# Using Schubert toolkit to compute with polynomials in several variables

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**Abstract**

We give the Monk formula for double Schubert polynomials and we show how to use Schubert polynomials to compute in the ring of polynomials in $n$ variables regarded as a free module of rank $n!$ over the ring of symmetric polynomials.

## 1 Introduction

Computations in the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in several variables are not so easy because dimensions grow very quickly with the number of variables and with the degree. The problem is to find linear bases adapted to some specific problems without losing track of the multiplicative structure. For $n = 1$, such bases are provided by interpolation theory. In higher dimensions, a polynomial can have some symmetry, which cannot be efficiently exploited if the polynomial is just written as a sum of monomials.

For example, if a polynomial is totally symmetrical, we can have recourse to many combinatorial objects which render computations independent of the number of variables. To be able to use symmetry methods in the case of general polynomials, we shall look at the ring of polynomials in $n$ variables $x_1, \ldots, x_n$ as a module over the ring $\mathrm{Sym}[x_1, \ldots, x_n]$ of symmetric polynomials. It is easy to show that the set of monomials of degree less than $(n-1, \ldots, 2, 1, 0)$ (componentwise) is a linear basis of this module, which is free of rank $n!$.

We shall use a more interesting basis for this free module over $\mathrm{Sym}[x_1, \ldots, x_n]$, consisting of Schubert polynomials in the $x_i$ and $y_i$ (where the $y_i$ are a second set of indeterminates). These Schubert polynomials are the universal coefficients in the generalized Newton interpolation formula (they generalize Newton's interpolation polynomials $(x_1 - y_1) \ldots (x_1 - y_n)$). Moreover they are the natural basis in the geometric interpretation of the ring of polynomials as the equivariant cohomology ring of the flag manifold. Last, but not least, their combinatorics extends the one of Schur functions (the natural basis of the ring $\mathrm{Sym}[x_1, \ldots, x_n]$), with constructions on permutations replacing the handling of partitions and Young tableaux.

## 2 Background

Let $\mathbb{Z}[\mathrm{x}] := \mathbb{Z}[x_1, \ldots, x_n]$ be the ring of polynomials in the independent variables $x_1, \ldots, x_n$. We denote by $s_i, i = 1, \ldots, n-1$ the elementary (simple) transposition that interchanges

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the variables $x_i$ and $x_{i+1}$ and by $\partial_i, i = 1, \ldots, n-1$ the divided difference operator on the polynomial ring $\mathbb{Z}[\mathrm{x}]$:

$$\mathbb{Z}[\mathrm{x}] \ni p \quad \longrightarrow \quad \partial_i(p) := \big(p - s_i(p)\big) \big/ (x_i - x_{i+1}) \ . \tag{1}$$

In 1973, Bernstein, Gelfand & Gelfand [1] and Demazure [2] made the fundamental observation that divided differences satisfy, besides elementary transpositions, braid relations:

$$\begin{array}{rclrcll} s_i s_j &=& s_j s_i \ , & \partial_i \partial_j &=& \partial_j \partial_i \ , & |i-j| > 1 \ , \\ s_i s_{i+1} s_i &=& s_{i+1} s_i s_{i+1} \ , & \partial_i \partial_{i+1} \partial_i &=& \partial_{i+1} \partial_i \partial_{i+1} \ . & \end{array} \tag{2}$$

together with the extra relation $\partial_i^2 = 0$ replacing $s_i^2 = 1$.

Let $\mathfrak{S}_n$ be the symmetric group of degree $n$. $\mathfrak{S}_n$ is embedded into $\mathfrak{S}_{n+1}$ by adding a fixed point $n+1$. It allows one to define the group:

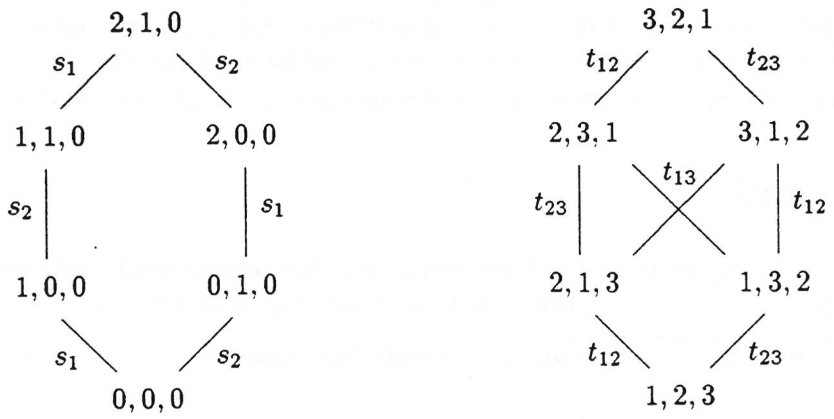
$$\mathfrak{S}_\infty := \lim_{i \to \infty} \mathfrak{S}_i \ , \tag{3}$$

of permutations of the set of positive integers fixing all but a finite number of them. A permutation $w \in \mathfrak{S}_n$ will be denoted by $w := (w(1), \ldots, w(n))$. An inversion of $w \in \mathfrak{S}_n$ is a couple $(i, j)$ of indices such that $1 \leq i < j \leq n$ and $w(i) > w(j)$. This gives a coding for a permutation $w$: the *code* $c(w)$ of $w$ is set to be a vector of non-negative integers, the $i$-th component being the number of positions $j > i$ such that $w(j) < w(i)$. There is a one-to-one correspondence between elements of $\mathfrak{S}_\infty$ and vectors of $\mathbb{N}^{(\mathbb{N}^*)}$.

