Counting constrained domino tilings of Aztec diamonds

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An *Aztec diamond* may be defined as the union of those 1-by-1 closed lattice squares that lie completely inside the region $\{(x, y) : |x|+|y| \leq n+1\}$. A *domino* is a 1-by-2 rectangle, and a *tiling* of an Aztec diamond by dominoes is a collection of dominoes whose interiors are disjoint and whose union is the Aztec diamond. Figure 1 shows a tiling of an Aztec diamond of order 4 by dominoes. It was shown in [3] that the Aztec diamond of order n has exactly $2^{n(n+1)/2}$ tilings by dominoes. In this article, we establish two other results concerning domino tilings of Aztec diamonds.

The first result is a formula governing the number of domino tilings of Aztec diamonds, subject to the constraint that a particular pair of adjacent lattice squares must be covered by a domino. This formula is expressed in terms of a three-variable generating function whose coefficients are the respective probabilities that a domino tiling, chosen uniformly at random from the set of all tilings of some particular Aztec diamond, contains a domino covering some particular pair of adjacent lattice squares. We prove that this generating function is in fact a rational function of its variables.

The second result is a formula (first conjectured by William Jockusch in private correspondence) for the number of domino tilings of an Aztec diamond of order n from which the central 2-by-2 square has been removed, referred to here as a *holey Aztec diamond.* This number is easily seen to be equal to the number of domino tilings of an Aztec diamond subject to the constraint that the central 2-by-2 square must be covered by a pair of horizontal dominoes. Hence proving the formula is equivalent to proving a formula for the probability that a randomly-chosen domino tiling of an Aztec diamond of order n contains not one particular domino, as in the preceding paragraph, but rather a pair of dominoes in two particular locations.

In general, it is not possible to determine the probability that a random tiling contains some *pair* of dominoes from the probabilities that a random tiling contains each domino *individually*, but in the case where the dominoes happen to form a 2-by-2 square (as happens here), then a general graph-theoretic lemma on perfect matchings of plane graphs allows us to derive the probability of such a compound event from the probabilities of four simple events. This reduces proving Jockusch's conjecture to proving a formula for the number of order-n tilings containing a horizontal domino in the center of the nth row, which we are able to derive from the generating function described above.

In give a more precise statement of the formula involving the three-variable generating function, we will find it convenient to pass to a dual picture in which domino tilings of an Aztec diamond of order n are replaced by perfect matchings of the dual graph G_n, which we blow up by a factor of 2 so as to make all coordinates integers. Specifically, the vertices of G_n are lattice points in $\mathbf{Z}^2$ with a, b odd and $|a| + |b| \leq 2n$, and the edges of G_n connect vertices at distance 2. We represent each edge in this graph by its midpoint (i, j), where i is even and j is odd for a horizontal edge and vice versa for a vertical edge. A *perfect matching* of G_n is a collection of edges such that each vertex of G_n is an endpoint of exactly one edge in the collection.

Domino tilings of the order-n diamond correspond to perfect matchings of G_n, so that the probability that a randomly-chosen tiling of the Aztec diamond will contain a domino in a particular position is equal to the probability that a randomly-chosen perfect matching of G_n will contain the corresponding edge of G_n. We call this quantity an *edge-probability*, and denote it by $E_{i,j}^{(n)}$, where (i,j) is the midpoint of the edge of G_n being considered.

Appealing to the four-fold rotational symmetry of G_n, we may without loss of generality focus on the horizontal edges centered at vertices (i,j) with $i-j \equiv 2n+1 \pmod 4$. We will show that

$$\sum_{n=1}^{\infty}\left(\sum_{i,j} E_{i,j}^{(n)} x^i y^j\right) z^n = \frac{yz/2}{(1-y^2z)(1-(x^2+x^{-2}+y^2+y^{-2})z/2+z^2)},$$

where i,j range over all pairs with $|i|+|j| \le 2n$, i even, j odd, and $i-j \equiv 2n+1 \pmod 4$. This algebraic relation encodes a recurrence relation for the numbers $E_{i,j}^{(n)}$, which allows us to calculate them efficiently for fairly large n (with storage space being more of the critical resource than computation time). For instance, the unnumbered figure that appears at the end of this abstract shows the $E_{i,j}^{(n)}$'s for $n = 256$, where intensity-level (black to white) corresponds to the value of $E_{i,j}^{(n)}$ (0 to 1), and where the upper-left corner of the figure corresponds to the edge centered at $(0, 2000)$. (For an explanation of the circularity of the boundary of the northern black region, see [2] or [1].)

