Invariants of cubical spheres

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Abstract

Let I be an invariant of cubical complexes which may be expressed as a linear combination of the number of faces of different dimensions. We prove that I is a nonnegative linear combination of the entries of the Ron Adin h-vector, if it does not decrease when we add a new facet in a shelling. It is known that the entries of the toric h-vector have this property, and we show that the same holds for the "triangulation h-vector" which arises from the Hilbert series of the Stanley ring of a cubical complex. Thus the nonnegativity of the Ron Adin h-vector implies the nonnegativity of all other cubical h-vectors. We prove this nonnegativity for all cubical complexes obtained as a barycentric subdivision of a simplicial sphere.

Résumé

Soit I un invariant des complexes cubiques qui peut être exprimé comme une combinaison linéaire des nombres de faces des différentes dimensions. Nous démontrons que si I ne diminue pas quand on ajoute une nouvelle facette à un effeuillage, alors I est une combinaison linéaire positive des éléments du h-vecteur de Ron Adin. Il est bien connu que les éléments du h-vecteur torique possèdent cette propriété, et nous montrons qu'elle est aussi vraie pour le "h-vecteur de triangulation" qui provient de la série de Hilbert–Poincaré de l'anneau de Stanley d'un complexe cubique. Ainsi, la positivité du h-vecteur de Ron Adin implique la positivité de tous les autres h-vecteurs cubiques connus. Nous démontrons cette positivité pour tout complexe cubique obtenu comme subdivision barycentrique d'une sphère simpliciale.

Introduction

In the past few years more and more efforts were made to find the cubical analogues to the upper and lower bound theorems on the *f-vectors* $(f_{-1}, f_0, \ldots, f_{d-1})$ of $(d-1)$-dimensional simplicial spheres. (Here f_i stands for the number of i-dimensional faces.) In the simplicial case, these results may be formulated in the most compact way by using the *h-vector* $(h_0, \ldots, h_d)$ of the simplicial complex, which is a vector of linear combinations of the f_i's. In terms of the h-vector the Upper Bound Theorem tells us that we have

$$h_i \leq \binom{f_0 - d + i - 1}{i} \quad \text{for } 0 \leq i < \left[\frac{d}{2}\right]$$

for every simplicial $(d-1)$-sphere, and the Generalized Lower Bound Theorem is equivalent to saying that the h_i's form a unimodal sequence. Moreover, the h-vector is nonnegative for simplicial spheres, and the *Dehn-Sommerville equations*, which generate all linear relations holding for the f-vector of a simplicial sphere are equivalent to the relations

$$h_i = h_{d-i} \quad \text{for } i = 0, 1, \ldots, d.$$

References to all these results may be found in [12]. In the proof of the Upper Bound Theorem, the h-vector occured in the numerator of the Hilbert series of the *Stanley–Reisner ring* of a simplicial complex (see [11]). The *cd-index* of a simplicial Eulerian partially ordered set was described by Stanley in [15, Theorem 3.1] in terms of its h-vector, such that the nonnegativity of the *cd*-index of Gorenstein* simplicial posets became a consequence of the nonnegativity of their h-vector.

In the focus of the study of cubical complexes stood the search for the "right" cubical analogue of the simplicial h-vector. Three h-vectors were introduced, each of which preserved some properties of the simplicial original. The first was the *toric h-vector*, defined by Stanley in greater generality for Eulerian partially ordered sets in [14], which is nonnegative and unimodal for cubical rational polytopes. Unfortunately, this unimodality does not seem to yield the strongest possible lower bound results for the f-vector of a cubical sphere. Clara Chan has proved in [4] that the toric h-vector of shellable cubical complexes is nonnegative.

The second h-vector, which we call here *triangulation h-vector*, was studied by Hetyei in [7]. This h-vector occurs in the numerator of the Hilbert series of the *Stanley ring* of a cubical complex, which is a cubical analogue of the Stanley–Reisner ring of a simplicial complex. It is also nonnegative for shellable cubical complexes, and in the case of cubical polytopes it arises as the (simplicial) h-vector of the triangulation via pulling the vertices. Although for a large class of cubical convex polytopes the triangulation h-vector has the exotic property of being the f-vector of a simplicial complex, and this fact provides many examples to an interesting commutative algebraic conjecture of Eisenbud, Green and Harris, this h-vector seems also to be too large to help expressing the strongest lower and upper bound inequalities for the f-vector of a cubical sphere.