A reduced decomposition of a permutation $w$ is a decomposition of $w$ as a product of elementary transpositions, of minimal length which is called the *length* $l(w)$ of the permutation. The length is also equal to the number of inversions and to the sum of the components of the code.

There are two natural orders on the symmetric group which makes it a poset ranked by the length. The *permutohedron* is the poset with permutations as vertices and edges between all pairs $(w, ws), w \in \mathfrak{S}_n$, $s$ a simple transposition. The corresponding order is called the *weak* (right) *order*. We denote by $t_{ij}$ the transposition $(i, j)$. The *Bruhatohedron* is the poset with permutations as vertices and edges between all pairs $(w, wt), w \in \mathfrak{S}_n$, $t$ a transposition such that $|l(w) - l(wt)| = 1$.

Here we display the permutohedron for $\mathfrak{S}_3$ writing codes instead of permutations, and on its right, the Bruhatohedron:



Braid relations (2) allow to define $\partial_w$ for any permutation $w$: taking any reduced decomposition $s_{i_1}\ldots s_{i_k}$ of $w$, these relations imply that the product $\partial_{i_1}\ldots\partial_{i_k}$ does not depend upon the choice of the reduced decomposition, and therefore can be denoted $\partial_w$.

Instead of computing in the algebra of divided differences, Lascoux and Schützenberger introduced in 1982 *Schubert polynomials* [12, 13] as follows. Let $\mathbf{x},\mathbf{y}$ be two sets of indeterminates $\mathbf{x} := \{x_1,\ldots,x_n\}$, $\mathbf{y} := \{y_1,\ldots,y_n\}$. For any $n$ and $w_0$ the element of $\mathfrak{S}_n$ of length $\binom{n}{2}$, we define the maximal (double) Schubert polynomial $\mathbb{X}_{w_0}$ to be:

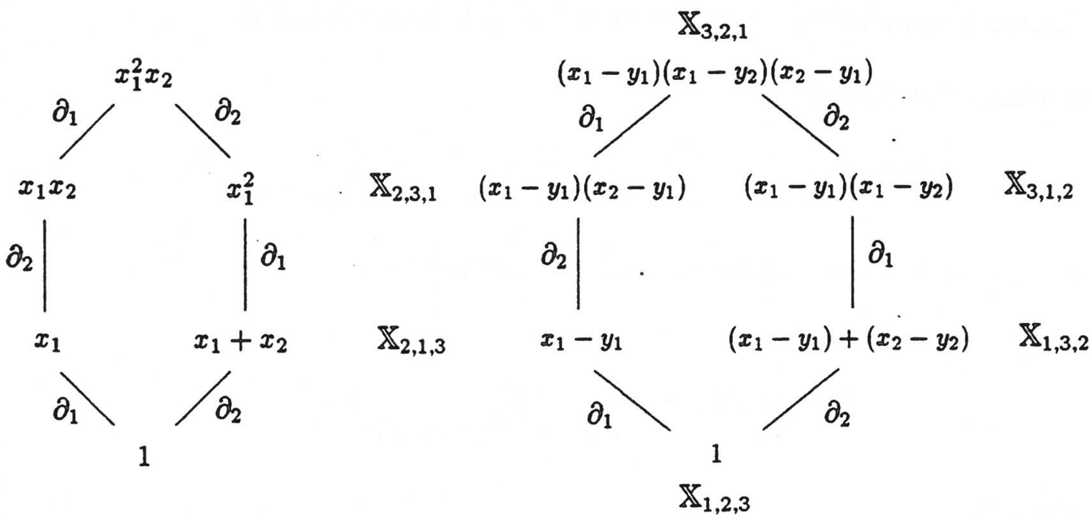
$$\mathbb{X}_{w_0} := \prod_{i+j\le n} (x_i - y_j) , \tag{4}$$

and in general, for any permutation $w \in \mathfrak{S}_n$,

$$\mathbb{X}_w := \partial_{w^{-1}w_0}(\mathbb{X}_{w_0}) , \tag{5}$$

in which divided differences act only on the $x_i$'s. The specializations $y_i = 0, \forall i$ of the $\mathbb{X}_w$ are called *simple Schubert polynomials* and denoted $X_w$.

The combinatorics of Schubert polynomials is closely linked with the combinatorics of divided differences and of the two orders on the symmetric group. For $\mathfrak{S}_3$, simple Schubert polynomials and Schubert polynomials are:




## 3 Schubert bases

We refer to Macdonald [14] for the algebraic theory of Schubert polynomials. Let $w$ be a permutation in $\mathfrak{S}_n$ and $c(w) = (c_1, c_2, \ldots, c_n)$ its code, then:

$$X_w = x_1^{c_1}x_2^{c_2}\ldots x_n^{c_n} + \sum \alpha_i\, x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n} , \tag{6}$$

where $\alpha_i > 0$, $i_1 \le n-1, i_2 \le n-2, \ldots, i_n \le 0$, and $(i_1, i_2, \ldots, i_n)$ lexicographically smaller than $(c_1, c_2, \ldots, c_n)$.

