Murnaghan-Nakayama Rules for Characters of Iwahori-Hecke Algebras of the Complex Reflection Groups $G(r, p, n)$

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Summary

The finite irreducible complex reflection groups come in three infinite families: the symmetric groups $\mathcal{S}_n$ on n letters; the wreath product groups $\mathbb{Z}_r \wr \mathcal{S}_n$, where $\mathbb{Z}_r$ denotes the cyclic group of order r; and a series of index-p subgroups $G(r,p,n)$ of $\mathbb{Z}_r \wr \mathcal{S}_n$ for each positive integer p that divides r. In the classification of finite irreducible reflection groups, besides these infinite families $\mathcal{S}_n$, $\mathbb{Z}_r$, and $G(r,p,n)$, there exist only 34 exceptional irreducible reflection groups, see [ST].

A formula for the irreducible characters of the Iwahori-Hecke algebras for $\mathcal{S}_n$ is known [Ram], [KW], [vdJ]. This formula is a q-analogue of the classical Murnaghan-Nakayama formula for computing the irreducible characters of $\mathcal{S}_n$. Similar formulas for the characters of the groups $G(r,p,n)$ are classically known, see [Mac], [Ste], [AK], [Osi] and the references there. Formulas of this type are also known for the Iwahori-Hecke algebras of Weyl groups of types B and D [HR], [Pfe1], [Pfe2]. Recently, Iwahori-Hecke algebras have been constructed for the groups $\mathbb{Z}_r \wr \mathcal{S}_n$ and $G(r,p,n)$ [AK], [BM], [Ari]. In this paper we derive Murnaghan-Nakayama type formulas for computing the irreducible characters of the Iwahori-Hecke algebras that correspond to $\mathbb{Z}_r \wr \mathcal{S}_n$ and $G(r,p,n)$.

Hoefsmit [Hfs] has given explicit analogues of Young's seminormal representations for the Iwahori-Hecke algebras of types A_{n-1}, B_n, and D_n. Ariki and Koike, [AK] and [Ari], have constructed "Hoefsmit-analogues" of Young's seminormal representations for Iwahori-Hecke algebras $H_{r,p,n}$ of the groups $G(r,p,n)$. Our approach is to derive the Murnaghan-Nakayama rules by computing the sum of diagonal matrix elements in an explicit "Hoefsmit" representation of each algebra. We are motivated by Curtis Greene [Gre], who takes this approach using the Young seminormal form of the irreducible representations of the symmetric group and gives a new derivation of the classical Murnaghan-Nakayama rule. Greene does this by using the Möbius function of a poset that is determined by the partition

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

which indexes the irreducible representation. We generalize Greene's poset theorem so that it works for our cases. In this way we are able to compute the characters of the Hecke algebras $H_{r,n} = H_{r,1,n}$.

To compute the characters of the Iwahori-Hecke algebra $H_{r,p,n}$ of $G(r,p,n)$, $p > 1$, we use double centralizer methods (Clifford theory methods) to write these characters in terms of a certain bitrace on the irreducible representations of $H_{r,n} = H_{r,1,n}$. We then compute this bitrace in terms of the irreducible character values of $H_{r,n}$.

The character formulas given in this paper contain the Murnaghan-Nakayama rules for the complex reflection groups $G(r,p,n)$ and the Iwahori-Hecke algebras of classical type as special cases.

Extended Abstract

Let r, p, d, and n be positive integers such that $pd = r$. The complex reflection group $G(r,p,n)$ is the set of $n \times n$ matrices such that

(a) The entries are either 0 or rth roots of unity.
(b) There is exactly one nonzero entry in each row and each column.
(c) The dth power of the product of the nonzero entries is 1.

The following are important special cases of $G(r,p,n)$.

(1) $G(1,1,n) = S_n$, the symmetric group.
(2) $G(r,1,n) = \mathbb{Z}_r \wr S_n$.
(3) $G(2,1,n) = WB_n$ the Weyl group of type B.
(4) $G(2,2,n) = WD_n$ the Weyl group of type D.

Characters of Iwahori-Hecke Algebras of $G(r,1,n) = (\mathbb{Z}/r\mathbb{Z}) \wr S_n$.

