SOME POLYNOMIALS ASSOCIATED WITH UP-DOWN PERMUTATIONS

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ABSTRACT. Up-down permutations, introduced many years ago by André under the name alternating permutations, were studied by Carlitz and coauthors in a series of papers in the 1970s. We return to this class of permutations and discuss several sets of polynomials associated with them. These polynomials allow us to divide up-down permutations into various subclasses, with the aid of the exponential formula. We find combinatorial interpretations, and explicit—albeit complicated—expressions for the coefficients of these polynomials. One of our interpretations leads to a new type of sequence that is equinumerous with the up-down permutations, and we give a bijection.

I Introduction

We recall that an **up-down permutation** of length n is a string of numbers $a_1 a_2 \cdots a_n$, where $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$ and, for each i, a_{2i} is greater than both a_{2i-1} and a_{2i+1}. Thus, for example, 263514 is an up-down permutation of length 6. These were first studied by André **[An1]**, **[An2]**, who called them alternating permutations. Although Netto devoted a section to them in his early treatise on combinatorics [Ne], they were rediscovered by Entringer in the 1960s **[E1]**. A subsequent paper of Entringer **[E2]** apparently piqued the interest of Leonard Carlitz, who (with occasional coauthors) made an extensive study of these and related objects in a series of papers in the 1970s. (See **[C1]**, **[C2]**, **[C3]** and **[CS]**, in particular, as well as other papers cited in these works.) Carlitz objects to André's terminology, on the ground that it implies a (spurious) connection with the alternating group, and calls them up-down permutations, a name we will adhere to. At around the same time, Melzak treated them briefly in his book **[Me]**. With the same objection, he referred to them as zigzag sequences.

The basic result is due to André: if E_n denotes the number of up-down permutations of $\{1, 2, \ldots, n\}$, and $E_0 = 1$, then

$$\sec t + \tan t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{1.1}$$

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In particular,

$$\sec t = \sum_{n=0}^{\infty} E_{2n} \frac{t^{2n}}{(2n)!} \qquad \text{and} \qquad \tan t = \sum_{n=0}^{\infty} E_{2n+1} \frac{t^{2n+1}}{(2n+1)!} \tag{1.2}$$

The E_n are often called **Euler numbers**, although this name is used about as often instead for the coefficients of $\operatorname{sech} x$, either explicitly or implicitly, *e.g.*, in **[GKP]**, **[No]**, **[Kn]**, or in some of Carlitz's other papers. For this reason, it is also common to refer to the E_n of even suffix as **secant numbers** and to those of odd suffix as **tangent numbers**. These numbers have arisen recently in work of Arnold on singularities of functions **[Ar]**, and they can be given by something like a Pascal's triangle (see **[Ar]** or **[Du]**).

In addition to the Carlitz school, there is a good deal of other work on enumerating permutations with respect to various patterns. Much of this is summarized in **[GJ]**. We particularly wish to mention a q-analogue of (1.1), due to Stanley **[Sta]** and Gessel **[G1]**. Two interesting recent papers relating to these matters are **[Stg]** and **[RZ]**. Our work begins with an extension of (1.1).

II Some polynomials

We consider

$$(\sec t + \tan t)^x =: \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!} \tag{2.1}$$

Differentiating with respect to t we find that

$$(\sec t + \tan t)^x \, x \sec t = \sum_{n=1}^{\infty} a_n(x) \frac{t^{n-1}}{(n-1)!} \, .$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+1}(x) \frac{t^n}{n!} &= x \sum_{k=0}^{\infty} a_k(x) \frac{t^k}{k!} \sum_{j=0}^{\infty} E_{2j} \frac{t^{2j}}{(2j)!} \\ &= x \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k+2j=n} \binom{n}{2j} E_{2j} \, a_k(x) \end{aligned}$$

and hence

$$a_{n+1}(x) = x \sum_{j} \binom{n}{2j} E_{2j} \, a_{n-2j}(x) \tag{2.2}$$

Since $a_0(x) = 1$, (2.2) implies that $a_n(x)$ is a monic polynomial of degree n in x, which moreover is an even function of x if n is even, and an odd function of x if n is odd. (The latter fact also follows from $(\sec t + \tan t)^{-1} = \sec(-t) + \tan(-t)$.)