Schubert polynomials are compatible with the embedding $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$. Thus, $\{X_w, w \in \mathfrak{S}_\infty\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[x_1, x_2, \ldots]$. More precisely, let $\eta : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}$ be the constant term homomorphism. Then [14, (4.14)], one has:

$$\forall p \in \mathbb{Z}[x_1, x_2, \ldots], \qquad p = \sum_{w\in\mathfrak{S}_\infty} \eta(\partial_w p)\, X_w . \tag{7}$$

**Example 3.1** Let $p := x_1^5 + x_1 x_2^3 x_3$. Then $p$ is equal to:

$$\begin{aligned} p &= \mathrm{X}_{6,1,2,3,4,5} - \mathrm{X}_{3,4,2,1} - \mathrm{X}_{4,2,3,1} + \mathrm{X}_{2,5,3,1,4} \\ &= x_1^5 \quad - x_1^2x_2^2x_3 - x_1^3x_2x_3 + (x_1x_2^3x_3 + x_1^2x_2^2x_3 + x_1^3x_2x_3) \,. \end{aligned}$$

It should be noted that this expression of $p$ involves simple Schubert polynomials indexed by permutations in $\mathfrak{S}_4, \mathfrak{S}_5$ and $\mathfrak{S}_6$ while the original polynomial only depends upon the three variables $x_1, x_2$ and $x_3$. It is possible to give an expression of $p$ involving only simple Schubert polynomials in $\mathfrak{S}_3$, but the coefficients are now symmetric functions in $x_1, x_2, x_3$:

$$\begin{aligned} p \;=\; & \left(x_1x_2x_3^2 + x_1x_2^2x_3 + x_1^2x_2x_3 - x_1^2x_2^2x_3 - x_1^2x_2x_3^2 - x_1x_2^2x_3^2\right)\mathrm{X}_{1,2,3} - \left(x_1^3x_2\right. \\ & + x_1^3x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 \\ & \left. + x_2^2x_3^2 + x_2x_3^3\right)\mathrm{X}_{2,1,3} + \left(x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^3 + x_2^2x_3 + x_2x_3^2\right. \\ & \left. + x_3^3\right)\mathrm{X}_{3,1,2} - \left(x_1x_2x_3\right)\mathrm{X}_{2,3,1} + \left(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2\right)\mathrm{X}_{1,3,2} \,. \end{aligned}$$

Such expansions will be studied in the last part of this paper.

The multiplicative structure of the ring $\mathbb{Z}[x_1, \ldots, x_n]$, in the basis of simple Schubert polynomials is described by *Monk's formula* [16], [14, formula (4.15')]:

**Proposition 3.2 (Monk)**

$$x_r \mathrm{X}_w = \sum_{\nu = w\, t_{ri},\, i>r,\, l(\nu)=l(w)+1} \mathrm{X}_\nu - \sum_{\mu = w\, t_{ir},\, i<r,\, l(\mu)=l(w)+1} \mathrm{X}_\mu \,. \tag{8}$$

Since $\mathrm{X}_{s_r} = x_1 + \cdots + x_r$, proposition (3.2) is equivalent to:

**Proposition 3.3 (Monk)**

$$\mathrm{X}_{s_r}\mathrm{X}_w = \sum_{\nu = w\, t_{ij},\, i \le r < j,\, l(\nu)=l(w)+1} \mathrm{X}_\nu \,. \tag{9}$$

For example,

$$x_3\, \mathrm{X}_{3,1,5,4,2} = \mathrm{X}_{3,1,6,4,2,5} - \mathrm{X}_{3,5,1,4,2} - \mathrm{X}_{5,1,3,4,2} \,.$$

Formulas (7) and (8) have the disadvantage of expressing polynomials in $x_1, \ldots, x_n$ in terms of Schubert polynomials possibly indexed by permutations belonging to $\mathfrak{S}_m$ such that $m > n$. These two formulas are easily proved by decreasing induction on the length of $w$ using the *Leibnitz formula*:

$$\partial_k(pq) = (\partial_k p)q + (s_k p)\partial_k q \,, \tag{10}$$

in which $p$ and $q$ are two polynomials. Leibnitz's formula generalizes to any $\partial_w$. Instead of (10), we write:

$$\partial_k(pq) = \begin{array}{|c|c|}\hline \partial_k & p \\ \hline 1 & q \\ \hline \end{array} + \begin{array}{|c|c|}\hline s_k & p \\ \hline \partial_k & q \\ \hline \end{array} \tag{11}$$

where $\begin{array}{|c|c|}\hline \partial_k & p \\ \hline 1 & q \\ \hline \end{array} := (\partial_k p)q$ and $\begin{array}{|c|c|}\hline s_k & p \\ \hline \partial_k & q \\ \hline \end{array} := (s_k p)(\partial_k q)$. It is now clear that:

$$\partial_j\partial_k(pq) = \begin{array}{|c|c|c|}\hline \partial_j & \partial_k & p \\ \hline 1 & 1 & q \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline s_j & \partial_k & p \\ \hline \partial_j & 1 & q \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \partial_j & s_k & p \\ \hline 1 & \partial_k & q \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline s_j & s_k & p \\ \hline \partial_j & \partial_k & q \\ \hline \end{array} \tag{12}$$