We actually prove a more general result that includes an additional parameter. This result concerns random selection of matchings in which one orientation of edge (horizontal or vertical) is favored over the other. More specifically, we can select a perfect matching of G_n so that the probability of any particular matching M being chosen is equal to $p^{h(M)/2}(1-p)^{v(M)/2}$, where $h(M)$ (resp. $v(M)$) is the number of horizontal (resp. vertical) edges in M, necessarily an even integer. Here p is an arbitrary real number between 0 and 1; the case $p = \frac{1}{2}$ corresponds to the unbiased case discussed earlier. Let $E_{i,j}^{(n)}(p)$ be the probability that a randomly-chosen perfect matching of G_n (chosen in accordance with the p-biased distribution) will contain the edge centered at (i,j); then Theorem 1 of our paper states that

$$\sum_{n=1}^{\infty}\left(\sum_{i,j} E_{i,j}^{(n)}(p) x^i y^j\right) z^n = \frac{pyz}{(1-y^2z)(1-(px^2+px^{-2}+qy^2+qy^{-2})z+z^2)},$$

where we have put $q = 1-p$ for convenience, and where i,j range over the same pairs as in the preceding formula.

Our proof of this theorem makes use of the "shuffling algorithm" introduced in [3]. In the earlier article, shuffling was construed as a deterministic operation that gave a bijective proof of the formula for the number of domino tilings of Aztec diamonds; specifically, given a tiling of the diamond of order $n-1$ and a string of n bits, one can construct a tiling of the diamond of order n, such that each possible tiling of the larger region arises once and only once. Under this correspondence, the positions of dominoes in a tiling of the larger region are determined by the positions of dominoes in a tiling of the smaller region, plus extra information coming from the n bits. Here, we show how one can view shuffling as a random process, by treating both the tiling of the smaller region and the bit-string as indeterminate (i.e. random); shuffling gives rise to a scheme that determines probabilities associated with the larger diamond from probabilities associated with the smaller diamond. Naturally, we express this in terms of perfect matchings rather than tilings.

To prove Theorem 1, it turns out to be very useful to introduce quantities pertaining to the shuffling process itself. These are the *net creation rates* (the reason for this terminology will be explained in the full article). These rates are associated not with edges but with certain cells in the square grid. Our proof of Theorem 1 is essentially based on a double induction, in which the edge-probabilities in the graph G_n determine the creation rates for the graph G_n which in combination with the edge-probabilities for G_n allow one to compute the edge-probabilities for G_{n+1}. The creation rates occur as coefficients of the generating function

$$\frac{z}{1-(px^2+px^{-2}+qy^2+qy^{-2})z+z^2},$$

which is similar to the formula for the edge-probabilities but simpler. This generating function is related to the Krawtchouk polynomials, and in some sense it may be even more fundamental than the formula governing the edge-probabilities; for instance, in the article [1], which obtains $o(1)$-estimates for the edge-probabilities as $n \to \infty$, the fundamental formula from the current paper that is used is actually the formula for the generating function associated with creation rates.

Theorem 2 of the current paper is the formula of Jockusch mentioned earlier. It states that the number of perfect matchings of the graph obtained from G_n by deleting its four central vertices is

$$2^{\left(\binom{n+1}{2}-2n-3\right)}\left(2^n-\binom{n/2}{n/4}^2\right)^2$$

for $n \equiv 0 \pmod 4$,

$$2^{\left(\binom{n+1}{2}-2(n-1)-3\right)}\left(2^{n-1}+\binom{(n-1)/2}{(n-1)/4}^2\right)^2$$

for $n \equiv 1 \pmod 4$, and

$$2^{(n(n+1)/2)-3}$$

for $n \equiv 2$ or $3 \pmod 4$. We derive this from Theorem 1 by means of a simple contour integration (to calculate individual coefficients of the generating function) and the following general result (Lemma 1): If G is any bipartite plane graph with edges a, b, c, d forming a 4-cycle, then the probability that a randomly-chosen perfect matching of G (chosen uniformly at random from the set of all perfecting matchings) contains edges a and c is just $p_a p_c + p_b p_d$, where p_a, p_b, p_c, and p_d are the probabilities that a random matchings contains the individual edges a, b, c, and d, respectively.