The first several polynomials are $a_0(x) = 1$, $a_1(x) = x$, $a_2(x) = x^2$, $a_3(x) = x^3 + x$, $a_4(x) = x^4 + 4x^2$, $a_5(x) = x^5 + 10x^3 + 5x$, and $a_6(x) = x^6 + 20x^4 + 40x^2$. Note that $a_n(1) = E_n$; thus the coefficients of $a_n(x)$ represent a division of the up-down permutations of an n-set into a number of subclasses. We shall have more to say about this presently, but let us first return to the generating function (2.1). If we differentiate with respect to x, we obtain

$$\sum_{n=1}^{\infty} a_n'(x)\,\frac{t^n}{n!} = (\sec t + \tan t)^x \log(\sec t + \tan t) \tag{2.3}$$

Evidently

$$\log(\sec t + \tan t) = \sum_{n=0}^{\infty} E_{2n}\,\frac{t^{2n+1}}{(2n+1)!} \tag{2.4}$$

and substituting this into (2.3) we find that

$$a_n'(x) = \sum_j \binom{n}{2j+1} E_{2j}\, a_{n-2j-1}(x)$$

The polynomials $a_n(x)$ are also of **binomial type**, *i.e.*,

$$a_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} a_k(x)\, a_{n-k}(y)$$

If we knew a combinatorial interpretation of $\log(\sec t + \tan t)$, we could invoke the exponential formula (an excellent reference for this is [Wi]) to find a combinatorial interpretation of our polynomials. But this is easy from (2.4): $\log(\sec t + \tan t)$ is the exponential generating function for up-down permutations that end with 1. (Note that these are necessarily of odd length.) For these permutations are clearly equinumerous with up-down permutations of even length one less—we need only cut the 1 off and diminish all the other elements by 1—and we know that E_{2n} enumerates these.

The exponential formula then tells us that $a_n(x)$ counts combinatorial objects comprising up-down permutations that end in 1. Since $a_n(1) = E_n$, these objects are going to be ordinary up-down permutations. To illustrate what is happening, let us consider an example.

III Up-down permutations of length 5: an example

We seek to explain the fact that $a_5(x) = x^5 + 10x^3 + 5x$. Clearly this represents a division of the 16 up-down permutations of length 5 into classes of size 10, 5 and 1, but on what basis?

The term $5x$ is easy to explain. This term represents the 5 up-down permutations of length 5 that end with 1, which are 24351, 25341, 34251, 35241 and 45231. The significance of the x is that only one up-down permutation ending in 1 is necessary to make these 5—thus x only appears to the first power. The other 11 up-down permutations of length 5 will be made up from several shorter alternating permutations that end with 1.

Moving to the other extreme case, the term x^5 corresponds to the up-down permutation 15243. We read any up-down permutation from left to right until we encounter the element 1. We cut off the 1 and everything that precedes it, and we relabel the remaining k (say) elements with $\{1, 2, \ldots, k\}$ in an order-preserving fashion. In this case this leaves us with 4132; in general we will always have a **down-up** permutation of some length k at this stage (where if $k = 0$, as in the preceding paragraph, then we are already done). There is a simple involution between down-up and up-down permutations of length k: subtract every element from $k + 1$. In the present case we have now arrived at the up-down permutation 1423, and we repeat the procedure as many times as necessary to deal with all the elements. We will have to use it 5 times here, since when we apply it to 1423 we will get 132, and then 12, and then 1. So 15243 comprises 5 copies of the up-down permutation 1. From our point of view, this sort of behavior is the archetype—any part of an up-down permutation made up of copies of 1 goes from the smallest available element to the largest and back.

(This algorithm may be described more simply as reading from left to right looking first for the 1, then for the largest element to the right of 1, then for the smallest element to the right of that, and so forth. The description in the preceding paragraph seems to be preferable for a q-analogue of this theory.)

The other 10 up-down permutations of length 5 comprise two copies of 1 and one copy of 231 (which is the unique up-down permutation of length 3 that ends with 1). For example, 34152 decomposes into 341-5-2; 341 is a 231-type permutation, and 52 would be relabeled as 21 and subtracted to 12, which decomposes into two copies of 1. Since these 10 permutations decompose into three pieces each, we get a term $10x^3$.