reading each tableau as the product of its rows. More generally, given any reduced decomposition $s_{i_1} \dots s_{i_k}$ of a permutation $w$, $p$ and $q$ two polynomials, then $\partial_w(pq)$ is represented by the following sum of $2^{l(w)} = 2^k$ terms:

$$\partial_w(pq) := \sum_{j=1}^{k} \begin{array}{|c|c|c|c|}\hline a_1 & \cdots & a_k & p \\ \hline b_1 & \cdots & b_k & q \\ \hline \end{array}$$

where the symbol $\begin{array}{|c|}\hline a_m \\ \hline b_m \\ \hline \end{array}$ stands for either $\begin{array}{|c|}\hline \partial_{i_m} \\ \hline 1 \\ \hline \end{array}$ or $\begin{array}{|c|}\hline s_{i_m} \\ \hline \partial_{i_m} \\ \hline \end{array}$.

For now on, we fix the integer $n$ and use only Schubert polynomials $\mathbf{X}_w$ or $\mathbb{X}_w$, $w \in \mathfrak{S}_n$. Let $\mathrm{Sym}[\mathrm{x}] := \mathrm{Sym}[x_1, \dots, x_n]$ be the subring of symmetric polynomials in $x_1, \dots, x_n$. In fact, one has [14, Chap. V] the following proposition:

**Proposition 3.4** *The space* $\mathbb{Z}[\mathrm{x}]$ *is a free module of dimension* $n!$ *over the ring* $\mathrm{Sym}[\mathrm{x}]$ *with basis the simple Schubert polynomials* $\mathbf{X}_w$, $w \in \mathfrak{S}_n$*:*

$$\mathbb{Z}[x_1, \dots, x_n] \quad = \quad \bigoplus_{w \in \mathfrak{S}_n} \mathrm{Sym}[x_1, \dots, x_n]\, \mathbf{X}_w \,. \tag{13}$$

In other words, any polynomial $p \in \mathbb{Z}[\mathrm{x}]$ may be uniquely expressed as a sum:

$$p = \sum_{w \in \mathfrak{S}_n} c_w \mathbf{X}_w \,,$$

where the $c_w$'s belong to $\mathrm{Sym}[\mathrm{x}]$, as illustrated by the previous expansion of $x_1^5 + x_1 x_2^3 x_3$.

## 4 Double Monk's formula

Geometrical problems [5] impose to take double Schubert polynomials $\mathbb{X}_w$ as a linear basis of the space of polynomials in $\mathrm{x}$ with coefficients in $\mathrm{y}$. To recover the multiplicative structure of this space, we need to describe the products of Schubert polynomials by the single variables $x_i$. This is given by the following formula which extends Monk's formula (8):

**Proposition 4.1**

$$x_r \mathbb{X}_w \quad = \quad y_{w(r)} \mathbb{X}_w + \sum_{\nu = w\, t_{rj},\, j>r,\, l(\nu)=l(w)+1} \mathbb{X}_\nu - \sum_{\mu = w\, t_{jr},\, j<r,\, l(\mu)=l(w)+1} \mathbb{X}_\mu \,. \tag{14}$$

**Proof.** We need only consider two cases. First, we check the proposition in the case $w = w_0$: $(x_r - y_{n+1-r})\mathbb{X}_{w_0}$ is equal to the Schubert polynomial $\mathbb{X}_\mu$ where $\mu = w_0 t_{r,n+1}$ because $\mu$ is a *dominant* permutation (that is a permutation whose code is a partition in weakly decreasing order) [14, formula (6.14)]. This can be rewritten:

$$x_r \mathbb{X}_{w_0} = y_{w_0(r)} \mathbb{X}_{w_0} + \mathbb{X}_{w_0 t_{r,n+1}} \,,$$

which is exactly (14). Now, the case of any permutation $w$ is easily proved by decreasing induction on the length of $w$ using the Leibnitz formula to compute $\partial_i(x_r \mathbb{X}_w)$, $i$ being such that $l(ws_i) = l(w) + 1$. The initial case contains the extra term $y_{w_0(r)} \mathbb{X}_{w_0}$ and this product gives the term $y_{w(r)} \mathbb{X}_w$ in formula (14). ∎

The Monk formula (8) corresponds to the specialization $y_i = 0, \forall i$ of the formula (14).

Schur functions appear in algebra and geometry as cohomology classes related to determinantal varieties. A more general situation arises when taking a vector bundle $V$ and two flags of vector bundles: $A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = V$ and $V = B_n \twoheadrightarrow B_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow B_1$. Classes of determinantal varieties are now polynomials in the Chern classes $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ of the two flags, and one has the conditions that $f(\mathbf{x}) = f(\mathbf{y})$ for all symmetric functions $f$, because in that case $f$ is a function of $V$ only [6]. We shall show now that under the identification of symmetric functions in $\mathbf{x}$ and symmetric functions in $\mathbf{y}$, *i.e.* modulo the ideal $\mathcal{I}(\mathbf{x},\mathbf{y})$ generated by the $e_k(\mathbf{x}) - e_k(\mathbf{y})$ ($e_k(\mathbf{x})$ being the $k$-th elementary symmetric function in the variables $x_1, \ldots, x_n$), the eventual extra term in double Monk's formula (14) vanishes. In other words, the product $x_r \mathbb{X}_w$, $w \in \mathfrak{S}_n$ possibly involves $\mathbb{X}_{w'}$, $w' \in \mathfrak{S}_{n+1} \backslash \mathfrak{S}_n$ in proposition 4.1, but this term vanishes if one works in the ring $\mathbb{Z}[\mathbf{y}][\mathbf{x}]/\mathcal{I}(\mathbf{x},\mathbf{y})$. Thus, we now take the ring of coefficients to be:

$$\mathbb{K} := \frac{\mathrm{Sym}[x_1, \ldots, x_n] \otimes \mathbb{Z}[y_1, \ldots, y_n]}{\mathcal{I}(\mathbf{x},\mathbf{y})} .$$