Let q and $u_1, u_2, \ldots, u_r$ be indeterminates. Let $H_{r,n}$ be the associative algebra with 1 over the field $\mathbb{C}(u_1, u_2, \ldots, u_r, q)$ given by generators $T_1, T_2, \ldots, T_n$ and relations

$$
\begin{array}{lll}
(1) & T_iT_j = T_jT_i, & \text{for } |i-j| > 1, \\
(2) & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & \text{for } 2 \le i \le n-1, \\
(3) & T_1T_2T_1T_2 = T_2T_1T_2T_1, & \\
(4) & (T_1 - u_1)(T_1 - u_2)\cdots(T_1 - u_r) = 0, & \\
(5) & (T_i - q)(T_i + q^{-1}) = 0, & \text{for } 2 \le i \le n.
\end{array}
$$

Upon setting $q = 1$ and $u_i = \xi^{i-1}$, where ξ is a primitive rth root of unity, one obtains the group algebra $\mathbb{C}[\mathbb{Z}_r \wr S_n]$ of the wreath product group $\mathbb{Z}_r \wr S_n$. When $r = 1$ and $u_1 = 1$, we have $T_1 = 1$, and $H_{1,n}$ is isomorphic to an Iwahori-Hecke algebra of type A_{n-1}. The case $H_{2,n}$ when $r = 2$, $u_1 = p$, and $u_2 = p^{-1}$, is isomorphic to an Iwahori-Hecke algebra of type B_n.

Representations. An *r-partition* of size n is an r-tuple, $\mu = \left(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}\right)$ of partitions such that $|\mu^{(1)}| + |\mu^{(2)}| + \cdots + |\mu^{(r)}| = n$. If $\nu = \left(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(r)}\right)$ is another r-partition, we write $\nu \subseteq \mu$ if $\nu^{(i)} \subseteq \mu^{(i)}$ for $1 \le i \le r$. In this case, we say that $\mu/\nu = (\mu^{(1)}/\nu^{(1)}, \nu^{(2)}/\mu^{(2)}, \ldots, \mu^{(r)}/\nu^{(r)})$ is an r-skew shape. We refer to r-skew shapes and r-partitions collectively as *shapes*. If λ is a shape of size n, a *standard tableau* $L = \left(L^{(1)}, L^{(2)}, \ldots, L^{(r)}\right)$ of shape λ is a filling of the Ferrers diagram of λ with the numbers $1, 2, \ldots, n$ such that the numbers are increasing left to right across the rows and increasing down the columns of each $L^{(i)}$. For any shape λ, let $\mathcal{L}(\lambda)$ denote the set of standard tableaux of shape λ and, for each standard tableau L, let $L(k)$ denote the box containing k in L.

Define the *content* of a box b of a (possibly skew) shape $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ by

$$\mathrm{ct}(b) = u_k q^{2(j-i)}, \qquad \text{if } b \text{ is in position } (i,j) \text{ in } \lambda^{(k)}.$$

For each standard tableau L of size n, define the scalar $(T_i)_{LL}$ by

$$(T_i)_{LL} = \frac{q - q^{-1}}{1 - \frac{\mathrm{ct}(L(i-1))}{\mathrm{ct}(L(i))}}, \qquad \text{for } 2 \le i \le n.$$

Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a (possibly skew) shape of size n, and for each standard tableau $L \in \mathcal{L}(\lambda)$, let v_L denote a vector indexed by L. Let V^λ be the $\mathbb{C}(u_1, \ldots, u_r, q)$-vector space spanned by $\{v_L \mid L \in \mathcal{L}(\lambda)\}$, so that the vectors v_L form a basis of V^λ. Define an action of $H_{r,n}$ on V^λ by defining

$$\begin{aligned} T_1 v_L &= \mathrm{ct}(L(1)) v_L, \\ T_i v_L &= (T_i)_{LL} v_L + (q^{-1} + (T_i)_{LL}) v_{s_i L}, \qquad 2 \le i \le n, \end{aligned}$$

where $s_i L$ is the same standard tableau as L except that the positions of i and $i-1$ are switched in $s_i L$. If $s_i L$ is not standard, then we define $v_{s_i L} = 0$. The modules V^λ, where λ runs over all r-partitions of size n, form a complete set of nonisomorphic irreducible modules for $H_{r,n}$ ([You],[Hfs],[AK]).