The 10 is really a $\binom{5}{3}$, in the sense that the coefficient of x^{n-2} in $a_n(x)$ is always $\binom{n}{3}$. One may prove this by induction using (2.1) without difficulty, but it is more interesting to give a combinatorial explanation, and this is not hard. If we break an up-down permutation of length n into $n - 2$ pieces of odd length, we must get one piece of length 3 and the rest of length 1. That is, we must get one piece of 231-type and $n - 3$ 1's. We may choose any three elements to be in the 231 piece, and this determines the permutation completely. Let us give an example to show how.

Suppose we look at alternating permutations of length 11, and we choose 2,6 and 8 to be in the piece of length 3. Then the permutation must begin with 1, and 11 must come next since it will become 1 after the original 1 is cut off. After 11, the next candidate to be a new 1 is the element 2, so the piece of length 3 must come next since it contains 2. We put 2, 6 and 8 into a 231 pattern, *i.e.*, 682, and then continue zigzagging. The result is 1-11-6-8-2-10-3-9-4-7-5. The reader may check that, after applying the algorithm to this three times, we obtain 162534, which is the length 6 counterpart of 15243 and hence comprises six 1's.

IV General expression for the coefficients

In view of the even/oddness property of $a_n(x)$, we have

$$a_n(x) = \sum_k \alpha(n, 2k)\, x^{n-2k} \tag{4.1}$$

for some coefficients $\alpha(n, 2k)$, where we have seen already that $\alpha(n, 0) = 1$ and $\alpha(n, 2) = \binom{n}{3}$. Substituting (4.1) into (2.2), we get

$$\alpha(n+1, 2m) = \sum_j \binom{n}{2j} E_{2j}\, \alpha(n-2j, 2m-2j) \tag{4.2}$$

We may attempt to find a general expression for $\alpha(n, 2k)$; either by induction, using (4.2), or by counting, informed by the exponential formula. Let us first calculate $\alpha(n, 4)$, and then tackle the general case. Here we are breaking up-down permutations of length n into $n-4$ pieces, where "pieces" means throughout this section "up-down permutations ending with the smallest element" (we note again that these are necessarily of odd length). There are two general ways of doing this: either one piece has length 5, and the others length 1; or two pieces have length 3 and the rest length 1. The first case is similar to the argument for $\alpha(n, 2)$. We may choose any 5 elements and arrange them in any of the 5 patterns 24351, 25341, 34251, 35241, or 45231, and this determines the permutation completely. For example if we pick $n = 9$, the elements 2,3,4,6,9 and the pattern 34251, then the permutation begins with 1, and the block containing 9 comes next, with 9 at the end. Subtracting the pattern 34251 from 6 we get 32415, and putting 2,3,4,6,9 in the appropriate locations we find that the permutation so far is 143629. The smallest remaining element is 5 and the largest 8, so the whole permutation is 143629587. Thus in general there are $5\binom{n}{5} = E_4\binom{n}{5}$ permutations of this sort.

In the second case, the decomposition into 2 pieces of length 3 and the rest of length 1, we have $\binom{n}{6}$ ways to choose the elements that will be in the length 3 pieces. Suppose for example that $n = 12$ and we have chosen 3,4,6,8,9,11. The first of these numbers that would appear at the end of a piece is 11 (the hierarchy being 1,12,2,11,3,10 and so forth), and we can choose any two of the other 5 elements to be in the same piece as 11. Suppose we take 3 and 6. The permutation is now determined. It begins 1-12-2. Then 3,6 and 11 appear in a 213 pattern, so we have 1-12-2-6-3-11 so far. The smallest remaining element is 4, and it therefore is the next thing to become 1. But 4 was one of the elements in our original choice of six elements, the other unused ones being 8 and 9. So they come next in a 231 pattern; thus we are up to 1-12-2-6-3-11-8-9-4 and finally 1-12-2-6-3-11-8-9-4-10-5-7.

It follows from the above considerations that $\alpha(n, 4) = E_4\binom{n}{5} + \binom{5}{2}\binom{n}{6}$; this may also be proved by induction. $\alpha(n, 6)$ may be written neatly as $61\binom{n}{7} + 280\binom{n+1}{9}$.