The ring $\mathbb{Z}[\mathbf{y}][\mathbf{x}]/\mathcal{I}(\mathbf{x},\mathbf{y})$ is still a free module of dimension $n!$ over the ring $\mathbb{K}$, with basis either the $\mathrm{X}_w$, or the $\mathbb{X}_w$, $w \in \mathfrak{S}_n$. We shall denote this ring by $\mathbb{K}[\mathbf{x}]$.

**Proposition 4.2** *Let $w \in \mathfrak{S}_n$ and $1 \le r \le n$. In the ring $\mathbb{K}[\mathbf{x}]$, one has:*

$$x_r \mathbb{X}_w \equiv y_{w(r)} \mathbb{X}_w + \sum_{\nu = w\, t_{rj},\, j>r,\, l(\nu)=l(w)+1} \mathbb{X}_\nu - \sum_{\mu = w\, t_{jr},\, j<r,\, l(\mu)=l(w)+1} \mathbb{X}_\mu , \qquad (15)$$

*the summation being limited to permutations belonging to $\mathfrak{S}_n$.*

**Proof.** In formula (14), there exists at most one term implying a permutation in $\mathfrak{S}_{n+1}$. This term comes from the case $w = w_0$. Therefore, the proof of proposition amounts to check the following nullity:

$$\forall r,\ 1 \le r \le n, \qquad (x_r - y_{n+1-r}) \mathbb{X}_{w_0} \equiv 0 .$$

In fact, we are going to prove a more precise statement (*i.e.* that a certain factor of the previous expression vanishes).

**Lemma 4.3** *Let $r$ be an integer $1 \le r \le n$. Then, modulo the ideal $\mathcal{I}(\mathbf{x},\mathbf{y})$, one has:*

$$\prod_{1 \le i \le r,\, 1 \le j \le n+1-r} (x_i - y_j) \equiv 0 . \qquad (16)$$

**Proof.** Given two finite alphabets $A$ and $B$ of respectively $l$ and $m$ elements, then $R(A,B) := \prod_{a \in A,\, b \in B} (a-b)$ is some vexillary Schubert polynomial for the two alphabets $A$ and $B$. Since it is vexillary, it also has a determinantal expression [14, page 94]:

$$R(A,B) = \left| h_{m+j-i}(A-B) \right|_{1 \le i,j \le l} ,$$

where by definition the $h_k(A-B)$ are the coefficients of the rational function in $z$:

$$f(A,B) := \prod_{b \in B} (1 - zb) \Big/ \prod_{a \in A} (1 - za) .$$

This function is therefore equal to $R(\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_{n+1-r}\})$ and is a determinant in the $h_k(\{x_1, \ldots, x_r\} - \{y_1, \ldots, y_{n+1-r}\})$.

Take now an extra variable $u$. The product $g := \prod_{1 \le i \le n} (1 - zx_i)(1 - zu)$ is equal to $\prod_{1 \le j \le n} (1 - zy_j)(1 - zu)$ because the elementary symmetric functions in the $x_1, \ldots, x_n$

are equal to the elementary symmetric functions in the $y_1, \ldots, y_n$. Thus multiplying the numerator and denominator of $f(\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_{n+1-r}\})$ by $g$, we transform $f$ into the rational function $f(\{y_{n+2-r}, \ldots, y_n, u\}, \{x_{r+1}, \ldots, x_n, u\})$, without changing its value. This implies that the $h_k(\{x_1, \ldots, x_r\} - \{y_1, \ldots, y_{n+1-r}\})$ coincide with the $h_k(\{y_{n+2-r}, \ldots, y_n, u\} - \{x_{r+1}, \ldots, x_n, u\})$. This transformation shows that the determinant is equal to:

$$R(\{y_{n+2-r}, \ldots, y_n, u\}, \{x_{r+1}, \ldots, x_n, u\}) ,$$

that vanishes because of the factor $(u - u)$. ■

The proof of proposition 4.2 follows immediately. ■

**Example 4.4** Take $\mathfrak{S}_3$. The product $(x_2 - y_2)\mathbb{X}_{\omega_0}$ is equal to $R(\{x_1, x_2\}, \{y_1, y_2\})$. Modulo the ideal $\mathcal{I}(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$, one has:

$$\begin{aligned} R(\{x_1, x_2\}, \{y_1, y_2\}) &= \begin{vmatrix} h_2(x_1 + x_2 - y_1 - y_2) & h_3(x_1 + x_2 - y_1 - y_2) \\ h_1(x_1 + x_2 - y_1 - y_2) & h_2(x_1 + x_2 - y_1 - y_2) \end{vmatrix} \\ &= \begin{vmatrix} h_2(y_3 - x_3) & h_3(y_3 - x_3) \\ h_1(y_3 - x_3) & h_2(y_3 - x_3) \end{vmatrix} = \begin{vmatrix} y_3\,(y_3 - x_3) & y_3^2\,(y_3 - x_3) \\ (y_3 - x_3) & y_3\,(y_3 - x_3) \end{vmatrix} , \end{aligned}$$

which clearly vanishes.

