# Rook Theory and Hypergeometric Series

by
James Haglund

Mathematics Department
The University of Illinois at Urbana-Champaign
Urbana, IL 61801
jhaglund@math.uiuc.edu

*Abstract: The number of ways of placing $k$ non-attacking rooks on a Ferrers board is expressed as a hypergeometric series, of a type originally studied by Karlsson and Minton. Known transformation identities for series of this type translate into new theorems about rook polynomials.*

## 1. Summary.

Since its introduction in the 1940's by Riordan and Kaplansky, *rook theory* has continued to find application to an ever-expanding list of topics in Enumerative Combinatorics. In this article we establish a connection between rook polynomials and certain types of *hypergeometric series*, and explore the consequences.

Section 2 begins with an overview of some historical results in rook theory, and highlights recent work, which counts permutations by cycle type. We also introduce an extra parameter into Chow's "path-cycle" symmetric function of a digraph, and show how his reciprocity theorem extends to this more general function.

In section 3 we explicitly express the cycle-counting rook numbers as hypergeometric series, and show how some special cases of the Karlsson-Minton Summation formulas have a very simple combinatorial interpretation. Then we use Gasper's transformation for series of this type to derive a transformation identity for rook polynomials, which specializes to an interesting Stirling number identity. By combining Gasper's transformation with familar facts from rook theory, we are led to a new expression for Karlsson-Minton type series with parameter $z$.

In section 4 we demonstrate how a generating function for rook polynomials yields a generating function for terminating, balanced hypergeometric series of all orders. By differentiating in the standard way we obtain recurrence relations (which are essentially "iterated" contiguous relations). By a different method we derive a recurrence for rook numbers which contains Whipple's transformation for terminating, balanced ${}_4F_3$'s and Saalschütz summation as special cases. Some choices of the parameters in this recurrence yield identities which have simple interpretations involving permutations of multisets.

Section 5 contains $q$-versions of some of the previous results. *Notation* : LHS and RHS are abbreviations for "left hand side" and "right hand side" respectively. $\mathbb{N}$ = the nonnegative integers, $\mathbb{Z}$ = the integers, $\mathbb{C}$ = the complex numbers, "COEF($z^k$) in" means "the coefficient of $z^k$ in".

## 2. Rook Theory.

Consider an infinite grid of squares, with the same labelling as the points in the first quadrant having positive integral coordinates; the lower left-hand square has (column,row) coordinates (1,1), etc.. A *board* $B$ is a finite subset of these squares, together with a value of $n$, called the number of columns. The squares of $B$ must satisfy $(i,j) \in B \implies 1 \le i \le n,\ 1 \le j$. If in addition $(i,j) \in B \implies j \le n$ (all the squares of $B$ are contained in the $n \times n$ grid) then $B$ is called *admissible*. See Figure 1.

Let $r_k(B)$ be the number of ways of placing $k$ rooks on the squares of $B$ (throughout the article, all placements are assumed to be non-attacking, i.e. no two rooks in the same row, and no two in the same column). If $B$ is admissible, let $a_k(B)$ be the number of ways of placing $n$ non-attacking rooks on the square $n \times n$ grid with exactly $n-k$ rooks on $B$. The $a_k$ are usually called "hit" numbers. Of particular interest is $a_n$, which equals the number of permutations on $n$ letters which avoid the "forbidden" positions encoded by the squares of $B$ (we can identify a rook on square $(i,j)$ with the condition that $i$ is sent to $j$ in the associated permutation). The $a_k(B)$ can be expressed in terms of the $r_k(B)$ via an identity of Riordan and Kaplansky [KaRi];

$$\sum_k k! r_{n-k}(B)(z-1)^{n-k} = \sum_k z^k a_{n-k}(B) \tag{1}$$

If $B$ is not admissible, define $a_k(B)$ via (1) (although they no longer count permutations).

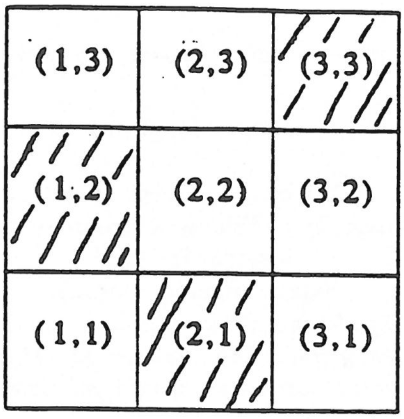


Figure 1: The shaded squares (1,2), (2,1), and (3,3)
of the $3 \times 3$ grid form an admissible board $B$.

A *Ferrers board* $B$ is a board with the property that $(i,j) \in B$ implies all squares to the right and below $(i,j)$ are also in $B$. More formally, $(i,j) \in B \implies (k,p) \in B$ for $i \leq k \leq n$ and $1 \leq p \leq j$. These boards can be identified with the Ferrers graphs of partitions. They were introduced by Foata and Schützenberger, who proved that every Ferrers board is *rook equivalent* (has the same rook numbers) to a unique board with strictly increasing column heights. Ferrers boards satisfy the important *factorization theorem* of Goldman, Joichi, and White [GJW1];

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k} = \prod_{i=1}^{n}(x+c_i-i+1), \tag{2}$$

with $c_i$ = the height of the $i^{th}$ column of $B$.

Throughout this article, if $B$ is a Ferrers board it will represent the board of Figure 2, indicated by the following notation: $B = B(h_1, d_1; h_2, d_2; \ldots; h_t, d_t)$.