The third h-vector, which we call *the Ron Adin h-vector*, was defined by Ron Adin in [1]. ¿From the very beginning, this h-vector seemed to be winning over the other two. Just like the toric and the triangulation h-vectors, it is nonnegative for shellable cubical complexes, and the cubical Dehn–Sommerville equations are equivalent to the relations $h_i = h_{d-i}$ for $i = 0, 1, \ldots, d$. But this h-vector has also further properties reminiscent of the simplicial one. First, the (normalized) Ron Adin h-vector of a cube is $(1, \ldots, 1)$, just like in the simplicial case. Second, the conjectured unimodality of the Ron Adin h-vector includes for $h_0 \leq h_1$ the first nontrivial inequality proved by Blind and Blind in [2], and for $h_1 \leq h_2$ the cubical lower bound conjecture made by Jockusch in [8]. Third, as Ehrenborg and Hetyei have shown in [6], the Ron Adin h-vector occurs in a cubical analogue of Stanley's result [15, Theorem 3.1] on the *cd*-index of simplicial Eulerian posets.

Now we add one more result indicating that the Ron Adin h-vector is the "right" cubical analogue to prove strong inequalities about the f-vector of cubical spheres. Our Theorem 2.2 in Section 2 implies that every linear combination of the f_i's which does not decrease when we add a new facet in a shelling of a shellable cubical complex, is a nonnegative linear combination of the Ron Adin h_i's. As a consequence, by Clara Chan's proof of the nonnegativity of the toric h-vector of shellable cubical complexes, the toric h-vector is nonnegative whenever the same holds for the Ron Adin h-vector. In Section 3 we prove that also the triangulation h-vector has the required nondecreasing property for shellings. (In [7] only the nonnegativity of the triangulation h-vector of an entire shellable cubical complex is proved). Therefore both toric and triangulation h-vectors are nonnegative for cubical spheres, if the same holds for the Ron Adin h-vector.

Finally, in Section 4 we prove the nonnegativity of the Ron Adin h-vector for a special class of cubical spheres: for those cubical complexes which are obtained from simplicial spheres via barycentric subdivision. The cubical spheres belonging to this class are not necessarily shellable.

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1 Preliminaries

Definition 1.1 *A* cubical complex $\mathcal{C}$ *is a family of finite sets (called faces) on a vertex set* V*, such that* $\mathcal{C}$ *is closed under intersection,* $\{v\} \in \mathcal{C}$ *holds for all* $v \in V$*, and for every face* $\sigma \in \mathcal{C}$ *the set of faces contained in* σ *and ordered by inclusion is isomorphic to the lattice of faces of a cube.*

A maximal face of a cubical complex is called *a facet*, and the number $\dim(\sigma) \stackrel{\text{def}}{=} \text{rank}([\emptyset, \sigma]) - 1$ is called the *dimension* of the face σ. The dimension of a cubical complex $\mathcal{C}$ is the maximum of the dimensions of its faces. Given a $(d-1)$-dimensional cubical complex $\mathcal{C}$, we denote the number of its faces of dimension k by f_k, and we call the vector $(f_{-1}, f_0, \ldots, f_{d-1})$ the *f-vector* of $\mathcal{C}$.

The simplest example of a cubical complex is the collection $\mathcal{C}^d$ of faces of a standard d-cube. This complex may be geometrically realized as the family of vertex sets of the faces of $[0,1]^d \subset \mathbb{R}^d$. We call a bijection $\phi : V(\mathcal{C}^d) \longrightarrow \{0,1\}^d$ sending faces into vertex sets of faces of $[0,1]^d$ a *standard geometric realization* of $\mathcal{C}^d$. Using this realization, we may define balls and spheres as follows.