# 5 Multiplicative structure

We study the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials seen as a free module of dimension $n!$ over the ring $\mathrm{Sym}[x_1, \ldots, x_n]$ of symmetric polynomials, with basis the simple Schubert polynomials $\mathrm{X}_w, w \in \mathfrak{S}_n$. For this purpose, we have to recover the multiplicative structure on this module. As explained previously, Monk's formula gives the multiplicative structure of the ring in an infinite number of variables, and does not preserve the subspace with basis the $\mathrm{X}_w, w \in \mathfrak{S}_n$. This formula does not use the fact that coefficients can belong to $\mathrm{Sym}[x_1, \ldots, x_n]$. As there is at most one term (in Monk's formula) indexed by a permutation $\mu$ outside of $\mathfrak{S}_n$, we are thus led to express certain simple Schubert polynomials as a sum with coefficients in $\mathrm{Sym}[x_1, \ldots, x_n]$ of simple Schubert polynomials indexed by permutations of $\mathfrak{S}_n$. More generally, every simple Schubert polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ has an expression in the Schubert basis $\mathrm{X}_w, w \in \mathfrak{S}_n$.

Instead of giving identities on Schubert polynomials, we shall give identities in the *algebra of divided differences* [13], also called *nilCoxeter algebra* [4], which imply the identities we need. In fact, one could have chosen to give stronger identities on sums of words modulo *nilplactic* relations $(i < j < k)$:

$$\partial_k \partial_i \partial_j \equiv \partial_i \partial_k \partial_j ,$$

$$\partial_j \partial_k \partial_i \equiv \partial_j \partial_i \partial_k ,$$

$$\partial_l \partial_{l+1} \partial_l \equiv \partial_{l+1} \partial_l \partial_{l+1} ,$$

but we prefer to work in the simpler nilCoxeter algebra, which is a quotient of the nilplactic algebra (two words that are nilplactic equivalent, are equivalent modulo Coxeter relations). Recall that symmetric polynomials in $x_1, \ldots, x_n$ are scalars for $\partial_1, \ldots, \partial_{n-1}$. Let $k, m$ be integers and $A$ be the ordered alphabet $A := \{\partial_1, \ldots, \partial_m\}$. We consider the noncommutative symmetric functions of $A$ in the sense of [7]. Thanks to a result of Fomin and Greene [3], these functions commute in fact with each other. We denote by $\Lambda^k(A)$ the $k$-th elementary symmetric function:

$$\Lambda^k(A) := \sum_{i_1 > \cdots > i_k} \partial_{i_1} \ldots \partial_{i_k} ,$$

and by $S^k(A)$ the $k$-th complete symmetric function:

$$S^k(A) := \sum_{i_1 \leq \cdots \leq i_k} \partial_{i_1} \ldots \partial_{i_k} = \sum_{i_1 < \cdots < i_k} \partial_{i_1} \ldots \partial_{i_k} ,$$

since $\partial_i^2 = 0$. We also consider $\lambda_t(A) = \sum \Lambda^i(A)\, t^i$ and $\sigma_t(A) = \sum S^i(A)\, t^i$.

**Proposition 5.1** *Let $m, n$ and $k$ be integers, $A := \{\partial_1, \ldots, \partial_m\}$ and $B := \{\partial_{m+1}, \ldots, \partial_{m+n}\}$. We write $A + B := \{\partial_1, \ldots, \partial_m, \partial_{m+1}, \ldots, \partial_{m+n}\}$. Then, in the nilCoxeter algebra, one has:*

$$\Lambda^k(B) = \Lambda^k(A+B)S^0(A) - \Lambda^{k-1}(A+B)S^1(A) + \Lambda^{k-2}(A+B)S^2(A) - \cdots . \tag{17}$$

**Proof.** One has $\sigma_t(A+B) \stackrel{\text{def}}{=} \sigma_t(A)\, \sigma_t(B)$, so that $\lambda_t(A+B) = \lambda_t(B)\, \lambda_t(A)$. Thus, $\lambda_t(B) = \lambda_t(A+B)\, \lambda_t^{-1}(A) = \lambda_t(A+B)\, \sigma_{-t}(A)$. Extracting the coefficient of $t^k$ in both sides, one obtains (17). ■

Given a word $w := \partial_i \partial_j \ldots \partial_k$, we denote by $w^+$ the word $\partial_{i+1} \partial_{j+1} \ldots \partial_{k+1}$ and we extend this morphism to the nilCoxeter algebra by linearity.