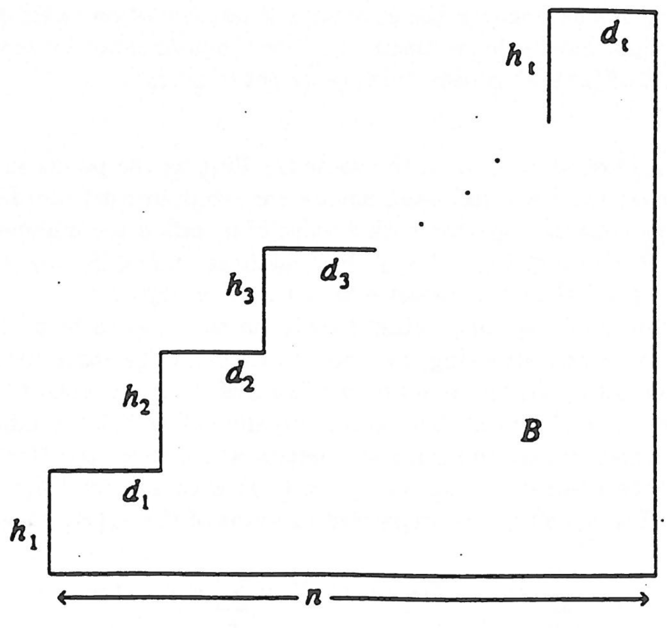


Figure 2: The Ferrers board $B = B(h_1, d_1; \ldots; h_t, d_t)$.

In order to allow leading columns of height zero and for other technical reasons we allow the $h_i$ to be nonnegative integers, but the $d_i$ will be strictly positive integers. Note that the RHS of (2) can be written as

$$\prod_{i=1}^{t}(x + H_i - D_i + 1)_{d_i}, \tag{3}$$

where $H_i := h_1 + \ldots + h_i$, $D_i := d_1 + d_2 + \ldots + d_i$ (this notation will be used often), and $(x)_k := x(x+1)\cdots(x+k-1)$.

For some time researchers have sought a $q$-version of (1), the inclusion-exclusion identity of Riordan and Kaplansky. For arbitrary boards this problem has never been completely solved, although partial solutions occur in [ChRo] and [JoRo]. For Ferrers boards Garsia and Remmel [GaRe] introduced a $q$-version which has found a number of applications [Din1], [Din2], [Hag1]. In particular, Ding has shown that the Poincaré polynomials of cohomolgy for certain algebraic varieties associated to Ferrers boards are expressable as $q$-rook polynomials of Garsia and Remmel.

Other recent work in rook theory incorporates the cycle structure of *simple directed graphs* associated to rook placements. This idea originated in a 1989 paper of Gessel [Ges1]; if a rook occupies square $(i, j)$, draw an edge from $i$ to $j$ in the associated digraph (otherwise do not draw such an edge). The resulting digraph (on $n$ vertices) will consist of a certain number of cycles and a certain number of directed paths (vertices with no incident edges count as a directed path of length one). See Figure 3.

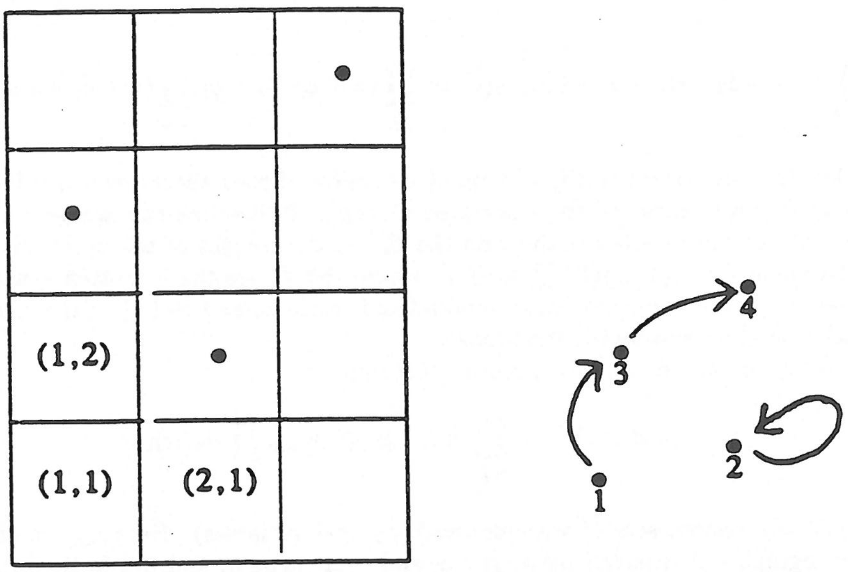


Figure 3: A rook placement and the associated digraph.

It should be mentioned that the special problem of determining $r_n(B)$, which can be viewed as the permanent of a matrix, has been studied in great detail by Shevelev. His work also contains some results on determining $r_n(y, B)$; see [Shev] and the references therein.

Let

$$r_k(y) := \sum_{\text{placements of } k \text{ rooks on } B} y^{\text{number of cycles}},$$

so for the placement of Figure 3 we associate $y^2$. If $B$ is admissible, we can define

$$a_k(y, B) := \sum_{\substack{\text{placements of } n \text{ rooks on } n\times n \text{ square} \\ n-k \text{ rooks on } B}} y^{\text{number of cycles}}. \tag{4}$$

Chung and Graham introduced the function

$$C(B; x, y) := \sum_k x(x-1)\cdots(x-k+1) r_{n-k}(y). \tag{5}$$

One of their results can be expressed as follows [ChG];

$$C(B;x,y) = \sum_{\substack{S \\ \text{0 to } n \text{ rooks on } B \\ \text{each rook in a cycle}}} (y-1)^{\text{number of cycles of S}} \sum_{\substack{T \\ n \text{ rooks on } n\times n \text{ square} \\ S\subseteq T}} \binom{x+|T\cap B|-|S|}{n-|S|}$$

where the inner sum is over all placements $T$ of $n$ non-taking rooks which contain the rooks in $S$, and $|T\cap B|$ is the number of rooks in $T$ on $B$. In the outer sum, each rook in $S$ must be in a cycle.