Definition 1.2 *A collection* $\{F_1, F_2, \ldots, F_k\}$ *of facets of the boundary of a d-cube is called a* $(d-1)$-dimensional ball *or a* $(d-1)$-dimensional sphere *respectively, if the set* $\bigcup_{i=1}^{k} conv(\phi(F_i))$ *is homeomorphic to a* $(d-1)$*-dimensional ball or sphere respectively.*

This definition is combinatorial because of the following observation, originally due to Ron Adin and Clara Chan. Given a collection of facets of $\partial\mathcal{C}^d$, let r be the number of facets F_i such that the facet opposite to F_i does not belong to $\{F_1, F_2, \ldots, F_k\}$, and let s be the number of pairs of opposite facets $\{F_i, F_j\} \subseteq \{F_1, F_2, \ldots, F_k\}$. (Obviously we have $r + 2s = k$.) We call (r, s) the *type* of the collection $\{F_1, F_2, \ldots, F_k\}$. Using the notion of type we may characterize balls and spheres in the following way.

Lemma 1.3 *The collection of facets $\{F_1, \ldots, F_k\}$ of the boundary of a d-cube is a $(d-1)$-sphere if and only if it has type $(0, d)$ and it is a $(d-1)$-ball if and only if its type (r, s) satisfies $r > 0$.*

Given a face σ of a cubical complex C we call the cubical complex $\{\tau \in C : \tau \subseteq \sigma\}$ the *restriction of C to σ*, and we denote it by $C|_\sigma$.

Definition 1.4 *A cubical complex C is* pure *if all facets of C have the same dimension. We define* shellable cubical complexes *as follows.*

1. *The empty set is a ((-1)-dimensional) shellable cubical complex.*
2. *A point is a (zero-dimensional) shellable complex.*
3. *A d-dimensional pure complex C is shellable if its facets can be listed in a linear order $F_1, \ldots, F_n$ (called a* shelling*), such that for each $k \in \{2, \ldots, n\}$ the subcomplex $C|_{F_k} \cap (C|_{F_1} \cup \cdots \cup C|_{F_{k-1}})$ is a pure complex of dimension $(d-1)$ and its maximal dimensional faces form a $(d-1)$-dimensional ball or sphere.*

By abuse of notation we say that the attachment of $C|_{F_k}$ to $C|_{F_1} \cup \cdots \cup C|_{F_{k-1}}$ in a shelling $F_0, F_1, \ldots, F_k$ has type (r, s) if the set of facets of $C|_{F_k} \cap (C|_{F_1} \cup \cdots \cup C|_{F_{k-1}})$ considered as a collection of facets of $C|_{F_k}$ has type (r, s).

Lemma 1.5 *When we add a facet of attachment type (r, s) to the shelling of a $(d-1)$-dimensional cubical complex, the number of j-faces increases by*

$$\sum_{\substack{u+v=j-s \\ u,v \geq 0}} \binom{r}{u} \binom{d-1-s}{v} \cdot 2^{d-1-s-v}$$

for $s \leq j \leq d-1$, and remains unchanged for all other j's.

Definition 1.6 *A* simplicial complex *Δ is a family of subsets of a finite set V such that $\{v\} \in \Delta$ for all $v \in V$ and Δ is closed under taking subsets. The elements of Δ are called* faces.

Observe that this definition may be rephrased in a way analogous to Definition 1.1.

We introduce the notion of face, dimension, f-vector and shellability similarly to the cubical case, and we define the *h-vector* $(h_0, \ldots, h_d)$ of a $(d-1)$-dimensional simplicial complex by the formula

$$\sum_{i=0}^{d} h_i \cdot t^i \stackrel{\text{def}}{=} \sum_{j=-1}^{d-1} f_j \cdot t^{j+1} \cdot (1-t)^{d-j-1}. \tag{1}$$

2 Invariants in terms of the Ron Adin h-vector

Definition 2.1 *Let C be a cubical complex of dimension $d-1$ with f-vector $(f_{-1}, f_0, \dots, f_{d-1})$. The* (normalized) Ron Adin h-vector *of C is defined by*

$$h^{RA}(x) = \sum_{i=0}^{d} h_i^{RA} \cdot x^i \stackrel{def}{=} \sum_{j=0}^{d} f_{j-1} \cdot c_j(x),$$

where the polynomials $c_j(x)$ are given by

$$c_j(x) = \begin{cases} \dfrac{1-(-x)^{d+1}}{1+x} & \text{if } j = 0, \\[2ex] \dfrac{2^{j-d} \cdot x^j \cdot (1-x)^{d-j} + (-1)^{d-j} \cdot x^{d+1}}{1+x} & \text{if } 1 \le j \le d. \end{cases}$$