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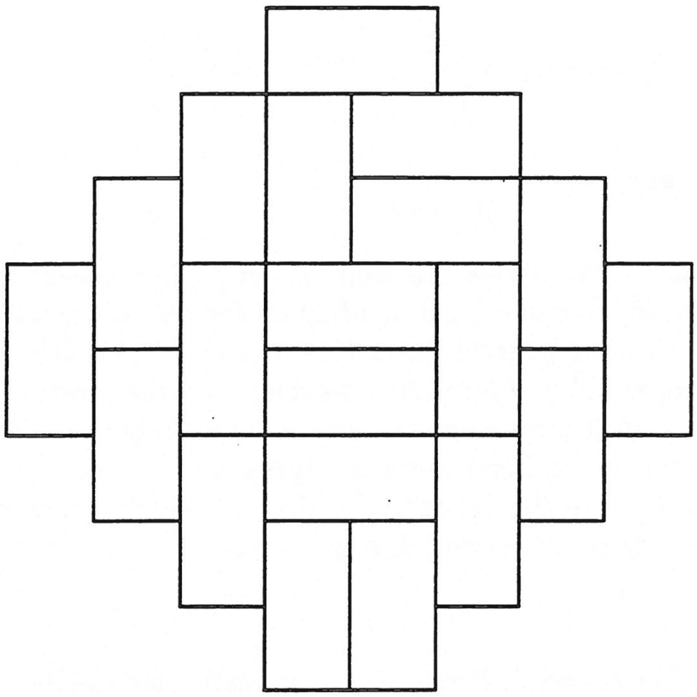
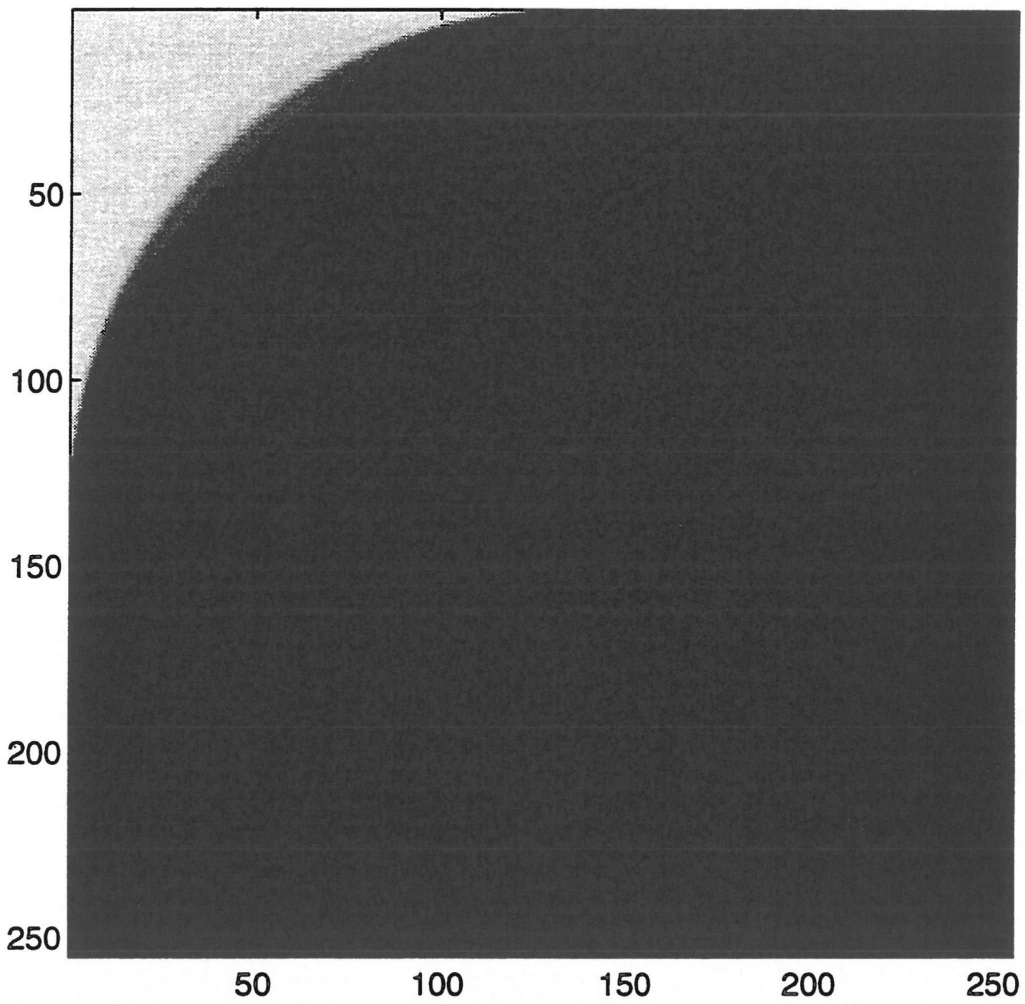


Figure 1



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