Gessel [Ges2] found a more compact expansion for $C(B;x,y)$;

$$C(B;x,y) = \sum_k a_{n-k}(y,B)\frac{(x+y)_k x(x-1)\cdots(x-n+k+1)}{(y)_n}. \tag{6}$$

He also noted that

$$\sum_k (y)_k r_{n-k}(y,B)(z-1)^{n-k} = \sum_k z^k a_{n-k}(y,B). \tag{7}$$

Shortly after a preprint of Chung and Graham's influential work became available, the author and Dworkin noticed independently that a version of the factorization theorem for Ferrers boards held for $r_k(y)$ [EHR], [Dwo];

$$\sum_k x(x-1)\cdots(x-k+1)r_{n-k}(y) = \prod_{c_i\geq i}(x+c_i-i+y)\prod_{c_i<i}(x+c_i-i+1). \tag{8}$$

Earlier Stanley and Stembridge [StS] developed a version of rook theory which takes into account the cycle structure of rook placements and the associated digraph. To describe this we need two partitions $\alpha$, $\beta$. The $\alpha_i$ are the lengths of the directed paths, and the $\beta_i$ are the lengths of the cycles. In their theory they weight a given placement by $f_\alpha(Y)p_\beta(Y)\prod_i m_i(\alpha)!$, where the $f_\alpha$ are the forgotten symmetric functions in the set of variables Y, $p_\beta$ are the power-sum symmetric functions, and $m_i(\alpha)$ is the multiplicity of $i$ in $\alpha$ (see [Mac] for background on symmetric functions).

Chow has recently considered a more general function;

$$C(B;X,Y) := \sum_{\alpha,\beta} m_\alpha(X)p_\beta(Y)r_{\alpha,\beta}\prod m_i(\alpha)!$$

(in this section X,Y will denote sets of variables and x,y real variables). Here $r_{\alpha,\beta}$ is the number of rook placements whose digraph has directed path type $\alpha$ and cycle type $\beta$, and $m_\alpha$ is the monomial symmetric function. If X is chosen so that $\sum_i x_i^k = p_k(X) = (-1)^{k+1}p_k(Y)$, $C(B;X,Y)$ reduces to the Stanley-Stembridge function. If $p_k(X)\equiv x$, and $p_k(Y)\equiv y$, we get Chung and Graham's $C(B;x,y)$. Here we are using the well-known fact that identities involving symmetric functions can be interpreted as polynomial identities in the $p_k$.

Chow proved a "reciprocity" theorem for $C(B;X,Y)$, which says that for admissible boards $B$,

$$C(B;X,Y) := \sum_{\substack{\text{0 to } n \text{ rooks on } B^c \\ \text{0 rooks on } B}} f_\alpha(X,Y)p_\beta(Y)\prod_i m_i(\alpha)!(-1)^{n+\ell(\alpha)} \tag{9}$$

where $\ell(\alpha)$ is the number of parts of $\alpha$, and $X,Y$ indicates the union of the two sets of variables $X$ and $Y$ (so $p_k(X,Y) = p_k(X)+p_k(Y)$). $B^c$ is the complement board consisting of those squares in the $n\times n$ grid not a part of $B$.

We now introduce another parameter into Chow's function;

$$C(B;X,Y;z) := \sum_{\alpha,\beta} m_\alpha(X)p_\beta(Y)r_{\alpha,\beta}(1-z)^{n-\ell(\alpha)}\prod m_i(\alpha)!. \tag{10}$$

The next theorem shows that this more general function satisfies a version of reciprocity which contains both (1) and (9) as special cases.
**Theorem 2.1** Let $B$ be an admissible board. Then

$$\sum_{\alpha,\beta} m_\alpha(X)p_\beta(Y)r_{\alpha,\beta}(1-z)^{n-\ell(\alpha)}\prod m_i(\alpha)! = (-1)^n\sum_k z^k C_k(B;X,Y), \qquad \text{where}$$

$$C_k(B;X,Y) = \sum_{\substack{0 \text{ to } n-k \text{ rooks on } B^c \\ k \text{ rooks on } B}} f_\alpha(X,Y)p_\beta(Y)\prod_i m_i(\alpha)!(-1)^{\ell(\alpha)}.$$

Note that if $z=0$ this reduces to (9).
*Remarks on the Proof* : Mimics Chow's proof of (9) [Cho,pp.7-8]. ■

The extent to which theorems about rook placements and applications of reciprocity extend to the symmetric functions $C_k(B;X,Y)$ is an interesting topic for research in its own right; the main focus of this article, however, is to study the following special case of $C_k(B;x,y)$, a new two-parameter version of the hit numbers.
**Definition.** For any board $B$, define $a_k(x,y,B)$ by

$$\sum_k (x)_k r_{n-k}(y)(z-1)^{n-k} = \sum_k z^k a_{n-k}(x,y,B). \tag{11}$$

If $B$ is the triangular board (see Example 3.3), the $a_k(x,1,B)$ have been introduced independently in recent work of Steingrimsson [Ste]. His approach is different from ours, involving partially ordered sets, and there is little duplication between our results.

Using known facts about symmetric functions [Chol], one finds that if $p_k(X)\equiv -x$ and $p_k(Y)\equiv y$, $C_k(B;X,Y)$ reduces to $a_k(x,y,B)$. This same choice for X,Y in Theorem 2.1 then gives (for admissible $B$)

$$a_k(x,y,B) = \sum_{\substack{n-k \text{ rooks on } B \\ 0 \text{ to } k \text{ on } B^c}} (-1)^{\ell(\alpha)}(y-x)_{\ell(\alpha)}y^{\ell(\beta)} \tag{12}$$

where $\alpha$,$\beta$ are the directed path type and cycle type of the associated rook placement. Note that

$$a_k(y,y) = \sum_{\substack{n-k \text{ rooks on } B \\ k \text{ on } B^c}} y^{\text{number of cycles}} = a_k(y,B) \tag{13}$$

since $(y-y)_{\ell(\alpha)}=0$ unless there are no directed paths, which means there are $n$ rooks on $B$.