Standard Elements. Define elements $t_i \in H_{r,n}$, for $1 \le i \le n$, by

$$t_i = T_i T_{i-1} \cdots T_2 T_1 T_2 \cdots T_{i-1} T_i.$$

For $1 \le k < \ell \le n$ and $0 \le i \le r-1$, define

$$R_{k\ell}^{(i)} = (t_k)^i \, T_{k+1} T_{k+2} \cdots T_\ell$$

and, for each $1 \le k \le n$, define $R_{kk}^{(i)} = (t_k)^i$. We say that an $\mathcal{S}_n$*-sequence* of length m is a sequence $\vec{\ell} = (\ell_1, \ldots, \ell_m)$ satisfying $1 \le \ell_1 < \ell_2 < \cdots < \ell_m = n$, and we say that a $\mathbb{Z}_r$*-sequence* of length m is a sequence $\vec{i} = (i_1, \ldots, i_m)$ satisfying $0 \le i_j \le r-1$ for each j. For an $\mathcal{S}_n$-sequence $\vec{\ell} = (\ell_1, \ldots, \ell_m)$ and a $\mathbb{Z}_r$-sequence $\vec{i} = (i_1, \ldots, i_m)$, define

$$T_{\vec{\ell}}^{\vec{i}} = R_{1,\ell_1}^{(i_1)} R_{\ell_1+1,\ell_2}^{(i_2)} \cdots R_{\ell_{m-1}+1,\ell_m}^{(i_m)} \in H_{r,n}.$$

Characters. For each standard tableau L of size n and for $1 \le k < \ell \le n$ and $0 \le i \le r-1$, define

$$\Delta^{(i)}_{k\ell}(L) = \mathrm{ct}(L(k))^i (T_{k+1})_{LL}(T_{k+2})_{LL}\cdots(T_\ell)_{LL},$$

and for $1 \le k \le n$, we define $\Delta^{(i)}_{kk}(L) = \mathrm{ct}(L(k))^i$. Let $\Delta^{(i)}(L) = \Delta^{(i)}_{1,n}(L)$, and for any shape λ (possibly skew), define

$$\Delta^{(i)}(\lambda) = \sum_{L \in \mathcal{L}(\lambda)} \Delta^{(i)}(L).$$

Let $\chi^\lambda_{H_{r,n}}$ denote the character of the irreducible $H_{r,n}$-representation V^λ. The following theorem is our analogue of the Murnaghan-Nakayama rule.

Theorem 1. *Let $\vec{\ell} = (\ell_1, \ldots, \ell_m)$ be an S_n-sequence, $\vec{i} = (i_1, \ldots, i_m)$ be a $\mathbb{Z}_r$-sequence, and suppose that λ is an r-partition of size n. Then*

$$\chi^\lambda_{H_{r,n}}(T^{\vec{i}}_{\vec{\ell}}) = \sum_{\emptyset=\mu^{(0)}\subseteq\mu^{(1)}\subseteq\cdots\subseteq\mu^{(m)}=\lambda} \Delta^{(i_1)}(\mu^{(1)})\Delta^{(i_2)}(\mu^{(2)}/\mu^{(1)})\cdots\Delta^{(i_m)}(\mu^{(m)}/\mu^{(m-1)}),$$

where the sum is over all sequences of shapes $\emptyset = \mu^{(0)} \subseteq \mu^{(1)} \subseteq \cdots \subseteq \mu^{(m)} = \lambda$ such that $|\mu^{(j)}/\mu^{(j-1)}| = |\ell_j|$.

To give an explicit formula for the value of $\Delta^{(i)}(\lambda)$, we say that the shape λ is a *border strip* if it is connected and does not contain any 2×2 block of boxes. The shape λ is a *broken border strip* if it simply does not contain any 2×2 block of boxes. Therefore, a broken border strip is a union of connected components, each of which is a border strip. A *sharp corner* in a border strip is a box with no box above it and no box to its left. A *dull corner* in a border strip is a box that has a box to its left and a box above it but has no box directly northwest of it.