Observe that we divided the h-vector given by R. Adin in [1] by 2^{d-1} in order to obtain $h_0 = 1$. Due to this normalization, [1, equation (19)] takes the following form.

$$f_{j-1} = 2^{d-j} \cdot \sum_{i=1}^{j} \binom{d-i}{d-j} \cdot (h_i^{RA} + h_{i-1}^{RA}) \quad \text{for } 1 \le j \le d.$$

Equivalently, after replacing j with $j+1$, we have

$$f_j = \begin{cases} 2^{d-j-1} \cdot \left(\binom{d-1}{d-j-1} + \sum_{i=1}^{j} h_i^{RA} \cdot \left(\binom{d-i}{d-j-1} + \binom{d-i-1}{d-j-1} \right) + h_{j+1}^{RA} \right) & \text{for } 1 \le j \le d-1, \\ 2^{d-1} \cdot (1 + h_1^{RA}) & \text{for } j = 0. \end{cases} \tag{2}$$

Theorem 2.2 *Let I be an invariant of $(d-1)$-dimensional cubical complexes which may be expressed as a linear combination of the f_i's. Then I is a nonnegative linear combination of the h_i^{RA}'s if and only if the following hold:*

(i) $I(C^{d-1}) \ge 0$, i.e., the value of I on the face complex of a $(d-1)$-cube is nonnegative.

(ii) in any shelling of any $(d-1)$-dimensional shellable complex, adding a facet of attachment type $(1, i)$ (where $i = 0, 1, \dots, d-2$) or of attachment type $(0, d-1)$ does not decrease I.

Proof: Assume first that the conditions (i)–(ii) hold, and we have $I = \sum_{j=-1}^{d-1} \alpha_j \cdot f_j$. Then, using equation (2), for every $(d-1)$ dimensional cubical complex C we may write

$$\begin{aligned} I(C) &= \sum_{j=-1}^{d-1} \alpha_j \cdot f_j = \alpha_{-1} + \alpha_0 \cdot 2^{d-1} \cdot (1 + h_1^{RA}) \\ &+ \sum_{j=1}^{d-1} \alpha_j \cdot 2^{d-j-1} \cdot \left(\binom{d-1}{d-j-1} + \sum_{i=1}^{j} h_i^{RA} \cdot \left(\binom{d-i}{d-j-1} + \binom{d-i-1}{d-j-1} \right) + h_{j+1}^{RA} \right). \end{aligned}$$

Observe that $\alpha_{-1} + \sum_{j=0}^{d-1} \alpha_j \cdot 2^{d-j-1} \cdot \binom{d-1}{d-j-1}$ is the value of I on the face complex of a $(d-1)$-cube. Hence we have

$$
\begin{aligned}
I(C) &= I(C^{d-1}) + \alpha_0 \cdot 2^{d-1} \cdot h_1^{RA} \\
&\quad + \sum_{j=1}^{d-1} \alpha_j \cdot 2^{d-j-1} \cdot \left(\sum_{i=1}^{j} h_i^{RA} \cdot \left(\binom{d-i}{d-j-1} + \binom{d-i-1}{d-j-1} \right) + h_{j+1}^{RA} \right) \\
&= h_0^{RA} \cdot I(C^{d-1}) + h_1^{RA} \cdot \left(\alpha_0 \cdot 2^{d-1} + \sum_{j=1}^{d-1} \alpha_j \cdot 2^{d-j-1} \cdot \left(\binom{d-1}{d-j-1} + \binom{d-2}{d-j-1} \right) \right) \\
&\quad + \sum_{i=2}^{d-1} h_i^{RA} \cdot \left(\sum_{j=i}^{d-1} \alpha_j \cdot 2^{d-j-1} \cdot \left(\binom{d-i}{d-j-1} + \binom{d-i-1}{d-j-1} \right) + \alpha_{i-1} \cdot 2^{d-i} \right) + h_d^{RA} \cdot \alpha_{d-1}.
\end{aligned}
$$

It is easy to verify that for $1 \le i \le d-1$, the coefficient of h_i^{RA} in the last sum is the double of the change of I when we add a facet of attachment type $(1, i-1)$ to a $(d-1)$-dimensional shellable cubical complex. In fact, for $j = i, i+1, \ldots, d-2$ the number of j-faces added is

$$
\binom{d-i-1}{d-j-2} \cdot 2^{d-j-2} + \binom{d-i-1}{d-1-j} \cdot 2^{d-1-j} = 2^{d-j-2} \cdot \left(\binom{d-i-1}{d-j-2} + 2 \cdot \binom{d-i-1}{d-1-j} \right),
$$

and, using the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, we may write this as

$$
2^{d-j-2} \cdot \left(\binom{d-i}{d-1-j} + \binom{d-i-1}{d-1-j} \right).
$$

The number of $(i-1)$-faces added is 2^{d-i-1}, the number of $(d-1)$-faces added is 1.