We end this section by listing extensions of some of the known algebraic identities satisfied by $a_k(B)$.
**Theorem 2.2** (For Ferrers boards, a $q$-version of the case $j=n, x=-1, y=1$ of this identity occurs in work of Garsia and Remmel [GaRe]) Let B be any board, and assume $j$ is a nonnegative integer. Then

$$\sum_{k=0}^{\infty}\binom{x+k-1}{k}a_j(-k,y,B)z^k = \frac{(-1)^j}{(1-z)^{j+x}}\sum_{k=0}^{j}\binom{n-k}{n-j}a_k(x,y,B)z^k.$$

**Theorem 2.3** (The $j=n$ case of this is due to Gessel [Ges2]). For any board $B$,

$$a_j(x,y,B) = \sum_{k=0}^{j} a_k(y,y,B)\binom{n-k}{n-j}\frac{(x)_k(y-x)_{j-k}}{(y)_j}(-1)^{j-k}.$$

### 3. Ferrers Boards and Hypergeometric Series.

The remainder of this article will focus on Ferrers boards, for which there are well-known explicit formulas for $r_k$ and $a_k$. These formulas extend easily to $r_k(y)$ and $a_k(x,y)$, and will be used to express these functions as hypergeometric series.

**Definition.** Let

$$PR(x,y,B) := \prod_{i=1}^{t}(H_i - D_i + x + y)_{d_i}.$$

*Remark* : Clearly $PR$ depends only on the sum $x+y$; we choose to view it as a function of both $x$ and $y$ in order to keep the connection with cycle-counting clear in what follows.

**Lemma 3.1** Let $B = B(h_1, d_1; h_2, d_2; ...; h_t, d_t)$ be a *regular* Ferrers board, by which we mean $B$ satisfies $H_i \geq D_i$ for $1 \leq i \leq t$. Then for $k \in \mathbb{N}$,

$$k! r_{n-k}(y) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(j,y,B), \text{ and} \tag{14}$$

$$a_k(x,y) = \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \binom{x+j-1}{j} PR(j,y,B). \tag{15}$$

*Proof* : Uses the Vandermonde convolution and (2). ■

*Remark* 1: Assume for the moment that $y \in \mathbb{N}$. Then clearly $PR(x,y,B) = PR(x,1,C)$, where $C = B(h_1 + y - 1, d_1; h_2, d_2; \ldots; h_t, d_t)$ is the board obtained from $B$ by replacing $h_1$ by $h_1 + y - 1$. Then by (14) and (15), we see that $r_k(y,B) = r_k(1,C)$, and $a_k(x,y,B) = a_k(x,1,C)$. Now say we have an algebraic identity involving the $r_k$'s or $a_k$'s. Typically this will be a polynomial or rational function identity in the $h_i$'s and $d_i$'s. Thus it is easy to translate back and forth between identites with the $y$ parameter and those without just by changing the value of $h_1$.

We now convert (14) and (15) into hypergeometric notation. Let $e_i := H_i - D_i + y$ (we will use this notation throughout the rest of the article!) and note that for $j \in \mathbb{N}$,

$$(e_i + j)_{d_i} = (e_i)_{d_i} \frac{(e_i + d_i)_j}{(e_i)_j} \qquad \text{assuming} \qquad e_i \neq 0,$$

hence

$$PR(j,y,B) = PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i + d_i)_j}{(e_i)_j} \qquad e_i \neq 0, 1 \leq i \leq t. \tag{16}$$

Plugging this into (14) and (15) we get

$$k! r_{n-k}(y) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i + d_i)_j}{(e_i)_j}$$

$$= PR(0,y,B)(-1)^k \ {}_{t+1}F_t \left[ \begin{matrix} -k, & e_1 + d_1, & \ldots, & e_t + d_t \\ & e_1, & \ldots, & e_t \end{matrix} \right], \text{ and} \tag{17}$$

$$a_k(x,y,B) = \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \frac{(x)_j}{(1)_j} PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i + d_i)_j}{(e_i)_j}$$

$$= PR(0,y,B) \binom{n+x}{k} (-1)^k \ {}_{t+2}F_{t+1} \left[ \begin{matrix} -k, & x, & e_1 + d_1, & \ldots, & e_t + d_t \\ & n+x-k+1, & e_1, & \ldots, & e_t \end{matrix} \right]. \tag{18}$$

Here we have used the standard notation

$${}_pF_q \left[ \begin{matrix} a_1, & a_2, & \ldots, & a_p \\ & b_1, & \ldots, & b_q \end{matrix} ; z \right]$$

for the series

$$\sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{k!(b_1)_k(b_2)_k\cdots(b_q)_k} z^k$$

(if the argument $z$ is 1 it will be omitted).

*Remark* 1: A version of Lemma 3.1, which holds for any board, can be obtained by replacing $PR(j,y,B)$ on the RHS of (14) and (15) by $\sum_k j(j-1)\cdots(j-k+1)r_{n-k}(y,B)$. The assumption that $B$ is a Ferrers board is only needed to write $r_k$ or $a_k$ in hypergeometric notation.

*Remark* 2: The formulas above assume $B$ is regular ($H_i \geq D_i$ for $1 \leq i \leq t$). As a general rule, any formula for Ferrers boards involving the $y$ parameter in sections 3, 4, or 5 will make this same assumption. If $H_i < D_i$ for some $i$, not all of the factors on the RHS of (8) have the parameter $y$ in them, but we can still write our $r_k$ and $a_k$ as hypergeometric series by first modifying the definition of $PR(x,y,B)$ appropriately and then shifting the index of summation.