Theorem 2. *Let λ be any shape (possibly skew) with n boxes. Let CC be the set of connected components of λ, and let $cc = |\mathrm{CC}|$ be the number of connected components of λ.*
(a) If λ is not a broken border strip, then $\Delta^{(k)}(\lambda) = 0$;
(b) If λ is a broken border strip, then

$$\Delta^{(0)}(\lambda) = (q-q^{-1})^{cc-1} \prod_{bs\in \mathrm{CC}} q^{c(bs)-1}(-q^{-1})^{r(bs)-1},$$

and, for $1 \le k \le r-1$,

$$\Delta^{(k)}(\lambda) = (-q+q^{-1})^{cc-1}\left(\prod_{s\in \mathrm{SC}} \mathrm{ct}(s)\right)\left(\prod_{d\in \mathrm{DC}} \mathrm{ct}(d)^{-1}\right)$$
$$\times \prod_{bs\in\mathrm{CC}} q^{c(bs)-1}(-q^{-1})^{r(bs)-1}\sum_{t=0}^{|\mathrm{DC}|}(-1)^t e_t(\mathrm{ct}(\mathrm{DC}))h_{k-t-cc}(\mathrm{ct}(\mathrm{SC})),$$

where SC *and* DC *denote the set of sharp corners and dull corners in λ, respectively, and if bs is a border strip, then $r(bs)$ is the number of rows in bs, and $c(bs)$ is the number of columns in bs. The function $e_t(\mathrm{ct}(\mathrm{DC}))$ is the elementary symmetric function in the variables $\{\mathrm{ct}(d), d \in \mathrm{DC}\}$, and the function $h_{k-t-cc}(\mathrm{ct}(\mathrm{SC}))$ is the homogeneous symmetric function in the variables $\{\mathrm{ct}(s), s \in \mathrm{SC}\}$.*

Characters of Iwahori-Hecke Algebras of $G(r,p,n)$, $p>1$.

Let $\varepsilon = e^{2\pi i/p}$ be a primitive pth root of unity, and let q and $x_1^{1/p},\dots,x_d^{1/p}$ be indeterminates. Then $H_{r,n}$ is the associative algebra with 1 over the field $\mathbb{C}(x_1^{1/p},\dots,x_d^{1/p},q)$ given by generators $T_1,\dots,T_n$, and relations

(1) $T_iT_j = T_jT_i$, for $|i-j|>1$,

(2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $2\le i\le n-1$,

(3) $T_1T_2T_1T_2 = T_2T_1T_2T_1$,

(4) $(T_1^p - x_1)(T_1^p - x_2)\cdots(T_1^p - x_d) = 0$,

(5) $(T_i - q)(T_i + q^{-1}) = 0$, for $2\le i\le n$.

This is the same as the earlier definition of the algebra $H_{r,n}$ except that we are using $\varepsilon^\ell x_k^{1/p}$, $1\le k\le d$, $0\le \ell\le p-1$, in place of $u_1,\dots,u_r$. Let $H_{r,p,n}$ be the subalgebra of $H_{r,n}$ generated by the elements

$$a_0 = T_1^p,\quad a_1 = T_1^{-1}T_2T_1,\quad \text{and}\quad a_i = T_i,\quad 2\le i\le n.$$

Ariki [Ari] shows that $H_{r,p,n}$ is an analogue of the Iwahori-Hecke algebra for the groups $G(r,p,n)$.

Representations. We organize each r-partition λ of size n into d groups of p partitions each, so that we can write

$$\lambda = (\lambda^{(k,\ell)}),\qquad \text{for } 0\le k\le d-1 \text{ and } 0\le \ell\le p-1,$$

where each $\lambda^{(k,\ell)}$ is a partition and $\sum_{k,\ell}|\lambda^{(k,\ell)}| = n$. It is convenient to view the partitions $\lambda^{(k,0)},\dots,\lambda^{(k,p-1)}$ as all lying on a circle so that we have d necklaces of partitions, each necklace with p partitions on it. In order to specify this arrangement, we shall say that λ is a (d,p)-partition.