Finally, the coefficient of h_d^{RA} is the change of I when we add a facet of attachment type $(0, d-1)$ to a $(d-1)$-dimensional shellable cubical complex. Therefore the coefficient of each h_i is nonnegative by our assumptions.

The converse is a corollary of the proof of [1, Theorem 5]. □

Remark 1 Observe that Theorem 2.2 provides an alternative definition for the Ron Adin h-vector.

Example *The toric h-vector.* In [4] Clara Chan proves that the toric h-vector of a shellable cubical complex is nonnegative. In her proof she shows that at the adding of each new facet in a shelling, the toric h_i's change by a nonnegative number. By Theorem 2.2 the h_i's may be expressed as nonnegative linear combinations of $h_0^{RA}, \ldots, h_d^{RA}$. Hence the toric h-vector of a cubical sphere is nonnegative if the same holds for the Ron Adin h-vector.

3 The h-vector of the Stanley ring

In this section we study the h-vector introduced in [7], and show that Theorem 2.2 is applicable to it.

Given a cubical complex $\mathcal{C}$ over vertex set V, the *Stanley ring* $K[\mathcal{C}]$ of this $\mathcal{C}$ over a field K is defined as the factor of the polynomial ring $K[x_v : v \in V]$ by the ideal $I(\mathcal{C})$ generated by all monomials $x_{v_1} \cdot x_{v_2} \cdots x_{v_k}$ such that $\{v_1, \ldots, v_k\}$ is not contained in any face of $\mathcal{C}$, and by all binomials $x_u \cdot x_v - x_{u'} \cdot x_{v'}$ such that $\{u, v\}$ and $\{u', v'\}$ are diagonals of the same face.

In [7, Theorem 3] we have shown that the Hilbert-series of the Stanley ring $K[\mathcal{C}]$ of a $(d-1)$-dimensional cubical complex $\mathcal{C}$ is given by

$$\mathcal{H}(K[\mathcal{C}], t) = 1 + \sum_{i=0}^{d-1} f_i \cdot \sum_{k=1}^{\infty} (k-1)^i \cdot t^k. \tag{3}$$

We have introduced

$$f_j^{\triangle} \stackrel{\text{def}}{=} \begin{cases} \sum_{i=j}^{d-1} f_i \cdot S(i,j) \cdot j! & \text{when } 0 \leq j \leq d-1 \\ 1 & \text{when } j = -1 \end{cases}, \tag{4}$$

and we have transformed (3) into the following equivalent form.

$$\mathcal{H}(K[\mathcal{C}], t) = \frac{\sum_{i=-1}^{d-1} f_i^{\triangle} \cdot t^{i+1} \cdot (1-t)^{d-i-1}}{(1-t)^d}. \tag{5}$$

The vector $(f_{-1}^{\triangle}, \ldots, f_{d-1}^{\triangle})$ is the f-vector of a simplicial complex associated to $\mathcal{C}$. When $\mathcal{C}$ is the boundary complex of a cubical polytope then $(f_{-1}^{\triangle}, \ldots, f_{d-1}^{\triangle})$ is the f-vector of a triangulation of the boundary. Thus we call the h-vector defined by

$$\sum_{i=0}^{d} h_i^{\triangle} \cdot x^i \stackrel{\text{def}}{=} \sum_{i=-1}^{d-1} f_i^{\triangle} \cdot t^{i+1} \cdot (1-t)^{d-i-1}$$

the *triangulation h-vector* of $\mathcal{C}$.

In the proof of the main result of this section we will apply the following lemma.