*Remark* 3: One of the more important types of ${}_{t+1}F_t\left[\begin{matrix} w_1, & \ldots, & w_t, & w_{t+1} \\ & b_1, & \ldots, & b_t \end{matrix}\right]$ series are *balanced* series, where the parameters satisfy $b_1+\ldots b_t = 1+w_1+\ldots w_{t+1}$. The $a_k(B)$ satisfy this property, while the $k!r_{n-k}(B)$ do not. For this reason, most of our attention will be focused on the $a_k$. Also, from results on the $a_k$ one can often deduce properties of the $r_k$ since from (17) and (18),

$$k!r_{n-k}(y) = \lim_{x\to\infty} \frac{a_k(x,y)}{\binom{n+x}{k}}.$$

Equation (8) shows that for Ferrers boards, $a_n(x,y)$ can be written as a product of linear factors in $x$ and $y$. Combining this with the $k=n$ case of (18) we get

$$\prod_{i=1}^{t}(e_i)_{d_i}\binom{n+x}{n}(-1)^n \;{}_{t+2}F_{t+1}\left[\begin{matrix} -n, & x, & e_1+d_1, & \ldots, & e_t+d_t \\ & x+1, & e_1, & \ldots, & e_t \end{matrix}\right] = \prod_{i}^{t}(-x+e_i)_{d_i}. \tag{19}$$

This is equivalent to an identity proved by Minton in 1970 for $n \in \mathbb{Z}$, and extended to complex $n$ by Karlsson in 1971 [Min], [Kar].

In 1981 Gasper [Gasp] showed that (19) is a special case of an interesting transformation identity;

$$\begin{aligned} {}_{t+2}F_{t+1}&\left[\begin{matrix} w, & x, & b_1+d_1, & \ldots, & b_t+d_t \\ & x+c+1, & b_1, & \ldots, & b_t \end{matrix}\right] = \frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)}\prod_{i=1}^{t}\frac{(b_i-x)_{d_i}}{(b_i)_{d_i}} \\ &\times {}_{t+2}F_{t+1}\left[\begin{matrix} -c, & x, & 1+x-b_1, & \ldots, & 1+x-b_t \\ & x+1-w, & 1+x-b_1-d_1, & \ldots, & 1+x-b_t-d_t \end{matrix}\right] \end{aligned} \tag{20}$$

where $w,c,x,b_i \in \mathbb{C}, d_i \in \mathbb{N}$, and $\Re(c-w) > n-1$. We can use this to derive the following result;

**Theorem 3.2** Let $B$ be a regular Ferrers board. Then

$$a_k(x,y,B) = a_{n-k}(x, 1+x-y+n-H_t-p, \widehat{B}_p),$$

where $\widehat{B}_p$ is obtained by first rotating the $n \times H_t$ grid containing $B$ 180 degrees, keeping the squares in this grid which were not in $B$, affixing a $p \times n$ rectangle to the bottom, and finally relabelling so that the square that was $(i,j)$ is now $(n+1-i, H_t+p+1-j)$. The parameter $p$ can be any positive integer, so long as $\widehat{B}_p$ is regular.

*Proof* : Uses (18) and (20). ■

**Example 3.3** Let $B$ be the triangular board of size $n$, so $B$ is regular and $H_t = n$. See Figure 4.
From [EHR],

$$r_{n+1-k}(y,B) = \sum_{\substack{\lambda \\ k \text{ blocks}}} y^{num(\lambda)} := S_2(n+1,k,y)$$

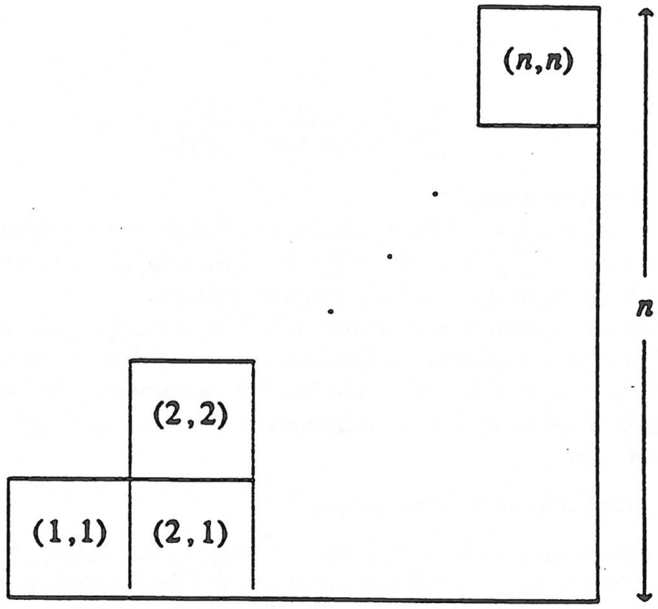


Figure 4: The triangular board of side $n$.

say, where the sum is over all set partitions $\lambda$ of $n+1$ elements into $k$ blocks, and $num(\lambda) :=$ the number of values of $i, 1 \leq i \leq n$, such that the $i^{th}$ and $(i+1)^{st}$ elements are in the same block. For example $S_2(3,2,y) = 2y+1$ since there are 3 set partitions of $\{a,b,c\}$ into 2 blocks:

$$\{a,b\}\{c\} \to y, \quad \{a,c\}\{b\} \to 1, \quad \{a\}\{b,c\} \to y.$$

Clearly $\widehat{B}_1 = B$ so after some simplification (11) and Theorem 3.2 imply

$$\sum_{k=1}^{n} (x)_{k-1} S_2(n,k,y)(z-1)^{n-k} = \sum_{k=1}^{n} (x)_{k-1} S_2(n,k,x-y) z^{k-1}(1-z)^{n-k}.$$

One of the well-known identities for Ferrers boards is

$$\sum_{k=0}^{\infty} PR(k,B) z^k = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^{n} z^k a_k(B).$$

Translating the $x$,$y$ version of this (the case $j=n$ of Theorem 2.2, together with (19)) into hypergeometric series notation led to a result on series of Karlsson-Minton type with parameter $z$.

**Theorem 3.4** Let $x, b_i, z \in \mathbb{C}$, $d_i \in \mathbb{N}$, and $D_t = n$. Also fix the branch of $\log z$ which is analytic for $z \in \mathbb{C}\backslash(-\infty,0]$, with $-\pi < arg(z) < \pi$ (the principal branch). Then for $z \in \mathbb{C}\backslash[1,\infty)$,

$$_{t+1}F_t\left[\begin{matrix} x, & b_1+d_1, & \ldots, & b_t+d_t \\ & b_1, & \ldots, & b_t \end{matrix}; z\right] =$$

$$\frac{1}{(1-z)^{n+x}} \sum_{k=0}^{n} \binom{n+x}{k} (-1)^k \; {}_{t+2}F_{t+1}\left[\begin{matrix} -k, & x, & b_1+d_1, & \ldots, & b_t+d_t \\ & n+x-k+1, & b_1, & \ldots, & b_t \end{matrix}\right] z^k.$$

*Remark* : Theorem 3.4 shows that series of Karlsson-Minton type with parameter $z$ , and one complex parameter $x$, are very close to being polynomials, albeit with complicated coefficients.