Since $H_{r,p,n}$ is a subalgebra of $H_{r,n}$, the irreducible $H_{r,n}$-representations V^λ are (not necessarily irreducible) representations of $H_{r,p,n}$. With the given specializations of the u_i, the content of a box b of λ is

$$\mathrm{ct}(b) = \varepsilon^\ell x_k^{1/p} q^{2(j-i)},\qquad \text{if } b \text{ is in position } (i,j) \text{ in } \lambda^{(k,\ell)}.$$

It follows that the action of $H_{r,p,n}$ on V^λ is given by

$$\begin{aligned} a_0 v_L &= \mathrm{ct}(L(1))^p v_L = x_k v_L,\quad \text{if } 1\in L^{(k,\ell)},\\ a_1 v_L &= (T_2)_{LL} v_L + \frac{\mathrm{ct}(L(1))}{\mathrm{ct}(s_2L(1))}(q^{-1} + (T_2)_{LL}) v_{s_2L},\\ a_i v_L &= (T_i)_{LL} v_L + (q^{-1} + (T_i)_{LL}) v_{s_iL}. \end{aligned}$$

Irreducible Representations. Let $\lambda = (\lambda^{(k,\ell)})$ be a (d,p)-partition. We define an operation σ that moves the partitions on each circle over one position. Given a box b in position (i,j) of the partition $\lambda^{(k,\ell)}$ then $\sigma(b)$ is the same box b except moved to be in position (i,j) of $\lambda^{(k,\ell+1)}$, where $\ell+1$ is taken modulo p. The map σ is an operation of order p and acts uniformly on the shape $\lambda = (\lambda^{(k,\ell)})$, on standard tableaux $L = (L^{(k,\ell)})$ of shape λ, and on the basis vector v_L of V^λ:

$$\sigma(\lambda) = (\lambda^{(k,\ell+1)}), \quad \sigma(L) = (L^{(k,\ell+1)}), \quad \text{and} \quad \sigma(v_L) = v_{\sigma(L)}.$$

In the last case, extend linearly to get the vector space homomorphism $\sigma: V^\lambda \longrightarrow V^{\sigma(\lambda)}$. We prove that the map $\sigma: V^\lambda \to V^{\sigma(\lambda)}$ is an $H_{r,p,n}$-module isomorphism, and thus the set of transformations $\{\sigma^\alpha \mid 0 \le \alpha \le p-1\}$ defines an action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on the set of (d,p)-partitions and on the set of vector spaces V^λ.

Now let K_λ be the stabilizer of λ under the action of $\mathbb{Z}/p\mathbb{Z}$. The group K_λ is a cyclic group of order $|K_\lambda|$ and is generated by the transformation σ^{f_λ} where f_λ is the smallest integer between 1 and p such that $\sigma^{f_\lambda}(\lambda) = \lambda$. The elements of K_λ are all $H_{r,p,n}$-module isomorphisms, and it follows that as an $H_{r,p,n} \times K_\lambda$-bimodule

$$V^\lambda \cong \bigoplus_{j=0}^{|K_\lambda|-1} V^{(\lambda,j)} \otimes Z_j,$$

where $V^{(\lambda,j)}$ is an *irreducible* $H_{r,p,n}$-module and Z_j is the irreducible K_λ-module with character η_j.

Characters. Let $\chi^{(\lambda,j)}$ denote the character of the irreducible $H_{r,p,n}$-module $V^{(\lambda,j)}$, and let χ^λ denote the $H_{r,p,n} \times K_\lambda$-bitrace on the module V^λ. By taking traces in the module equation above and applying orthogonality of characters for K_λ we derive the character formula:

$$\chi^{(\lambda,j)}(h) = \frac{1}{|K_\lambda|} \sum_{\alpha=0}^{|K_\lambda|-1} \varepsilon^{-j\alpha f_\lambda} \chi^\lambda(h\sigma^{\alpha f_\lambda}), \quad \text{where } f_\lambda = p/|K_\lambda|.$$

Thus, we are interested in computing the values of the bitrace $\chi^\lambda(h\sigma^{\alpha f_\lambda})$.