Lemma 3.1 *Let $\Phi_{a,b,c}(t)$ denote the formal power series $\sum_{k \geq 0} k^a \cdot (k+1)^b \cdot (k+2)^c \cdot t^k$, where a, b and c be natural numbers. Then the following hold.*

(i) We have $\Phi_{a,b,c}(t) = \frac{A_{a,b,c}(t)}{(1-t)^{a+b+c+1}}$, where $A_{a,b,c}(t)$ is a polynomial with integer coefficients, of degree at most $a+b+c$.

(ii) If $b > 0$ then the degree of $A_{a,b,c}(t)$ is at most $a+b+c-1$.

(iii) If $b > 0$ or $c = 0$ then $A_{a,b,c}(t)$ has only nonnegative coefficients.

The three statements of the lemma may be proved by induction on $a+b+c$, using differential equations for $\Phi_{a,b,c}(t)$.

Remark 2 The polynomials $A_{a,b,c}(t)$ are generalizations of the *Eulerian polynomials.* It is well known that for $b = c = 0$ the coefficient $A^i_{a,0,0}$ (where $i = 1, \dots, a$) is the number of those permutations σ of the set $\{1, \dots, a\}$, which have $(i-1)$ rises, i.e., which satisfy $|\{j \in \{1, 2, \dots, a\} : \sigma(j) < \sigma(j+1)\}| = i - 1$. See, e.g. [5, Section 6.5].

Remark 3 In the case when $a \geq b \geq c$, the polynomial $A_{a,b,c}(t)$ occurs as a special case of a generalization of Eulerian polynomials, see [13, Chapter 4, Exercise 26].

Theorem 3.2 *The invariants $h_0^\triangle, h_1^\triangle, \dots,$ and $h_d^\triangle$ are nonnegative linear combinations of the Ron Adin h-vector.*

Proof *(Sketch)*: We need only to show that the conditions of Theorem 2.2 are satisfied.

Let us compute the change of $\mathcal{H}(K[\mathcal{C}], t)$ when we add a shelling component of type (r, s) to a $(d-1)$-dimensional shellable cubical complex, where $r > 0$ or $s = d-1$. Using Lemma 1.5, after some calculation we obtain that this change is

$$\Delta\mathcal{H}(K[\mathcal{C}], t) = \frac{t \cdot A_{s,r,d-1-r-s}(t)}{(1-t)^d}.$$

Hence the change of $\sum_{j=0}^{d} h_j^\triangle \cdot t^j$ is $t \cdot A_{s,r,d-1-r-s}(t)$, which is a polynomial with nonnegative coefficients by part (iii) of Lemma 3.1.

Finally, similar calculations show that the value of $\mathcal{H}(K[\mathcal{C}^{d-1}], t)$, i.e., the Hilbert series of the Stanley ring of a standard $(d-1)$-cube is equal to

$$\mathcal{H}(K[\mathcal{C}^{d-1}], t) = \frac{A_{0,d-1,0}(t)}{(1-t)^d}.$$

Hence the triangulation h-vector of a standard $(d-1)$ cube satisfies

$$\sum_{j=0}^{d} h_j^\triangle \cdot t^j = A_{0,d-1,0}(t),$$

and is also nonnegative by part (iii) of Lemma 3.1. □

Remark 4 The proofs of the Theorems 2.2 and 3.2 allow us to express the triangulation h-vector of a $(d-1)$-dimensional cubical complex in terms of its Ron Adin h-vector. We have the following formula.

$$\sum_{i=1}^{d} h_i^{\triangle} \cdot t^i = h_0^{RA} \cdot A_{0,d-1,0}(t) + 2 \cdot \sum_{i=1}^{d-1} h_i^{RA} \cdot t \cdot A_{i-1,1,d-i-1}(t) + h_d^{RA} \cdot t \cdot A_{d-1,0,0}(t). \tag{6}$$

Remark 5 The polynomials $A_{a,b,c}(t)$ used in equation (6) are not covered in Remarks 2 and 3. It is an interesting problem to find a combinatorial interpretation of all nonnegative $A_{a,b,c}^i$'s.

4 Cubical barycentric subdivision of a simplicial sphere

In this section we prove that the Ron Adin h-vector of a cubical complex, which was obtained by barycentric subdivision of a simplicial sphere, is nonnegative.