## 4. Generating Functions and Recurrence Relations.

In this section we translate various recurrence relations for the hit numbers for Ferrers boards into statements about hypergeometric series. Some "iterated" contiguous relations for balanced series are obtained. We also show that special cases of Saalschütz summation and Whipple's ${}_4F_3$ transformation have simple combinatorial interpretations involving permutations of multisets.

By exploiting a connection between compositions of vectors and rook placements [Hag2, Thm. 22], the following generating function for rook polynomials of Ferrers boards is derived;

$$(\sum_{i=1}^{t} x_i + y_i + \sum_{1\leq i\leq j\leq t} x_i y_j)^k = \sum_{\mathbf{h},\mathbf{d}\in N^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} r_{H_t+D_t-k}(B(h_1,d_1;\dots;h_t,d_t))k!. \tag{21}$$

Using this, we can easily obtain the following generating function for $a_k(x,1,B)$;

$$(1-\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i + (1-z) \sum_{1\leq i\leq j\leq t} x_i y_j)^{-x} = \sum_{\mathbf{h},\mathbf{d}\in N^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!}(x)_{H_t} \sum_{k=0}^{n} a_{n-k}(x+H_t,1,B)z^k. \tag{22}$$

By differentiating (22) with respect to one of the variables we can derive recurrence relations for the $a_k$. The following result is typical;

$$(n-k)a_k(x,1,B) = \sum_{1\leq i\leq j\leq t} h_i d_j a_k(x,1,B-h_i-d_j). \tag{23}$$

It is worth noting that Pfaff-Saalschütz summation, which gives the sum of a terminating, balanced ${}_3F_2$, can be derived by iterating the $t=1$ case of (23) (after using (18)).

A different type of recurrence can be derived from Theorem 2.2.

**Theorem 4.1** Let $B$ be a regular Ferrers board. Let $B_j = B(h_1,d_1;\dots;h_p-j,d_p-j;h_{p+1},d_{p+1};\dots;h_t,d_t)$ be the board obtained from $B$ by decreasing $h_p$ and $d_p$ by $j$ (here we assume $j\geq h_p,d_p$). Then

$$a_k(x,y,B) = j! \sum_{s=k-j}^{k} a_s(x,y,B_j)\binom{H_p-D_{p-1}+y+s-1}{s-k+j}\binom{n+D_{p-1}-H_p-s+x-y}{k-s}.$$

*Proof* : By induction on $j$. The details are omitted. ■

**Corollary 4.2** Assume $d_p \leq h_p + h_{p+1}$, or that $p=t$. Let $B' = B(h_1,d_1;\dots;h_{p-1},d_{p-1};h_p+h_{p+1}-d_p,d_{p+1};\dots;h_t,d_t)$ be the Ferrers board obtained from $B$ by removing the "$p^{th}$ step" (if $p=t$, $B' = B(h_1,d_1;\dots;h_{t-1},d_{t-1})$). Then

$$a_k(x,y,B) = d_p! \sum_{s=k-d_p}^{k} a_s(x,y,B')\binom{H_p-D_{p-1}+y+s-1}{s-k+d_p}\binom{n+D_{p-1}-H_p-s+x-y}{k-s}$$

where $a_s(x,y,\emptyset) = \delta_{s,0}$.

*Proof* : Set $j=d_p$ in Theorem 4.1. ■

Interpreted in terms of hypergeometric series, Corollary 4.2 is equivalent to the known fact that a terminating, balanced ${}_{t+2}F_{t+1}$ with unit argument can be expressed in terms of terminating, balanced ${}_{t+1}F_t$'s with unit argument. If we let $t=1$, then $B'=\emptyset$, and there is only one term on the RHS. After simplification, this reduces to the Pfaff-Saalschütz summation formula mentioned earlier. Letting $t=2$, and using the fact that $a_s(x,y,B')$ can be summed, the RHS turns out to be a terminating, balanced ${}_4F_3$ (as does the LHS). This theorem is known as *Whipple's transformation*.

The $x=1$, $y=1$ case of Corollary 4.2 was previously discovered by the author [Hag1], in connection with the study of *permutations of multisets*. A permutation $\sigma$ of a multiset $M$ is a linear list $\sigma_1\sigma_2\cdots\sigma_{|M|}$ of the elements of $M$. Let $N_k(\mathbf{v})$ be the number of permutations, of the multiset in which $i$ occurs $v_i$ times, having exactly $k-1$ descents. A descent is a value of $i$, $1\leq i\leq |M|-1$, such that $\sigma_i > \sigma_{i+1}$. A connection between $N_k(\mathbf{v})$ and rook placements was discovered by Riordan and Kaplansky and developed further by the author. Dillon and Roselle [DiR] derived a recurrence for the $N_k$ which is a special case of Corollary 4.2.

**5. q-versions.**

Throughout this section, let $q$ be a real variable, $0 < q < 1$. For Ferrers boards $B$, Garsia and Remmel [GaRe] define

$$R_k(B) := \sum_{\text{placements } C \text{ of } k \text{ rooks on } B} q^{invC}$$

where $inv\ C$ is a certain statistic. To calculate it, cross out all squares on $B$ below and all squares on $B$ to the right of each rook in $C$. The number of squares on $B$ not crossed out by this procedure is $inv\ C$.
*Example* : If $C$ consists of rooks on squares $(2,1)$ and $(4,3)$ of the $n = 4$ case of Figure 4, then $inv\ C = 5$.