**Proposition 5.2** *Let $m, k$ be integers, $A := \{\partial_1, \ldots, \partial_m\}$, $B := \{\partial_{m+1}, \ldots, \partial_{m+k}\}$. Then, in the nilCoxeter algebra:*

1. *There exist elements $T_{i,m}$ such that:*

$$\partial_{m+k} \ldots \partial_{m+1} = \sum_{i=k}^{0} (-1)^{k-i}\, \Lambda^i(A+B)\, T_{i,m} , \tag{18}$$

*and satisfying the recursion:*

$$T_{i,m+1} = T_{i,m} - T_{i+1,m}\, \partial_{m+1} . \tag{19}$$

2. *There exist elements $U_{i,m}$ satisfying the following relations:*

$$\partial_1 \ldots \partial_k = \sum_{i=k}^{0} (-1)^{k-i}\, S^i(A+B)\, U_{i,m} , \tag{20}$$

$$U_{i,m+1} = U_{i,m} - \partial_{m+1+k}\, U_{i+1,m} . \tag{21}$$

3. *There exist elements $V_{i,m}$ satisfying the following relations:*

$$\partial_{m+1} \ldots \partial_{m+k} = \sum_{i=k}^{0} (-1)^{k-i}\, S^i(A+B)\, V_{i,m} , \tag{22}$$

$$V_{i,m+1} = V_{i,m}^+ - V_{i+1,m}^+\, \partial_1 . \tag{23}$$

*4. Finally, there exist elements $W_{i,m}$, such that:*

$$\partial_1 \dots \partial_k = \sum_{i=k}^{0} (-1)^{k-i} W_{i,m} S^i(A+B) , \tag{24}$$

*and satisfying the recursion (21).*

**Proof.** (18) is the special case of proposition 5.1 for $k = n$. (20) is equivalent to it modulo reversing the order on the divided differences. (19) and (21) are equivalent for the same reason. In other words, we have two different recursions and we prove only the formula for $\partial_1 \dots \partial_k$ because in this case, the proof amounts to control that the extra terms, when passing from $m$ to $m+1$, cancel each other and thus, that the equality is preserved. But now, this is implied by the recursion on the set of words as we shall check for simplicity for (24), in the case $k = 2$. Recursions on the $W_i$ are simply recursions on words so we forget to write the $\partial$: for now, 12 stands for $\partial_1\partial_2$. Suppose by induction that we have the formula for $k + m = 4$ and let us prove it for $k + m = 5$. If one checks or assumes that:

$$12 = S^2(1,2,3,4) - (3+4)S^1(1,2,3,4) + 43 + 32 - 23 ,$$

then the statement says, by applying the rule $W_{i,m+1} = W_{i,m} - (m+1+k)\, W_{i+1,m}$, that we still have the following equality:

$$12 = S^2(1,2,3,4,5) - (3+4+5)S^1(1,2,3,4,5) + 5(3+4) + 43 + 32 - 23 .$$

Now, the extra expression involving **5** is $S^1(1,2,3,4)5 - 5S^1(1,2,3,4,5) - (3+4)5 + 53 + 54$, that is $15 + 25 + 35 + 45 - 51 - 52 - 53 - 54 - 55 - 35 - 45 + 53 + 54 = 0$. ■

Now, each identity on words of proposition 5.2 can be translated in terms of Schubert polynomials. The problem that we consider is to express Schubert polynomials $\mathbf{X}_{w'}, w' \in \mathfrak{S}_{n+1} \setminus \mathfrak{S}_n$ in terms of $\mathbf{X}_w, w \in \mathfrak{S}_n$. We need only consider the $w'$ of the type $(n+1, n, \dots, \widehat{m}, \dots, 2, 1, m)$ because using the divided differences $\partial_1, \dots, \partial_{n-1}$, we get all others. For these special permutations $w'$, one has the identity:

$$\partial_{m+k} \dots \partial_{m+1} \left(\mathbf{X}_{m+k+1,\dots,2,1}\right) = \mathbf{X}_{m+k+1,\dots,\widehat{k+1},\dots,2,1,k+1} ,$$

which involves the left-hand side of (23).

Finally, we also need the following lemma which connects multiplication of simple Schubert polynomials by the $e_i(x_1, \dots, x_n)$ and multiplication by the $\Lambda^i(\partial_1, \dots, \partial_n)$ in the nilCoxeter algebra.

**Lemma 5.3** *Let $\mu \in \mathfrak{S}_n$ and $w_n$ (resp. $w_{n+1}$) denote the maximal permutation of $\mathfrak{S}_n$ (resp. $\mathfrak{S}_{n+1}$). Then,*

$$e_i(x_1, \dots, x_n)\, \mathbf{X}_\mu = \partial_{\mu^{-1}w_n} \Lambda^{n-i}(\partial_1, \dots, \partial_n)\, \mathbf{X}_{w_{n+1}} . \tag{25}$$

**Proof.** Since $\mathbf{X}_\mu = \partial_{\mu^{-1}w_n} \mathbf{X}_{w_n}$ and since $\partial_{\mu^{-1}w_n}$ commutes with the multiplication by $e_i(x_1, \dots, x_n)$, it is sufficient to prove the formula for $\mu = w_n$ and we will do it for all the $e_i(x_1, \dots, x_n)$ at the same time. $\sum e_i(x_1, \dots, x_n)\, \mathbf{X}_{w_n}$ is equal to the sum of all monomials whose exponents are between $(n-1, n-2, \dots, 1, 0, 0)$ and $(n, n-1, \dots, 1, 0)$. But this is the same for $\sum \Lambda^i(\partial_1, \dots, \partial_n)\, \mathbf{X}_{w_{n+1}}$ because the order in which divided differences are performed is such that, at each stage, one has to perform $\partial_j (\dots x_j^{k+1} x_{j+1}^k \dots) = \dots x_j^k x_{j+1}^k \dots$ that is a monomial. ■