Theorem. *The general form of* $\alpha(n, 2k)$ *is*

$$\alpha(n, 2k) = \sum_{j=1}^{k} \binom{n}{2k+j} \delta(2k, j)$$

where $\alpha(n,0)=1$ *and*

$$\delta(2k,j)=\sum_{\substack{k_1+\cdots+k_j=k\\ k_i\geq 1}}\left\{\begin{array}{c}\binom{2k+j-1}{2k_1}E_{2k_1}\binom{2k+j-2k_1-2}{2k_2}E_{2k_2}\times\cdots\\ \times\binom{2k+j-2k_1-\cdots-2k_{j-1}-j}{2k_j}E_{2k_j}\end{array}\right\}$$

A few remarks on this expression are in order. Note that $2k$ is the amount by which the number of elements exceeds the number of pieces. Also j counts the number of pieces whose size exceeds 1, which we will refer to as "large pieces". Thus there are $2k+j$ elements in the j large pieces, and moreover, $2k_i$ is one less than the number of elements in the i^{th} large piece. Finally, the alert reader will have noticed that the last binomial coefficient in the expression for $\delta(2k,j)$ equals 1.

Proof. We give a combinatorial proof; an inductive proof is also possible. The parameter j is at least 1 since, if $k>0$, there is at least one large piece. j does not exceed k since the minimum size of a large piece here is 3. We choose $2k+j$ of our n elements to be in the large pieces; of course this can be done in $\binom{n}{2k+j}$ ways. Some one of these $2k+j$ elements must be encountered first as we build the permutation by zigzagging. This element is in a piece with some even number $2k_1$ of the other chosen elements. There are $\binom{2k+j-1}{2k_1}$ ways to choose the elements for this piece, and E_{2k_1} patterns that they may be arranged in. Of the remaining elements, one must be encountered next in the zigzag pattern (possibly after some singleton elements which are not among the $2k+j$ chosen ones), and some even number $2k_2$ of the chosen elements are in the same piece, arranged in one of E_{2k_2} patterns, so there are $\binom{2k+j-2k_1-2}{2k_2}E_{2k_2}$ choices for this piece, and so on.

V Some more polynomials

Another generating function that is amenable to this sort of treatment is

$$(1-\sin t)^{-x}=:\sum_{n=0}^{\infty}c_n(x)\frac{t^n}{n!}\tag{5.1}$$

We have

$$\frac{\partial}{\partial t}(1-\sin t)^{-x}=x(1-\sin t)^{-x-1}\cos t\tag{5.2}$$

$$=x(1-\sin t)^{-x}(\sec t+\tan t)\tag{5.3}$$

¿From (5.3) and (1.1) we get the recurrence

$$c_{n+1}(x)=x\sum_{k=0}^{n}\binom{n}{k}E_{n-k}\,c_k(x)\tag{5.4}$$

The first few polynomials are $c_0(x)=1$, $c_1(x)=x$, $c_2(x)=x^2+x$, $c_3(x)=x^3+3x^2+x$ and $c_4(x)=x^4+6x^3+7x^2+2x$.

In what follows we will require the expansions of $-\log(1-\sin t)$ and $(1-\sin t)^{-1}$. Since these functions are respectively the integral and derivative of $\sec t + \tan t$, we have

$$-\log(1-\sin t) = \sum_{n=1}^{\infty} E_{n-1} \frac{t^n}{n!} \tag{5.5}$$

$$(1-\sin t)^{-1} = \sum_{n=0}^{\infty} E_{n+1} \frac{t^n}{n!} \tag{5.6}$$

If we differentiate (5.1) with respect to x and equate coefficients using (5.5), we get the derivative formula

$$c_n'(x) = \sum_{j=1}^{n} \binom{n}{j} E_{j-1}\, c_{n-j}(x)$$

Moreover, the polynomials $c_n(x)$ are of binomial type.