This definition led them to a $q$-version of (2);

$$\sum_{k=0}^{n}[x][x-1]\cdots[x-k+1]R_{n-k}(B) = \prod_{i=1}^{n}[x+c_i-i+1],$$

where $[x] := \frac{1-q^x}{1-q}$ (which approaches $x$ as $q \to 1$) and $c_i$ is the height of the $i^{th}$ column of $B$.
They also define a $q$-version of $a_k(B)$ as follows;

$$\sum_{k=0}^{n} R_{n-k}(B)[k]!z^k \prod_{i=k+1}^{n}(1-zq^i) = \sum_{k=0}^{n} A_k(B)$$

where $[k]! := \prod_{i=1}^{k}[i]$. The polynomial $A_k(B)$ equals $a_k(B)$ when $q = 1$.

Garsia and Remmel conjectured that $R_k(B)$ is unimodal for all Ferrers boards (this is still open; a special case is the well-known conjecture that the $q$-Stirling numbers of the second kind are unimodal). They were able to show that for admissible $B$, $A_k(B) \in \mathbb{N}[q]$, and in [Hag1] it was demonstrated that their proof extends easily to show that for such boards $A_k(B)$ is a symmetric and unimodal polynomial in $q$.

A cycle-counting version of $R_k$ has been introduced in [EHR]. The following fact is used in its description: given a placement of $j$ non-attacking rooks in columns 1 through $i$ of $B$, where $0 \le j \le i$, then if $c_i$ (the height of the $i^{th}$ column of $B$) is $\ge i$, there is one and only one square in column $i$ where a rook placement will complete a cycle. If $c_i < i$, there is no such square.

Given a placement $C$ of $k$ rooks on $b$, define $s_i$ as follows; if $c_i < i$, $s_i =$ the square $(i, c_i+1)$, while if $c_i \ge i$, $s_i$ is the unique square such that, considering only the rooks from $C$ in columns 1 through $i-1$, a rook on square $s_i$ completes a cycle. Set $E = E(C) =$ the number of $i$ such that there is a rook from $C$ in column $i$ on or above square $s_i$. Then if we define

$$R_k(y,B) := \sum_{\substack{C \\ k\text{-rooks}}} [y]^{\text{number of cycles of C}} q^{invC+E}$$

we get [EHR];

$$\sum_{k=0}^{n}[x][x-1]\cdots[x-k+1]R_{n-k}(y,B) = \prod_{c_i\ge i}[x+c_i-i+y]\prod_{c_i<i}[x+c_i-i+1] = PR[x,y,B] \text{ say.} \tag{24}$$

We can use $R_k(y,B)$ to define a $q$-version of $a_k(x,y,B)$;

$$\sum_{k=0}^{n}[x][x+1]\cdots[x+k-1]R_{n-k}(y,B)z^k \prod_{i=k+1}^{n}(1-zq^{i+x-1}) := \sum_{k=0}^{n} A_k(x,y,B)z^k.$$

An easy calculation shows that if $q = 1$, $A_k(x,y,B) = a_k(x,y,B)$.

In a paper of Chu [Chu], a bilateral $q$-identity is developed which contains both of the $q$-versions of Gasper's transformation found in [Gasp] (the author has noticed that Chu's result can also be obtained by specializing a general result due to Sears and Slater). Together with (24), this can be used to find $q$-versions of almost all of the identities in sections 3 and 4. The exception is the generating function identity (21) and (12), which have no known $q$-versions at present. We use the standard notation

$$(w)_n := (1-w)(1-wq)\cdots(1-wq^{n-1}), [x] := \frac{1-q^x}{1-q}, \begin{bmatrix} x \\ k \end{bmatrix} := \frac{[x][x-1]\cdots[x-k+1]}{[k]!},$$

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1-zq^k), \quad {}_{t+1}\phi_t\left(\begin{matrix} a_1, & a_2, & \dots, & a_{t+1} \\ & b_1, & \dots, & b_t \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_{t+1})_n}{(q)_n(b_1)_n\cdots(b_t)_n} z^n.$$

**Lemma 5.1** For any regular Ferrers board $B$,

$$[k]!R_{n-k}(y) = PR[0,y,B](-1)^k q^{\binom{k}{2}} \; {}_{t+1}\phi_t\left(\begin{matrix} q^{-k}, & q^{e_1+d_1}, & \dots, & q^{e_t+d_t} \\ & q^{e_1}, & \dots, & q^{e_t} \end{matrix}; q, q\right), \text{ and}$$

$$A_k(x,y,B) = PR[0,y,B](-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+x \\ k \end{bmatrix} {}_{t+2}\phi_{t+1}\left(\begin{matrix} q^{-k}, & q^x, & q^{e_1+d_1}, & \dots, & q^{e_t+d_t} \\ & q^{n+x-k+1}, & q^{e_1}, & \dots, & q^{e_t} \end{matrix}; q, q\right).$$

**Theorem 5.2** For any regular Ferrers board $B$,

$$A_k(x,y,B) = A_{n-k}(x, 1+x-y+n-H_t-p, \widehat{B}_p)q^\alpha,$$

where $\alpha := n(-x+y-n)+k(n+x+1)+ area(B)$, with $area(B) =$ the number of squares in $B = \sum_i H_i d_i$. As in Theorem 3.2, $p$ is any positive integer for which $\widehat{B}_p$ is regular.

**Theorem 5.3** Let $j \in \mathbb{N}$, $j \le h_p, d_p$ for some $p$, with $1 \le p \le t$. Then if $B$ is a regular Ferrers board, and $B_j$ is the board described in Theorem 4.1,

$$A_k(x,y,B) = [j]! \sum_{s=k-j}^{k} A_s(x,y,B_j) \begin{bmatrix} e_p+d_p-1+s \\ j-k+s \end{bmatrix} \begin{bmatrix} n-e_p-d_p+x-s \\ k-s \end{bmatrix} q^{(k-s)(e_p+d_p+k-j-1)},$$

where $A_s(x,y,\emptyset) = \delta_{s,0}$, and $e_p = H_p - D_p + y$.