Definition 4.1 *Given a simplicial complex $\triangle$, its* first cubical barycentric subdivision *is the cubical complex $C(\triangle)$ defined as follows.*

(i) The vertex set of $C(\triangle)$ is the set of nonempty faces of $\triangle$.

(ii) The faces of $C(\triangle)$ are the intervals of the face poset of $\triangle$, i.e., all sets of the form

$$[\sigma, \tau] \stackrel{def}{=} \{\lambda \in \triangle : \sigma \subseteq \lambda \subseteq \tau\}.$$

Remark 6 The partially ordered set $\mathrm{Int}(P)$ of intervals of a partially ordered set P is studied in [13, Chapter 3, Exercises 7, 58]. Our definition is close to the special case, when P is the poset of faces of a simplicial complex, the only difference being that we do not consider the intervals containing the minimum element of P. When P is the face poset of a polytope, then we obtain *barycentric covers* studied by Babson, Billera, and Chan in [3].

It is easy to verify that $C(\triangle)$ is a cubical complex, and that a geometric realization of $\triangle$ may be extended to a geometric realization of $C(\triangle)$ by barycentrically subdividing the realization of $\triangle$ into a cubical complex.

We can express the f-vector $(f_{-1}^{\square}, \ldots, f_{d-1}^{\square})$ of $C(\triangle)$ in terms of the f-vector $(f_{-1}, \ldots, f_{d-1})$ of $\triangle$ as follows.

$$f_k^{\square} = \sum_{j=k}^{d-1} \binom{j+1}{k} \cdot f_j \quad \text{holds for } k = 0, 1, \ldots, d-1. \tag{7}$$

Theorem 4.2 *The entries of the Ron Adin h-vector of the cubical subdivision $\mathcal{C}(\Delta)$ of a simplicial complex Δ may be expressed as nonnegative linear combinations of the entries of the (simplicial) h-vector of Δ.*

Proof *(Sketch)*: By Definition 2.1 and equation (7), the Ron Adin h-vector $(h_0^{RA}, \ldots, h_d^{RA})$ of $\mathcal{C}(\Delta)$ satisfies the following.

$$\begin{aligned}\sum_{i=0}^{d} h_i^{RA} \cdot t^i &= \sum_{k=0}^{d} f_{k-1}^{\square} \cdot c_k(t) = c_0(t) + \sum_{k=1}^{d} \sum_{j=k-1}^{d-1} \binom{j+1}{k-1} \cdot f_j \cdot c_k(t) \\ &= c_0(t) + \sum_{k=1}^{d} \sum_{j=k-1}^{d-1} \binom{j+1}{k-1} \cdot f_j \cdot c_k(t) = c_0(t) + \sum_{j=0}^{d-1} f_j \cdot \sum_{k=1}^{j+1} \binom{j+1}{k-1} \cdot c_k(t).\end{aligned}$$

Substituting Definition 2.1, after some calculation we get

$$\sum_{k=1}^{j+1} \binom{j+1}{k-1} \cdot c_k(t) = \frac{2^{1-d} \cdot (1-t)^{d-j-2} \cdot t \cdot ((1+t)^{j+1} - (2 \cdot t)^{j+1}) + (-1)^{d-j+1} \cdot t^{d+1}}{1+t}.$$

Observe that (1) may be written as

$$\sum_{j=-1}^{d-1} f_j \cdot t^{j+1} = \sum_{i=0}^{d} h_i \cdot t^i \cdot (1+t)^{d-i}.$$

Putting all these observations together, after some calculation we get

$$\sum_{i=0}^{d} h_i^{RA} \cdot t^i = \frac{1 + h_d \cdot t^{d+1}}{1+t} + \frac{t}{1-t} \cdot \sum_{i=0}^{d} h_i \cdot \left(\left(\frac{1+t}{2}\right)^{i-1} - t^i \cdot \left(\frac{1+t}{2}\right)^{d-i-1} \right).$$

Using $h_0 = 1$ we get

$$\begin{aligned}\sum_{i=0}^{d} h_i^{RA} \cdot t^i &= h_0 \cdot \left(\frac{1}{1+t} + \frac{t}{1-t} \cdot \left(\frac{2}{1+t} - \frac{(1+t)^{d-1}}{2^{d-1}} \right) \right) \\ &\quad + h_d \cdot \left(\frac{t^{d+1}}{1+t} + \frac{t}{1-t} \cdot \left(\frac{(1+t)^{d-1}}{2^{d-1}} - \frac{2 \cdot t^d}{1+t} \right) \right) \\ &\quad + \frac{t}{1-t} \cdot \sum_{i=1}^{d-1} h_i \cdot \left(\left(\frac{1+t}{2}\right)^{i-1} - t^i \cdot \left(\frac{1+t}{2}\right)^{d-i-1} \right).\end{aligned}$$