We are now able to give an algorithm directly on permutations. For that purpose, we consider operators from $\mathfrak{S}_{n-1}$ into $\mathfrak{S}_n$, $P_0^n, \ldots, P_k^n$ defined by:

$$\begin{array}{rcl} P_0^n(\mu) & \longrightarrow & n\ \mu \\ P_1^n(\mu) & \longrightarrow & \mu_1\ n\ \mu_2 \ldots \\ \vdots & \longrightarrow & \vdots \\ P_k^n(\mu) & \longrightarrow & \mu_1 \ldots \mu_{k-1}\ n\ \mu_k \ldots \end{array}$$

**Proposition 5.4** *Let $n, m$ be two integers such that $m \leq n$. Let $\mathcal{R}(n,m)$ be the polynomial recursively defined by:*

$$\mathcal{R}(n,m) := \begin{cases} e_1 \mathbf{X}_{n,n-1,\ldots,2,1} - \sum_{k=1}^{n-1} P_{n-k}^n\Big(\mathcal{R}(n-1,k)\Big)_{e_j \leftarrow e_{j+1}}, & \text{if } m = n, \\ P_0^n\Big(\mathcal{R}(n-1,m)\Big)_{e_j \leftarrow e_{j+1}} - P_1^n\Big(\mathcal{R}(n-1,m)\Big)_{e_j \leftarrow e_{j+2}}, & \text{otherwise.} \end{cases}$$

*Then, $\mathcal{R}(n,m)$ is exactly equal to the Schubert polynomial which is indexed by the permutation* $(n+1, n, \ldots, \widehat{m}, \ldots, 2, 1, m)$.

Using the divided differences $\partial_1, \ldots, \partial_{n-1}$, which commute with the $e_i$, one can deduce from proposition 5.4 the expression of any Schubert polynomial in $\mathfrak{S}_{n+1}$.

## 6 Applications

Proposition 5.4 has modified Monk's formula (8) into a formula involving only permutations in $\mathfrak{S}_n$ and thus, allows to express the product of any Schubert polynomial $\mathbf{X}_w, w \in \mathfrak{S}_n$ by any polynomial in $x_1, \ldots, x_n$. More generally, it allows to multiply any linear combination of Schubert polynomials with coefficients in $\mathrm{Sym}[x_1, \ldots, x_n]$ by a polynomial with coefficients also in $\mathrm{Sym}[x_1, \ldots, x_n]$. To reduce the symmetric coefficients, one uses distinguished bases of symmetric functions and corresponding algorithms. These algorithms are furnished by combinatorial properties of partitions or Young tableaux and are independent of the number of variables.

As an application of the methods described in the preceeding section, we shall mention the cohomology ring of a relative *Grassmann* variety. More explicitely, let $A$ and $B$ be two disjoint alphabets, $A := \{x_1, \ldots, x_m\}$ and $B :=:= \{x_{m+1}, \ldots, x_n\}$ of respective cardinal $m$ and $n - m$. We shall denote by $A + B$ the alphabet $\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}$.

The cohomology space that we have just referred to is the space of polynomials in $x_1, \ldots, x_n$ which are symmetrical separately in $x_1, \ldots, x_m$ and $x_{m+1}, \ldots, x_n$. This space is a free module over the ring of symmetric polynomials $\mathrm{Sym}[x_1, \ldots, x_n]$. It has a canonical basis (the *Schubert cycles*) of cardinal $\binom{n}{m}$, consisting of *Schur functions* [15] indexed by partitions smaller than $(n-m)^m$. For example, in the case $m = 1$, the basis consists in $x_1^0, x_1^1, \ldots, x_1^{n-1}$ and the decomposition of a polynomial into this basis is given by *Euclid*'s algorithm. For a more general $m$, the product of two Schur functions belonging to the basis, given by the *Littlewood-Richardson* rule [15], will involve partitions not smaller than $(n-m)^m$.

However, one can express this product of Schur functions, thanks to the preceeding section, in terms of Schubert polynomials $\mathbf{X}_w, w \in \mathfrak{S}_n$. These Schubert polynomials must be symmetrical in $x_1, \ldots, x_m$ and therefore are Schur functions [14]. Because the indexing permutations are in $\mathfrak{S}_n$, then the corresponding Schur functions are indexed by partitions smaller or equal to $(n-m)^m$, *i.e.* belong to the canonical basis. In other words, section 5 gives a method to decompose any element of the cohomology ring in the basis of Schubert cycles. For instance, for $m = 3$, $n = 5$, the square of the Schur function $s_{21}(x_1, x_2, x_3)$ is equal to $e_4 s_2 - e_3 s_{21} + e_1^2 s_{22} + (e_4 - e_3 e_1) s_{11} + (e_4 e_1 + e_5) s_1 + (e_1^2 - e_2) s_{211} - (e_2 e_1 + e_3) s_{111} + s_{222} + 2 e_1 s_{221} + e_5 e_1$ while the Littlewood-Richardson rule gives $s_{42} + s_{411} + s_{33} + 2 s_{321} + s_{222}$ that involves partitions outside of (222).

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