Standard Elements. Define elements $S_i \in H_{r,p,n}$, $1 \le i \le n$, by

$$\begin{aligned} S_1 &= a_0 = t_1^p, \\ S_2 &= a_1 a_2 = t_1^{-1} t_2, \\ S_i &= a_i a_{i-1} \cdots a_4 a_3 a_1 a_2 a_3 a_4 \cdots a_{i-1} a_i = t_1^{-1} t_i, \qquad \text{for } 3 \le i \le n. \end{aligned}$$

For $1 \le k \le n$, define $S_{kk}^{(i)} = S_k^i$ and define $\tilde{S}_{12}^{(i)} = S_1^i a_1$. For all other $k < \ell$, define

$$S_{k\ell}^{(i)} = S_k^i a_{k+1} \cdots a_\ell, \quad \text{and} \quad \tilde{S}_{1\ell}^{(i)} = S_1^i a_1 a_3 \cdots a_\ell.$$

Let $(\ell_1, \ldots, \ell_m)$ be an S_n-sequence and let $(i_1, \ldots, i_m)$ be a $\mathbb{Z}_r$-sequence. We prove that is sufficient to compute the values of $\chi^\lambda(h\sigma^{\alpha f_\lambda})$, for elements $h \in H_{r,p,n}$ of the form $R_{1,\ell_1}^{(i_1)} R_{\ell_1+1,\ell_2}^{(i_2)} \cdots R_{\ell_{m-1}+1,\ell_m}^{(i_m)}$, where $i_1 + \cdots + i_m = 0 \pmod p$.

Our result is as follows:

Theorem 3. *Let λ be a (d,p)-partition, where $pd = r$. Let α be such that $0 \le \alpha \le |K_\lambda| - 1$. Define*

$$f_\lambda = p/|K_\lambda|, \quad \text{and} \quad \gamma = \frac{|K_\lambda|}{\gcd(\alpha, |K_\lambda|)}.$$

Let

$$h = R^{(i_1)}_{1,\ell_1} R^{(i_2)}_{\ell_1+1,\ell_2} \cdots R^{(i_m)}_{\ell_{m-1}+1,\ell_m}$$

where $(\ell_1, \cdots, \ell_m)$ is an S_n-sequence and $(i_1, \cdots, i_m)$ is a $\mathbb{Z}_r$-sequence such that $i_1 + \cdots + i_m = 0$ (mod p). The element h is an element of $H_{r,p,n} \subseteq H_{r,n}$. If all ℓ_i in the sequence $(\ell_1, \ldots, \ell_m)$ are divisible by γ then define

$$\begin{gathered} \bar{n} = n/\gamma, \quad \bar{r} = r/\gamma, \quad \bar{p} = p/\gamma, \\ (\bar{\ell}_1, \ldots, \bar{\ell}_m) = (\ell_1/\gamma, \ldots, \ell_m/\gamma), \\ \bar{\lambda}^{(k,\tau)} = \lambda^{(k,\tau)}, \quad \text{for } 0 \le \tau \le \bar{p} - 1 \quad \text{and} \quad \bar{h} = R^{(0)}_{1,\bar{\ell}_1} \cdots R^{(0)}_{\bar{\ell}_{m-1}+1,\bar{\ell}_m}. \end{gathered}$$

Then:

(a) If ℓ_i is not divisible by γ for some $1 \le i \le m$ then $\chi^\lambda(h\sigma^{\alpha f_\lambda}) = 0$.

(b) If all ℓ_i are divisible by γ and if $i_k \ne 0$ for some k, then $\chi^\lambda(h\sigma^{\alpha f_\lambda}) = 0$.

(c) If all ℓ_i are divisible by γ and if $i_k = 0$ for all k, then

$$\chi^\lambda(h\sigma^{\alpha f_\lambda}) = \frac{\gamma^{\bar{n}}}{[\gamma]^{\bar{n}-m}} \chi^{\bar{\lambda}}_{H_{\bar{r},\bar{n}}}(\bar{h}) \prod_{i=1}^{\gamma} \left(\frac{q}{1-\varepsilon^{-i}} + \frac{q^{-1}}{1-\varepsilon^{i}} \right)^{\bar{n}},$$

where $H_{\bar{r},\bar{n}}$ is with parameter q^γ, in place of q and with parameters $\varepsilon^{\gamma\tau} x_k^\gamma$, $0 \le k \le d-1$, $0 \le \tau \le \bar{p}-1$, in place of $u_1, \ldots, u_{\bar{r}}$. The element $\bar{h}$ is viewed as an element of the algebra $H_{\bar{r},\bar{n}}$ and $[\gamma] = (q^\gamma - q^{-\gamma})/(q - q^{-1})$.