Here h_0 is multiplied with

$$\phi_0(t) \stackrel{\text{def}}{=} \frac{2^{d-1} \cdot (1-t) + 2^d \cdot t - t \cdot (1+t)^d}{2^{d-1} \cdot (1+t) \cdot (1-t)} = \frac{2^{d-1} \cdot (1+t) - t \cdot (1+t)^d}{2^{d-1} \cdot (1+t) \cdot (1-t)} = \frac{2^{d-1} - t \cdot (1+t)^{d-1}}{2^{d-1} \cdot (1-t)}$$

and h_d is multiplied with

$$\phi_d(t) \stackrel{\text{def}}{=} \frac{2^{d-1} \cdot (1-t) \cdot t^{d+1} + t \cdot (1+t)^d - 2^d \cdot t^{d+1}}{2^{d-1} \cdot (1+t) \cdot (1-t)} = \frac{t \cdot (1+t)^{d-1} - 2^{d-1} \cdot t^{d+1}}{2^{d-1} \cdot (1-t)}.$$

Let also denote $\frac{t}{1-t} \cdot \left(\left(\frac{1+t}{2}\right)^{i-1} - t^i \cdot \left(\frac{1+t}{2}\right)^{d-i-1}\right)$ by $\phi_i(t)$ for $i = 1, 2, \ldots, d-1$. It is straightforward to verify that we have

$$t^d \cdot \phi_i\left(\frac{1}{t}\right) = \phi_{d-i}(t) \quad \text{for } i = 0, 1, \ldots d,$$

and thus it is sufficient to prove that $\phi_i(t)$ is a polynomial with nonnegative coefficients for $i \leq \frac{d}{2}$. For $i = 0$ we have

$$\phi_0(t) = \frac{2^{d-1} - t \cdot (1+t)^{d-1}}{2^{d-1} \cdot (1-t)} = \frac{2^{d-1} - \sum_{j=0}^{d-1} \binom{d-1}{j} \cdot t^{j+1}}{2^{d-1} \cdot (1-t)} = \sum_{j=0}^{d-1} \frac{\binom{d-1}{j}}{2^{d-1}} \cdot \frac{1-t^{j+1}}{1-t},$$

which is clearly a polynomial with nonnegative coefficients. Similarly, for $0 < i \leq \frac{d}{2}$ we have

$$\begin{aligned}\phi_i(t) &= \frac{t}{1-t} \cdot \left(\left(\frac{1+t}{2}\right)^{i-1} - t^i \cdot \left(\frac{1+t}{2}\right)^{d-i-1}\right) = \left(\frac{1+t}{2}\right)^{i-1} \cdot \frac{t}{1-t} \cdot \left(1 - t^i \cdot \left(\frac{1+t}{2}\right)^{d-2\cdot i}\right) \\ &= \left(\frac{1+t}{2}\right)^{i-1} \cdot t \cdot \left(\sum_{j=0}^{d-2\cdot i} \frac{\binom{d-2\cdot i}{j}}{2^{d-2\cdot i}} \cdot \frac{1-t^{i+j}}{1-t}\right),\end{aligned}$$

which is again a polynomial with nonnegative coefficients. □

Corollary 4.3 *If $\triangle$ is a Cohen-Macaulay simplicial complex (e.g. a simplicial sphere), then the Ron Adin h-vector of its cubical barycentric subdivision $C(\triangle)$ is nonnegative.*

It is known that there exist unshellable triangulations of the $(d-1)$-sphere for $d \geq 4$ ([9] gives a construction), but it is not known whether any cubical barycentric subdivision of a simplicial sphere is unshellable.

Conjecture 4.4 *For $d \geq 4$ there exist a simplicial complex $\triangle$ such that its geometric realization is homeomorphic to a $(d-1)$-sphere, and $C(\triangle)$ is not a shellable cubical complex.*

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