(5.5) has a simple combinatorial interpretation, as the exponential generating function for up-down permutations that begin with 1 (there are at least two natural bijections between these and general up-down permutations of length one less). Since $(1-\sin t)^{-x} = \exp(-x\log(1-\sin t))$, the exponential formula tells us that $(1-\sin t)^{-x}$ generates some sort of combinatorial structures comprising up-down permutations that begin with 1. To explain just what these are, let us set

$$c_n(x) = \sum_{k=0}^{n} \gamma(n,k)\, x^{n-k} \tag{5.7}$$

Then $n-k$ will be the number of "pieces" (*i.e.*, up-down permutations beginning with 1; let us also again refer to a piece whose size exceeds 1 as a "large piece"), and k is the excess of elements over pieces. For example, let us look at $c_3(x) = x^3 + 3x^2 + x$. There is only one up-down permutation of length 3 that begins with 1, namely 132, and it is generated by the x term. In general, $c_3(x)$ generates 3-digit strings containing one 1, one 2 and one 3, which one reads by first finding the 1. Anything from the 1 onwards (reading left to right) must be an up-down permutation, so that 123 is not a permissible string. To read 213, say, we find the 1, check that everything from there onwards is (or would be, if it were relabeled in an order-preserving fashion) an up-down permutation, and cut it off, relabel the rest in an order-preserving fashion, and repeat. So 213 is an up-down permutation of length 2 preceded by one of length 1, and so is 312. 231 is an up-down permutation of length 1 preceded by one of length 2, and these three strings are counted by the $3x^2$ term. 321 is three up-down permutations of length 1 strung together, and is generated by the x^3 term. So these strings are sequences in which everything from the 1 until the end is in an up-down pattern, and if we cut all that off and relabel the rest in an order preserving fashion, what remains still has this property, and continues to have it as we keep cutting pieces off and relabeling. We will refer to them as **zigzig** sequences, or zigzigs for short. In general, the coefficient of x^n in $c_n(x)$ (*i.e.* $\gamma(n,0)$) is always 1, and represents the zigzig sequence $n\,(n-1)\cdots 2\,1$.

As before, we can find $\gamma(n,k)$ in general by a combinatorial argument. We have $\gamma(n,n) = \delta_{n0}$, since zigzig sequences always have at least one piece, unless they are empty. $\gamma(n,n-1) = E_{n-1}$, since here we are simply counting the number of zigzigs made up of one up-down permutation that begins with 1. Working from the other end, $\gamma(n,1) = \binom{n}{2}$, since here we have one piece of size two and all the rest of size 1, and once we choose which two elements are to be in the size two piece, the zigzig is completely determined. Similarly $\gamma(n,2) = \binom{n}{3} + 3\binom{n}{4}$.

Let us work out $\gamma(n,3)$ with care, since it already contains most of the complexity of the general case. Here we have three more elements than pieces, and there are several ways in which the excess elements might be distributed. We might have one piece of size four and all the others of size 1. In this case we may choose any four elements to be in the size 4 piece, and we may arrange them in either a 1324 or a 1423 pattern, and this determines the zigzig completely. For definiteness, suppose $n = 8$ and we choose 2,4,5,7 and the pattern 1423. Then the zigzig ends with 1, and is immediately preceded by the piece of size 4, which is 2745. The remaining elements precede 2745 in increasing order from right to left. Thus the zigzig is 86327451. In general there are $2\binom{n}{4}$ such zigzigs, and the 2 arises because $E_3 = 2$.

We might instead have one piece of size three, one of size two, and the rest of size 1. It also makes a difference whether the piece of size 3 appears to the right of the piece of size two, or to the left. (Both possibilities can be handled at once, but the argument will be closer to the general case if we distinguish them.) Suppose first that the piece of size 3 is on the right. We choose 5 elements to be in the large pieces. One of these 5 elements is smaller than the others, and in this instance it must be put at the beginning of the size three piece. We may choose any two of the other four elements to complete this piece, and then everything is determined. For concreteness, suppose that $n = 9$ and we begin by choosing 1,3,5,6,8, and that we further choose 3 and 8 to complete the piece of size three. Then the zigzig sequence must be 975642183. In general we get $\binom{n}{5}\binom{4}{2}$ in this case. If the piece of size two is to the right of the piece of size three, we get instead $\binom{n}{5}\binom{4}{1}$. The two terms could be combined and simplified to $10\binom{n}{5}$; in other words, $\binom{n}{3,2}$.