The Method of Proof for Theorem 2.

In all of the original derivations [Ram],[KW],[vdJ] of the Murnaghan-Nakayama rules for Iwahori-Hecke algebras of types A_{n-1} the key was essentially to use the theory symmetric functions and Schur polynomials, and the Schur-Weyl duality between the Iwahori-Hecke algebras of type A_{n-1} and the Drinfel'd-Jimbo quantum groups $U_q(\mathfrak{gl}(m))$. This approach seems to be quite challenging for $H_{r,n}, r > 1$, although some progress has been made (see [ATY]).

Curtis Greene uses the theory of partially ordered sets and Möbius functions to prove a rational function identity ([Gre], Theorem 3.3) which can be used to derive the Murnaghan-Nakayama rule for symmetric group characters. We extend Greene's poset theorem so that it can be applied to computing Murnaghan-Nakayama rules, Theorem 2, for the irreducible characters of the Iwahori-Hecke algebras of $\mathbb{Z}_r \wr S_n$. The extended poset theorem stands on its own as a result for planar posets, so it is stated here.

Throughout this explanation $\hat{P}$ will denote a planar poset with unique minimal element u, and $P = \hat{P} - \{u\}$ will be the poset obtained by removing the minimal element u from $\hat{P}$. We let SC be the set of minimal elements of P, and we call these elements *sharp corners*. Two sharp corners s_1 and s_2 of SC are "adjacent" if they are not separated by another sharp corner as the boundary of P is traversed. If s_1 and s_2 are adjacent elements of SC and the least common multiple $s_1 \vee s_2$ exists, then we call $s_1 \vee s_2$ a *dull corner* of P. We let DC denote the set of all dull corners of P. Finally, we let cc denote the number of connected components of P, and note that $cc = |\mathrm{SC}| - |\mathrm{DC}|$.

Let $\{x_a, a \in \hat{P}\}$, be a set of commutative variables indexed by the elements of $\hat{P}$. For each $0 \le k \le r-1$ and each pair $a < b$ in $\hat{P}$, define a weight, $wt^{(k)}(a,b)$, by

$$\begin{aligned} wt^{(k)}(a,b) &= \frac{1 - x_a x_b^{-1}}{q - q^{-1}} && \text{for all } a, b \in P, \text{ and} \\ wt^{(k)}(u,a) &= x_a^{-k} && \text{for all } a \in P. \end{aligned}$$

Then for any planar poset $\hat{P}$ with unique minimal element u, define

$$\Delta^{(k)}(\hat{P}) = \prod_{\substack{a,b \in \hat{P} \\ a \neq b}} wt^{(k)}(a,b)^{\mu_{\hat{P}}(a,b)},$$

where $\mu_{\hat{P}}(a,b)$ is the Möbius function for the poset $\hat{P}$.

The following is our extension of the poset theorem:

Theorem 4. *Let $\hat{P}$ be a planar poset with unique minimal element u. Let $P = \hat{P} \setminus \{u\}$. Then*

$$\sum_{\hat{L} \in \mathcal{L}(\hat{P})} \Delta^{(0)}(\hat{L}) = (q - q^{-1})^{cc-1} \Delta^{(0)}(P),$$

and, for $1 \le k \le r-1$,

$$\begin{aligned} \sum_{\hat{L} \in \mathcal{L}(\hat{P})} \Delta^{(k)}(\hat{L}) = \Delta^{(k)}(P)(-q + q^{-1})^{cc-1} \left(\prod_{s \in \mathrm{SC}} x_s \right) \left(\prod_{d \in \mathrm{DC}} x_d^{-1} \right) \\ \times \sum_{t=0}^{|\mathrm{DC}|} (-1)^t e_t(x_{\mathrm{DC}}) h_{k-t-cc}(x_{\mathrm{SC}}) \end{aligned}$$

where cc is the number of connected components of P, $e_t(x_{\mathrm{DC}})$ is the elementary symmetric function in the variables $\{x_d, d \in \mathrm{DC}\}$, and $h_{k-t-cc}(x_{\mathrm{SC}})$ is the homogeneous symmetric function in the variables $\{x_s, s \in \mathrm{SC}\}$.

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