previous theorem, if $c_x = c_{\langle d \rangle}$, then V is called a *Green vector space.*

We show that there is a persistence theorem for Green vector spaces:

Theorem. *Let $V \subseteq Q_d$ be a Green vector space. Let x and y be general elements of Q_1 and let $\overline{\overline{V}}$ be the image of V under the projection $Q_d \to \overline{\overline{Q}}_d = Q_d/(xQ_{d-1} + yQ_{d-1})$. Let $c_{x,y} = \operatorname{codim}(\overline{\overline{V}}, \overline{\overline{Q}}_d)$. If $c = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_1}{1}$ with $m_1 \neq 1$, then*

$$c_{x,y} = (c_x)_{\langle d \rangle}.$$

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003

E-mail address: gasharov@math.lsa.umich.edu