Finally we could have three pieces of size two, and the rest of size 1. Here we must choose 6 elements to be in the pieces of size two. Of these, one is smallest, and any of the other 5 may be put in the same piece. One of the remaining four elements is the smallest, and any of the other three may be put with it, and the zigzig sequence is now determined. For example suppose that $n = 14$ and we choose 2,4,8,10,11,13. Let's pair 11 with 2 and 8 with 4. Then the zigzig is 14-12-10-13-9-7-6-5-4-8-3-2-11-1. In general there are $\binom{n}{6}\binom{5}{1}\binom{3}{1}$ such zigzig sequences. Thus $\gamma(n,3) = 2\binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6}$.

To write the general case, set

$$\gamma(n,k) = \sum_{j=1}^{k} \binom{n}{k+j} \sigma(k,j) \tag{5.8}$$

Then j is the number of large pieces, k is the excess of elements over pieces, and we have

$$\sigma(k,j) = \sum_{\substack{k_1+\cdots+k_j=k \\ k_i\geq 1}} \left\{ \begin{array}{c} \binom{k+j-1}{k_1} E_{k_1} \binom{k+j-k_1-2}{k_2} E_{k_2} \times \cdots \\ \times \binom{k+j-k_1-\cdots-k_{j-1}-j}{k_j} E_{k_j} \end{array} \right\}$$

The proof is just as in the case $k=3$. We first choose the $k+j$ elements that will be in the large pieces. One of these is smaller than all the others, so must appear at the beginning of the rightmost large piece. There is some number $k_1 \geq 1$ of other elements in this piece, and they may be arranged in E_{k_1} ways. Then one of the remaining $k+j-k_1-1$ elements is smaller than all the others, and so forth.

VI A bijection

In view of the generating function (5.6), zigzig sequences are equinumerous with up-down permutations that are one unit longer. In this section we describe a bijection between these two types of sequences. The bijection has its genesis in a recurrence for the E_n that we have not yet mentioned. If $f(x) = \sec x + \tan x$, then one easily finds that $f''(x) = f(x)\, f'(x)$, and on equating coefficients a recurrence relation that we shall write as

$$E_{n+1} = \sum_{j=1}^{n} \binom{n-1}{j-1} E_j\, E_{n-j} \tag{6.1}$$

One may interpret this by reading an up-down permutation from left to right until both 1 and the largest element $n+1$ have been encountered. What lies to the right after this is either an up-down or a down-up permutation of some length $n-j$, according to whether 1 precedes or succeeds $n+1$. This length may be any integer from 0 to $n-1$; there are $\binom{n-1}{n-j}$ choices for the elements in this permutation and E_{n-j} ways that they may be arranged. The other piece is an up-down permutation of length $j+1$ that either ends with 1 or with $n+1$. In either case there are E_j of these.

We now describe the bijection. To go from an up-down permutation of length $n+1$ to a zigzig sequence of length n, the basic idea is to bubble the largest element $n+1$ to the front of the sequence and then delete it. We illustrate this with the case $n=3$. Here we are mapping the 5 up-down permutations of length 4, namely 1324, 2314, 2413, 3412 and 1423, to the 5 zigzig sequences of length 3, which are 132, 231, 213, 312 and 321. We begin by asking whether 1 appears to the left of the maximal element $n+1$ (4 in this case), or to the right. If 1 is on the left, then we subtract every element from $n+2$; otherwise we do nothing. In the case $n=3$, this leaves us with 4231, 3241, 2413, 3412 and 4132. The rightmost piece of the zigzig sequence is now found by reading from the 1 to the end, so we get 1, 1, 13, 12 and 132 respectively, leaving behind 423, 324, 24, 34 and 4 respectively. If only the largest element remains, as in the last case, then we are done. Otherwise we repeat the procedure, now asking whether the smallest remaining element precedes $n+1$

or not. If it does not, then we do nothing. If it does, then we perform an operation which we will call **switching**, which is a slight generalization of our initial step of (in some cases) subtracting all elements from $n+2$. We can think of this in either of two ways. If we have a k-digit string and we want to switch it, we can either relabel the string with the elements $1, 2, \ldots, k$, preserving the order, then subtract every element from $k+1$, and then restore the original labels; or we can think of this as looking at the k-digit string and switching the smallest element with the largest, the second smallest with the second largest and so forth.

In the present case this means we leave 423 alone, change 324 to 342, change 24 to 42 and change 34 to 43. The next piece now starts at the smallest element and goes to the end, so we get 23, 2, 2 and 3 respectively with 4, 34, 4 and 4 respectively left behind. We repeat this procedure until all that is left is the largest element. In this case there is only one sequence where there is anything left to do, namely 34. We switch this to 43 and cut off the 3, and we are all done. Thus the up-down permutations 1324, 2314, 2413, 3412 and 1423 map respectively to the zigzig sequences 231, 321, 213, 312 and 132.

Let us try a more complex example, the up-down permutation 352817496. 1 precedes the maximal element 9, so we begin by subtracting all the elements from 10 (in other words, switching) to get 758293614. Then 14 is the rightmost piece of the corresponding zigzig sequence, and we are left with 7582936. Here 2 precedes 9, so we have to perform switching again. This entails interchanging 2 with 9, 3 with 8, 5 with 7, and 6 with itself, and brings us to 5739286. Now 286 is the next piece removed, and we are left with 5739. Again 3 precedes 9, so we switch to get 7593, and remove 3. In 759 the maximal element again appears to the right of the minimal one, so we switch to 795 and remove 5, then switch 79 to 97 and remove 7. Thus the zigzig sequence corresponding to 352817496 is 75328614.

We illustrate the procedure for going from a zigzig sequence of length n to an up-down permutation of length $n+1$ with the example 7625413. The first step is to put $n+1$ at the beginning of the sequence; in this case we get 87625413. The sequence will now have a down-up pattern at the beginning, which in this example only lasts for two elements. However long it persists, we switch that part of the sequence, leaving the rest alone, in this case arriving at 78625413. The sequence will now have an up-down pattern at the beginning which persists for at least one element more than the down-up pattern at the previous step. If the up-down pattern extends all the way to the end of the sequence, then we are done; otherwise we switch the up-down elements, leaving the others alone. In this case the up-down pattern goes as far as 786, so we switch this to 768 and have 76825413. We keep repeating these steps until we arrive at an up-down permutation. In this example the down-up pattern goes as far as 768254, which we switch to 452867, thus arriving at 45286713, which is an up-down permutation of length 8.

That these two procedures invert each other is fairly clear after doing the last two examples in the other direction. In the first case one is switching and removing up-down pieces at the right, and the minimal elements in the removed pieces increase. In the second case one is switching and adding up-down pieces on the right, and the minimal elements in the added pieces decrease. Thus we have a bijection.

VII Some concluding remarks

Here we briefly mention several related matters that we have either omitted to save space, or have not fully worked out as yet. We may treat the generating function $(\sec t)^x$ in a similar manner. In so doing we are going down a path previously trodden by Carlitz and Scoville [CS], but as our perspective is slightly different, we still get some new results. Specifically, we consider

$$\sec^x t =: \sum_{n=0}^{\infty} b_n(x) \frac{t^{2n}}{(2n)!}$$

Then $b_n(x)$ is a polynomial of degree n in x, with similar properties to $a_n(x)$ and $c_n(x)$, and with the further property that $b_n(x)$ interpolates between up-down permutations of even length and those of odd length, in the sense that $b_n(1) = E_{2n}$ and $b_n(2) = E_{2n+1}$. Our interpretation of the coefficients of $b_n(x)$ allows us to give a combinatorial explanation of this. (Indeed, this is our main contribution to the theory of these polynomials, which first arose in [No].) These polynomials are related to $a_n(x)$ and $c_n(x)$, since $\sec t + \tan t = \sec t\,(1 + \sin t)$, or, better yet, $\sec t\,(\sec t + \tan t) = (1 - \sin t)^{-1}$.

We could use (5.2) instead of (5.3) to obtain a recurrence relation for the polynomials $c_n(x)$. This would yield

$$c_{n+1}(x) = x \sum_j \binom{n}{2j} (-1)^j \, c_{n-2j}(x+1)$$

which implies that these polynomials might profitably be written in terms of rising factorials. We are also able to calculate the coefficients in this expansion; they may be related to the γ's of section V using Stirling numbers of the first kind.

Finally, we remark that, as we alluded to in section III, one can work out a q-analogue of at least some of these results. The main tools are the q-exponential formula introduced by Gessel in [G2], which we used previously in [Jo], and the q-analogue of (1.1) that we mentioned in the introduction.

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