There are also $q$-versions of Theorems 2.2 and 2.3;

**Theorem 5.4** For any Ferrers board $B$, with $0 \le j \le n$, and $|z| < 1$,

$$\sum_{k=0}^{\infty} \begin{bmatrix} x+k-1 \\ k \end{bmatrix} A_j(-k,y,B) q^{jk} z^k = (-1)^j q^{\binom{j}{2}} \frac{(q^{j+x}z)_\infty}{(z)_\infty} \sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix} A_k(x,y,B) z^k. \tag{25}$$

**Theorem 5.5** For any Ferrers board $B$, with $0 \le j \le n$,

$$A_j(x,y,B) = q^{\binom{j}{2}} \sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix} A_k(y,y,B) \frac{(q^x)_k (q^{y-x})_{j-k}}{(q^y)_j} (-q^x)^{j-k} q^{-\binom{k}{2}}. \tag{26}$$

**Acknowledgement.** The author would like to thank Professors Andrews, Bowman, and Stanton for supplying him with some of the references on hypergeometric series used in this article. He would also like to thank the referee s for supplying him with the following interesting identity, where $P$ is any polynomial with
$\Re(c+1-w-\deg(P)) > 0$

$$\sum_{j=0}^{\infty} \frac{(w)_j(x)_j}{j!(x+c+1)_j} P(j) = \frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)} \sum_{j=0}^{\infty} \frac{(-c)_j(x)_j}{j!(x+1-w)_j} P(-x-j). \tag{27}$$

Eq. (27) can be proven by proving it for the basis polynomials $P(u) = (-x-c-u)_d, d = 0,1,\dots,$ for which both sides of (27) can be evaluated by Gauss' theorem on the sum of a ${}_2F_1$. This can be used in place of Gasper's transformation in the proof of Theorem 3.2, the advantage being that part of this proof then extends to non-Ferrers boards. Other parts of the proof do not, however, so it doesn't appear that a version of Theorem 3.2 holds for arbitrary boards.

## REFERENCES


[Cho1] T. Y. Chow, *The path-cycle symmetric function of a digraph*, preprint (1994).

[Cho2] T. Y. Chow, *Symmetric Function Generalizations of Graph Polynomials*, Ph.D. Thesis, Mass. Inst. of Tech., Cambridge, Mass, 1995.

[Chu] W. Chu, *Partial fractions and bilateral summations*, J. Math. Phys. **35** (1994), 2036–2042.

[ChRo] Y.-C. Chen and G.-C. Rota, *q-Analogs of the principle of inclusion-exclusion and permutations with restricted position*, Disc. Math. **104** (1992), 7–22.

[ChG] F. Chung and R. Graham, *On digraph polynomials*, preprint (1993).

[Din1] K. Ding, *Invisible permutations and rook placements on a Ferrers board*, preprint (1992).

[Din2] K. Ding, *Rook Placements and Cellular Decomposition of Partition Varieties*, Ph. D. Thesis, University of Wisconsin - Madison, 1993.

[DiR] J. F. Dillon and D. P. Roselle, *Simon Newcomb's Problem*, SIAM J. Applied Math. **17** (1969), 1086–1093.

[Dwo] M. Dworkin, *Ph.D. Thesis*, Brandeis University, in preparation.

[EHR] R. Ehrenborg, J. Haglund, and M. Readdy, *Colored juggling patterns and weighted rook placements*, in preparation.

[FoSu] D. C. Foata and M. P. Schützenberger, *On the rook polynomials of Ferrers relations*, Colloq. Math. Janos Bolyai, 4, Combinatorial Theory and its Applications, vol. 2, North-Holland, Amsterdam, 1970, pp. 413–436.

[GaRe] A. M. Garsia and J. B. Remmel, *q-Counting rook configurations*, J. Combin. Theory (A) **41** (1986), 246–275.

[GaRa] G Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Math. and its Applications, 1990.

[Gasp] G. Gasper, *Summation formulas for basic hypergeometric series*, Siam. J. Math. Anal. **12** (1981), 196–200.

[Ges1] I. M. Gessel, *Generalized rook polynomials and orthogonal polynomials*, q-Series and Partitions (Dennis Stanton, ed.), IMA Volumes in Mathematics and its Applications, Springer Verlag, New York, 1989, pp. 159–176.

[Ges2] I. M. Gessel, private communication.

[GJW1] J. R. Goldman, J. T. Joichi, and D. E. White, *Rook theory I: Rook equivalence of Ferrers boards*, Proc. Amer. Math. Soc. **52** (1975), 485–492.

[Hag1] J. Haglund, *Compositions, Rook Placements, and Permutations of Vectors*, Ph.D. Thesis, University of Georgia, 1993.

[Hag2] J. Haglund, *Compositions and rook placements*, preprint (1994).

[JoRo] S. A. Joni and G.-C. Rota, *A vector space analog of permutations with restricted position*, J. Combin. Theor (A) **29** (1980), 59–73.

[KaRi] I. Kaplansky and J. Riordan, *The problem of the rooks and its applications*, Duke Math. J. **13** (1946), 259–268.

[Kar] P. W. Karlsson, *Hypergeometric functions with integral parameter differences*, J. Math. Phys. **12** (1971), 270–271.

[Mac] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1979.

[Min] B. M. Minton, *Generalized hypergeometric functions of unit argument*, J. Math. Phys. **11** (1970), 1375–1376.

[Rio] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley, New York, NY, 1958.

[Shev] V. S. Shevelev, *Some problems of the theory of enumerating the permutations with restricted positions*, J. Soviet Math. **61** (1992), 2272-2317.

[StS] R. Stanley and J. Stembridge, *On immanants of Jacobi-Trudi matrices and permutations with restricted position*, J. Combin. Theory (A) **62** (1993), 261–279.

[Ste] E. Steingrimsson, *A chromatic partition polynomial*, Proceedings of the Seventh International Conference on Formal Power Series and Algebraic Combinatorics, Marne-La-Vallée, France, 1995, pp. 511–521.


Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX