# ON GENERALIZED LEXICOGRAPHIC SHELLABILITY AND ORBIT ARRANGEMENTS

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ABSTRACT. We introduce a new poset property which we call EC-shellability. It is more general than lexicographic shellability, but still implies shellability. We prove that intersection lattices $\Pi_\lambda$ of orbit arrangements $\mathcal{A}_\lambda$ are EC-shellable for a very large class of partitions $\lambda$. This allows to compute the topology of the link and the complement for these arrangements. In particular, for this class of $\lambda$'s, we are able to settle the conjecture of A.Björner, stating that the cohomology groups of the complement of the orbit arrangements are torsion-free.

We also present a class of partitions, for which $\Pi_\lambda$ is not shellable.

## 1. INTRODUCTION

Let $\mathcal{A}$ be a central subspace arrangement and let $M_\mathcal{A}$ be its complement. The problem of computing the cohomology groups of $M_\mathcal{A}$ is one of the central questions in the theory of subspace arrangements. One usual way to do that is to prove that the intersection lattice $\mathcal{L}_\mathcal{A}$ of the arrangement $\mathcal{A}$ is shellable. Several combinatorial tools have been developed to handle this task. The most known are probably: lexicographic shellability (EL- and CL-shellability) and recursive atom ordering (see [B80, BWa82, BWa83, BWa94]). These methods are technically involved, but unfortunately, still unsufficient for many important cases.

We introduce a new poset property, which we call edge compatible shellability (or shortly EC-shellability). It is more general than the lexicographic shellability, but still implies shellability. The general idea is then to prove that $\mathcal{L}_\mathcal{A}$ is EC-shellable, which in particular means that $\mathcal{L}_\mathcal{A}$ is shellable, and so via formulae of M.Goresky, R.MacPherson, [GM], and G.M.Ziegler, R.Živaljević, [ZŽ], one can obtain the topology of the complement $M_\mathcal{A}$ and the link $V_\mathcal{A}^\circ$.

We believe that this new method has especially good potential in the case, when the poset is an intersection lattice of some subspace arrangement with a "nice" combinatorial description. For example, we are able to show that the intersection lattices $\Pi_\lambda$ of a large class of orbit arrangements $\mathcal{A}_\lambda$ are EC-shellable.

We shortly sketch the contents of the article.

**Section 2.** Most of the general notions used in the paper are introduced. If some unclarity still remains, the reader is refered to the standard textbooks: R.P.Stanley "Enumerative Combinatorics, vol. I", [St], for the combinatorial part, and J.R.Munkres "Elements of Algebraic topology", [Mu], for the topological part.

**Section 3.** Here all of the new theoretical tools are concentrated. The notion of EC-shellability is introduced. We also give a parallel definition of CC-shellability and prove that they both imply shellability. Finally, we consider an even more general notion of S-orderings.

**Section 4.** It is proved that $\Pi_\lambda$ is EC-shellable, whenever a certain condition on $\lambda$ (which we call condition (NES)) is fulfilled. We also come up with a class of partitions $\lambda$, for which $\Pi_\lambda$ is not shellable.

**Section 5.** The recursive atom ordering technique is applied to prove the shellability of the intersection lattice of orbit arrangements in a few more cases.

**Section 6.** The results of the previous two sections are applied for the special case $\lambda = (k^m, 1^t)$.

**Section 7.** Topological consequences of Theorems 4.1 and 5.1 are derived. We apply formulae of Goresky, MacPherson and Ziegler, Živaljević to the case of orbit arrangements.

**Section 8.** A formula for the Möbius function of the lattice $\Pi_\lambda$ is given. It is needed for the computation of the reduced Betti numbers of the complement $M_{\mathcal{A}_\lambda}$.

## 2. Basic notions and definitions

In this section we give a short summary of the standard notions used throughout the text.

A poset $P$ is called **bounded** if there exist a top element $\hat{1} \in P$ and a bottom element $\hat{0} \in P$ such that $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$. All the posets we will consider in this text will be finite and bounded. We say that $x$ **covers** $y$ if $x > y$ and there is no $z$ such that $x > z > y$, we denote that by $x \to y$. We call $x \in P$ an **atom** if $x$ covers $\hat{0}$. We say that $C \subseteq P$ is a **chain** if any two elements of $C$ are comparable. For a finite poset $P$ we will denote its chain complex by $\Delta(P)$. We say that $P$ is **pure** if $\Delta(P)$ is pure. Such posets are also often called graded.

**Definition 2.1.** *A simplicial complex $\Delta$ is called* **shellable** *if its facets can be arranged in linear order $F_1, F_2, \dots, F_t$, in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \dots, t$. Such an ordering of facets is called a* **shelling order**.

**Definition 2.2.** *A poset $P$ is said to be* **EL-shellable** *if one can label edges with elements from a poset $\Lambda$ so that for every interval $[x, y]$ in $P$,*

(i) *there is a unique rising maximal chain $c$ in $[x, y]$ (rising means that the associated labels form a strictly increasing sequence);*
(ii) *$c \prec c'$ for all other maximal chains $c'$ in $[x, y]$.*

*Here the symbol "$\prec$" means "lexicographically preceeding". We will often say "lexicographically less" or just "less".*

The notion of EL-shellability was first introduced in Chapter 2, [B80]. See also [BWa83] for further investigations and [BWa94] for the non-pure version.

Let $\mathcal{E}(P)$ denote the set of covering relations in $P$, i.e.

$$\mathcal{E}(P) = \{(x, y) | x \leftarrow y\},$$

then labeling the edges in the Hasse diagram of $P$ with elements from a poset $\Lambda$ is nothing else but a map $\mu : \mathcal{E}(P) \to \Lambda$. One can also consider a more general notion. Namely let

$$\mathcal{E}^*(P) = \{(x, y, m) | x \leftarrow y,\ x, y \in m,\ m \text{ is a maximal chain}\}.$$

Then a map $\mu : \mathcal{E}^*(P) \to \Lambda$ means that we are labeling the edges in our Hasse diagram "with respect" to the maximal chains, which they belong to. Such a map is called a **chain-edge labeling** if the following condition is satisfied:

**Condition (L).** *If two maximal chains $c = (\hat{0} = x_0 \to x_1 \to \cdots \to x_n = \hat{1})$ and $c' = (\hat{0} = x'_0 \to x'_1 \to \cdots \to x'_n = \hat{1})$ coincide along their first $d$ edges then their labels also coincide along these edges, that is, if $x_i = x'_i$ for $i = 0, \ldots, d$ then $\mu(x_{i-1}, x_i, c) = \mu(x'_{i-1}, x'_i, c')$ for $i = 1, \ldots, d$.*

If $[x, y]$ is an interval and $r$ is an unrefinable chain from $\hat{0}$ to $x$, then the pair $([x, y], r)$ will be called a **rooted interval** with root $r$, and will be denoted $[x, y]_r$. Let $c$ be any maximal chain of $[x, y]$, then $r \cup c$ is a maximal chain of $[\hat{0}, y]$. Take some maximal chain $m$ of $P$ which contains $r \cup c$ and let $\sigma_r(c)$ be the string of labels associated to the edges of the chain $c$ with respect to the chain $m$. The condition (L) guarantees that $\sigma_r(c)$ is independent on the choice of $m$.

**Definition 2.3.** *A poset $P$ is called* **CL-shellable** *if there exists a chain-edge labeling $\mu$ such that for every rooted interval $[x, y]_r$ in $P$,*

(i) *there is a unique rising maximal chain $c$ in $[x, y]_r$;*
(ii) *$c \prec c'$ for all other maximal chains $c'$ in $[x, y]_r$.*

The notion of CL-shellability was first introduced in Chapter 2 of [BWa82], it was effectively used there to prove the shellability of Bruhat orders of Coxeter groups.

**Definition 2.4.** *A pure poset $P$ is said to admit a* **recursive atom ordering** *if either $P$ consists of $\hat{0}$ and $\hat{1}$ or there is an ordering of its atoms $a_1, a_2, \ldots, a_t$, which satisfies:*

(R1) *for all $j = 1, 2, \ldots, t$, $[a_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}]$ that come first in the ordering are those that cover some $a_i$, where $i < j$;*
(R2) *for all $i < j$, if $a_i, a_j < y$ and $y$ does not cover $a_j$, then there is a $k < j$ and an element $z$ such that $a_k, a_j < z < y$.*

Recursive atom orderings were first considered in [BWa83]. It is proved there (Theorem 3.2) that a graded poset admits a recursive atom ordering if and only if it is CL-shellable.

We say that a poset $P$ is **semimodular** if for any $x, y \in P$, such that $x$ and $y$ both cover the same element $z$, there exists an element $t$, which covers both $x$ and $y$. A poset $P$ is called **totally semimodular** if all intervals are semimodular. Totally semimodular posets are interesting because according to Theorem 5.1 in [BWa83] *a graded poset $P$ is totally semimodular if and only if for every interval $[x, y]$ of $P$, every atom ordering in $[x, y]$ is a recursive atom ordering.*

We use the notation $\lambda = (\lambda_1, \ldots, \lambda_p)$ for the partition of the number $n = \sum_{i=1}^{p} \lambda_i$ into blocks of sizes $\lambda_1, \ldots, \lambda_p$ and we always have these blocks ordered after their sizes, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. By $\Pi_n$ we denote the **partition lattice** of the set $[n]$. It is a poset with elements all different partitions $[n]$ ordered under refinement.

A finite collection $\mathcal{A} = \{K_1, \dots, K_t\}$ of linear proper subspaces in $\mathbb{R}^n$ is called a **subspace arrangement**. A subspace arrangement is called **central** if $0 \in K_i$, $i = 1, \dots, t$. We will only consider central subspace arrangements. The **intersection semilattice** $\mathcal{L}_\mathcal{A}$ of an arrangement $\mathcal{A} = \{K_1, \dots, K_t\}$ is the collection of all non-empty intersections $K_{i_1} \cap \dots \cap K_{i_p}$, $1 \leq i_1 < \dots < i_p \leq t$, ordered by reverse inclusion: $x \leq y \Leftrightarrow y \subseteq x$. When $\mathcal{A}$ is central, $\mathcal{L}_\mathcal{A}$ is always a lattice.

In this paper we will deal with a special class of subspace arrangements which were first defined in the subsection 3.3 in [B94]. Here follows the definition. If $\pi = (B_1, \dots, B_p)$ is a nontrivial partition of the set $[n]$, then let $K_\pi = K_{B_1} \cap \dots \cap K_{B_p} = \{x \in \mathbb{R}^n | i, j \in B_k \Rightarrow x_i = x_j, \text{ for all } 1 \leq i, j, \leq n, 1 \leq k \leq p\}$. The type of $\pi$ is the sequence of block sizes $|B_i|$ arranged in non-increasing order. Given a non-trivial number partition $\lambda \vdash n$, let

$$\mathcal{A}_\lambda = \{K_\pi | \pi \in \Pi_n \text{ and type } (\pi) = \lambda\}$$

$\mathcal{A}_\lambda$ is called an **orbit arrangement**, expressing the fact that $\mathcal{A}_\lambda$ is the orbit of any single subspace $K_\pi$ under the natural action of $S_n$ on $\mathbb{R}^n$.

Let $\Pi_\lambda = \mathcal{L}_{\mathcal{A}_\lambda}$. Note that $\Pi_n = \Pi_{(2,1,\dots,1)}$. The main goal of this paper is to study the topological properties of $A_\lambda$ through the combinatorial analysis of its intersection lattice $\Pi_\lambda$.

## 3. Tools

We generalize the known definition of lexicographic shellability.

**Definition 3.1.** *We say that a pure poset $P$ has a* **edge compatible labeling**, *or for short just* **EC-labeling**, *if we can label edges with elements of some poset $\Lambda$ so that in any interval all the maximal chains have different labels and the following condition is satisfied:*

**Condition (EC).** *For any interval $[x, t]$, any maximal chain $c$ in $[x, t]$ and $y, z \in c$, such that $x < y < z < t$, if $c|_{[x,z]}$ is lexicographically least in $[x, z]$ and $c|_{[y,t]}$ is lexicographically least in $[y, t]$ then $c$ is lexicographically least in $[x, t]$.*

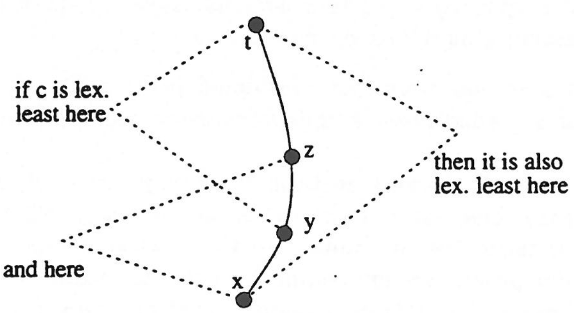


Figure 1.

**Proposition 3.2.** *For any pure poset $P$ and for any edge labeling the following conditions are equivalent.*

(1) *Condition (EC);*

(2) *Condition* ($EC'$). *For any interval* $[x,t]$, $c$ *a maximal chain in* $[x,t]$, *and* $y,z \in c$, *such that* $x \leftarrow y \leftarrow z$, *if* $c|_{[x,z]}$ *is lexicographically least in* $[x,z]$ *and* $c|_{[y,t]}$ *is lexicographically least in* $[y,t]$ *then* $c$ *is lexicographically least in* $[x,t]$;

(3) *Condition (BS) ("bad subchain"). For any interval* $[x,t]$, *any maximal chain* $c$ *in* $[x,t]$, *such that* $c$ *is not lexicographically least in* $[x,t]$ *and* $rk[x,t] > 2$, *there exist elements* $y,z \in c$, *such that* $c|_{[y,z]}$ *is a proper subchain of* $c$ *and* $c|_{[y,z]}$ *is not lexicographically least in* $[y,z]$.

(4) *Condition* ($BS'$). *For any interval* $[x,t]$, *any maximal chain* $c$ *in* $[x,t]$, *such that* $c$ *is not lexicographically least in* $[x,t]$, *there exist elements* $y,q,z \in c$, *such that* $y \leftarrow q \leftarrow z$ *and* $c|_{[y,z]}$ *is not lexicographically least in* $[y,z]$.

**Proof.** It is obvious that (1) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (4).

(2) $\Rightarrow$ (3). Condition ($EC'$) can be reformulated in the following way: if $c$ is not lexicographically least in $[x,t]$, then either $c|_{[x,z]}$ is not lexicographically least in $[x,z]$ or $c|[y,t]$ is not lexicographically least in $[y,t]$, that proves Condition (BS).

(4) $\Rightarrow$ (1) Consider an interval $[x,t]$, $c$ a maximal chain in $[x,t]$, $y,z \in c$, $x < y < z < t$, such that $c$ is not lexicographically least in $[x,t]$, but $c|_{[x,z]}$ is lexicographically least in $[x,z]$ and $c|_{[y,t]}$ is lexicographically least in $[y,t]$. Then there exist $p,q,r \in c$, such that $p \leftarrow q \leftarrow r$ and $c|_{[p,r]}$ is not lexicographically least in $[p,r]$. Obviously either $y \leq p$ or $r \leq z$. Assume $y \leq p$ (the other case goes along the same lines), then $p,q,r \in c|_{[y,t]}$. Since $c|[p,r]$ is not lexicographically least in $[p,r]$, we conclude that $c|_{[y,t]}$ is not lexicographically least in $[y,t]$, that gives a contradiction. $\square$

**Definition 3.3.** *A pure poset* $P$ *is said to have a* **compatible chain-edge labeling,** *or just* **CC-labeling,** *if there exists a map* $\mu : \mathcal{E}^*(P) \to \Lambda$, *which is a chain-edge labeling and such that in any interval all the maximal chains have different labels and the following condition is satisfied:*

**Condition (CC).** *For any rooted interval* $[x,t]_r$, *any maximal chain* $c$ *in* $[x,t]_r$ *and* $y,z \in c$, *such that* $x < y < z < t$, *if* $c|_{[x,z]}$ *is lexicographically least in* $[x,z]_r$ *and* $c|_{[y,t]}$ *is lexicographically least in* $[y,t]_{r'}$ *then* $c$ *is lexicographically least in* $[x,t]_r$, *where* $r' = r \cup c|_{[x,y]}$.

**Remark 3.4.** *Proposition* 3.2 *generalizes in a straightforward manner to the case of CC-labelings, one only has to consider rooted intervals instead of the usual ones. We leave the details to the reader.*

**Theorem 3.5.** *For any pure poset* $P$ *the following implications are satisfied:*

$$\begin{array}{ccccc} \textit{EC-labelable} & \Longrightarrow & \textit{CC-labelable} & \Longrightarrow & \textit{shellable} \\ \Uparrow & & \Uparrow & & \\ \textit{EL-shellable} & \Longrightarrow & \textit{CL-shellable} & & \end{array}$$

**Proof.** CC-labelable $\Rightarrow$ shellable. We will prove that the lexicographic ordering of the maximal chains in $P$ gives a shelling order on the facets of $\Delta(P)$.

Consider two maximal chains, $c$ and $d$, such that $c \prec d$. Say $c = (x_1, \dots, x_k)$, $d = (y_1, \dots, y_k)$ and let $p = \min\{i \mid x_{i+1} \neq y_{i+1}\}$, $q = \min\{j \mid j > i, x_j = y_j\}$. Denote $I = [x_p, x_q]$. $c|_I$ and $d|_I$ are two different maximal chains in $I_r$, where $r = (x_1, \dots, x_p) = (y_1, \dots, y_p)$. Hence one of them should be lexicographically preceeding to the other one. Since $c \prec d$ we get $c|_I \prec d|_I$ in $I_r$. So $d|_I$ is not lexicographically least in $I_r$, hence, according to the condition ($BS'$), there exist elements

$y_s \leftarrow y_{s+1} \leftarrow y_{s+2}$, such that $p \leq s \leq q-2$ and $d|_{[y_s, y_{s+2}]}$ is not lexicographically least in $[y_s, y_{s+2}]_{(y_1, \dots, y_s)}$. Let $y_s \leftarrow y' \leftarrow y_{s+2}$ be the lexicographically least chain in $[y_s, y_{s+2}]_{(y_1, \dots, y_s)}$. Take $c' = (y_1, \dots, y_s, y', y_{s+2}, \dots, y_k)$, then $c' \cap d \supseteq c \cap d$ and $c' \prec d$. This proves that the lexicographic ordering of the maximal chains in $P$ gives a shelling order.

The proof of the other implications in the theorem is a straightforward verification and is left to the reader. □

Because of the Theorem 3.5 and following the tradition we call a poset $P$ **EC-shellable** (resp. **CC-shellable**) if it has a EC-labeling (resp. CC-labeling).

It is known that shellability does not imply EL- or CL-shellability, see [VW, Wal] for counterexamples. If the other implications in Theorem 3.5 are strict is still open.

**Theorem 3.6.** *For any poset $P$ the following two statements are equivalent:*

(a) *$P$ is shellable and it is possible to label edges with elements of some poset so that the induced lexicographic ordering of the maximal chains of $P$ gives a shelling order;*
(b) *$P$ is EC-shellable.*

**Proof.** (b) $\Rightarrow$ (a) It is proved in the Theorem 3.5

(a) $\Rightarrow$ (b) The argument in the proof of Theorem 3.5 shows that the condition that the lexicographic ordering of the chains is the shelling order is equivalent to condition $(BS')$, which in its turn by Proposition 3.2 is equivalent to EC-shellability. □

All the discussion above can be easily generalized to the case of non-pure posets. Although, we have to demand one more condition (which is usually satisfied anyway): $\mu(x \leftarrow y) \neq \mu(x \leftarrow z)$ *for any $x, y, z \in P$ such that $y \neq z$.*

It is known that if the poset is CL-shellable, then the poset $\Lambda$, used for labeling, can be exchanged to $\mathbb{Z}$. The same is true for EC- and CC-shellable posets, both in pure and non-pure case. The reason for that is that one only needs to compare the edges between the same rank levels (in the pure case) or with the same smaller element (in the non-pure case). This question, however, is still open for the EL-shellable posets.

## 4. Main Theorems

Let $B$ to denote the multiset of positive integers, which are the sizes of the blocks in $\lambda$, i.e. $B = \{\lambda_1, \dots, \lambda_p\}$. Here and throughout this chapter we are using symbols "{" and "}" to denote multisets, hopefully no confusion with the usual sets should occur.

Let $\lambda = (\lambda_1, \dots, \lambda_p)$, let us define the numbers $s \geq 0$ and $k \geq 2$ by $\lambda_p = \lambda_{p-1} = \dots = \lambda_{p-s+1} = 1$ and $\lambda_{p-s} = k > 1$.

**Theorem 4.1.** *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ and $B$ be the multiset of the sizes of the blocks from $\lambda$ as described above. $\Pi_\lambda$ is EC-shellable if the following condition on $\lambda$ is satisfied:*

**Condition (NES)** *(no equal-sum subsets). It is impossible to partition $B$ into three disjoint multisets*

$$B = B_1 \cup B_2 \cup B_3,$$

*such that the following three conditions are satisfied:*

(a) $B_1$ *is not empty;*
(b) $B_1$ *and* $B_2$ *do not contain any number in common;*
(c) $\sum_{i \in B_1} i = \sum_{j \in B_2} j$.

**Remark.** Note that if the condition NES is satisfied for a partition $\lambda$, then, in particular, $\Pi_\lambda$ is pure.

**Proof.** □

Assume that $\lambda$ is a partition such that $t \geq k \geq 3$. Take $\Pi_\lambda$ and delete all the covering relations $x \leftarrow y$, where $x \neq \hat{0}$, corresponding to the mergings of $k$ singleton blocks into one $k$-block. After that delete all the elements which were not atoms, but which became atoms after the deletion of relations as above. In other words, we delete the elements which cannot be obtained without forming a $k$-block out of singletons. We call the obtained poset $Pure(\Pi_\lambda)$.

A different way to view this poset is just to say that it is a "union of the longest maximal chains" of $\Pi_\lambda$.

**Theorem 4.2.** *Let* $\lambda$ *be a partition and* $B$ *be the multiset of the sizes of the blocks in* $\lambda$. *Let* $P$ *be equal to the poset* $Pure(\Pi_\lambda)$ *if* $t \geq k \geq 3$ *and to just* $\Pi_\lambda$ *otherwise. Then* $P$ *is not Cohen-Macaulay (in particular not shellable) if it is possible to write* $B$ *as a union of three disjoint multisets:*

$$B = B_1 \cup B_2 \cup B_3,$$

*so that the following conditions are satisfied:*

(1) $\sum_{x \in B_1} x = \sum_{x \in B_2} x \neq 0$;
(2) $B_1$ *and* $B_2$ *are two minimal multisets saisfying (1), in other words,* $B_1$ *and* $B_2$ *have no number in common;*
(3) *if* $|B_1| = m_1$, $|B_2| = m_2$ *and* $m_1 \geq m_2$ *then either* $m_1 \neq 2$ *or* $m_2 \neq 1$;
(4) *none of the numbers from* $B_3$ *can be written as a sum of two numbers from* $B_1 \cup B_2$;
(5) *none of the multisets* $B_1$ *and* $B_2$ *contains only 1's.*

**Note.** If $P$ is not shellable then $\Pi_\lambda$ is not shellable either. If $t \leq k$ or $k = 2$ it is obvious, otherwise one needs the Rearrangement Lemma 2.6 from [BWa94].

**Proof.** □

## 5. The recursive atom ordering technique

**Theorem 5.1.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_{p-s}, 1^s)$ *then the following two conditions guarantee that* $\Pi_\lambda$ *is CL-shellable (hence shellable).*

(1) $s \leq 1$ *or* $\lambda_{p-s} = k = 2$;
(2) *Condition (NES).*

*Moreover the condition (2) can be weakened to allow* $\lambda$ *to have blocks of sizes* $u, v$ *and* $u + v$ *if there are no block sizes in* $\lambda$ *less than* $u + v$, *except for* $u$ *and* $v$.

**Proof.** □

## 6. Applications

In this section we will closer investigate shellability of $\Pi_\lambda$ for the special case $\lambda = (k^m, 1^t) = (\underbrace{k, \dots, k}_{m}, \underbrace{1, \dots, 1}_{t})$.

**Theorem 6.1.** *Let $\lambda = (k^m, 1^t)$, we have:*

(I) *if $t < k$ then $\Pi_\lambda$ has a EC-labeling (hence is shellable);*
(II) *if $k = 2$ then $\Pi_\lambda$ is CL-shellable;*
(III) *if $m = 1$ then $\Pi_\lambda$ is EL-shellable;*
(IV) *if $t \geq k, k \geq 3, m \geq 3$ then $\Pi_\lambda$ is not shellable;*

**Proof.**

(I) Follows from the Theorem 4.1.
(II) Follows from the Theorem 5.1.
(III) The homotopy type and Betti numbers of these lattices, also known as $\Pi_{n,k}$ were first computed in Theorem 1.5, [BWe]. Their EL-shellability was proved in [BWa94]. They were also important in connection with a problem in complexity theory, see [BLY], [BL].
(IV) Follows from the Theorem 4.2.

That completes the proof. $\square$

## 7. Topological consequences

In this section we will consider the topological consequences of Theorems 4.1 and 5.1.

First let us introduce some notations. Assume that we have a central subspace arrangement $\mathcal{A} = \{K_1, \dots, K_t\}$ in $\mathbb{R}^n$. We define $V_\mathcal{A} = K_1 \cap \dots \cap K_t$ and $M_\mathcal{A} = \mathbb{R}^n \setminus V_\mathcal{A}$, so $V_\mathcal{A}$ is the union of the subspaces in our arrangement and $M_\mathcal{A}$ is the complement to it. When the arrangement is central, $V_\mathcal{A}$ is contractible, so, in order to get hold on its topological properties, one often considers $V_\mathcal{A} \cap S^{n-1}$ instead. Also to determine cohomology of $M_\mathcal{A}$ is a central question in the theory of subspace arrangements. Shellability is one of the tools which has been proved to be usefull to do that.

The two following results are of fundamental importance for determining the topology of the arrangement $\mathcal{A}$ from the combinatorics of its intersection lattice $L_\mathcal{A}$.

**Proposition 7.1.** *(Goresky and MacPherson,* [GM]*). For every subspace arrangement $\mathcal{A}$ and all dimensions $i$:*

$$\tilde{H}^i(M_\mathcal{A}) \cong \bigoplus_{x \in L_\mathcal{A}^{>\hat{0}}} \tilde{H}_{\operatorname{codim}(x)-i-2}(\hat{0}, x). \tag{7.1}$$

**Proposition 7.2.** *(Ziegler and Živaljević,* [ZŽ]*). For every arrangement $\mathcal{A}$ in $\mathbb{R}^n$ there is a homotopy equivalence*

$$V_\mathcal{A} \cap S^{n-1} \simeq \operatorname*{wedge}_{x \in L_\mathcal{A}^{>\hat{0}}}(\Delta(\hat{0}, x) * S^{\dim(x)-1}) \tag{7.2}$$

The next proposition is an easy consequence of these results.

**Proposition 7.3.** *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$, such that $\mathcal{L}_{\mathcal{A}}$ is pure and shellable. Assume that for any $x \in \mathcal{L}_{\mathcal{A}}$ we have $codim(x) - rk(x) = c$, where $c$ is some constant. Let $r$ be the codimension of the highest dimensional subspaces in $\mathcal{A}$, then $r = c+1$. Then the cohomology groups of $M_{\mathcal{A}}$ are torsion-free and their Betti numbers are given by the following formulae:*

$$\tilde{\beta}^i(M_{\mathcal{A}}) = \begin{cases} \sum_{x \in \mathcal{L}_{\mathcal{A}}^{>\hat{0}}} |\mu(\hat{0}, x)|, & \text{if } i = r-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.3}$$

*Furthermore, $V_{\mathcal{A}} \cap S^{n-1}$ has the homotopy type of the wedge of $(n-r-1)$-spheres.*

**Proof.** Let $\Delta = \Delta(\hat{0}, x)$, for $x \in \mathcal{L}_{\mathcal{A}}$. We know that $\mathcal{L}_{\mathcal{A}}$ is shellable hence we can conclude that $\Delta$ is also shellable. In particular it means that

$$\tilde{\beta}_i(\Delta) = \begin{cases} |\mu(\hat{0}, x)|, & \text{if } i = \mathrm{rk}(x) - 2 \\ 0, & \text{otherwise} \end{cases}$$

and that $\Delta$ has the homotopy type of a wedge of $(\mathrm{rk}(x)-2)$-spheres. Insert that into the formulae in Theorem 7.1. The term on the right hand side of 7.1 does not disappear iff $\mathrm{codim}(x) - 2 - i = \mathrm{rk}(x) - 2$, that is if $i = \mathrm{codim}(x) - \mathrm{rk}(x) = c = r-1$.

If we look at equation 7.2, we see that because

$$(\mathrm{rk}(x) - 2) + (\dim(x) - 1) + 1 = n - r - 1$$

we can conclude that $V_{\mathcal{A}} \cap S^{n-1}$ has the homotopy type of the wedge of $(n-r-1)$-spheres. □

**Corollary 7.4.** *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition, which satisfies condition (NES) in Theorem 4.1 or conditions in Theorem 5.1. Then the cohomology groups of $M_{\mathcal{A}_\lambda}$ are torsion-free and their Betti numbers are given by the following formulae:*

$$\tilde{\beta}^i(M_{\mathcal{A}_\lambda}) = \begin{cases} |\mu(\Pi_\lambda)|, & \text{if } i = n-p-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

*Further, $V_{\mathcal{A}_\lambda} \cap S^{n-1}$ has the homotopy type of the wedge of $(p-1)$-spheres.*

**Proof.** If the partition $\lambda$ satisfies the condition (NES) in the Theorem 4.1 or the conditions in Theorem 5.1, then it is easy to see that $\mathcal{A}_\lambda$ (and hence $\mathcal{L}_{\mathcal{A}_\lambda}$) is pure, and it was proved that it is shellable. Also one can see that for any $x \in \mathcal{L}_{\mathcal{A}_\lambda}$ we have $\mathrm{codim}(x) - \mathrm{rk}(x) = n-p-1$. So all the assumptions of the proposition 7.3 are fulfilled. $\mathcal{A}_\lambda$ is obviously an arrangement in $\mathbb{R}^n$, where $n = \sum_{i=1}^p \lambda_i$. The dimension of the highest dimensional subspaces is $p$ and so the formulae 7.4 follow immediatelly from 7.3 and we also conclude that $V_{\mathcal{A}_\lambda} \cap S^{n-1}$ has the homotopy type of the wedge of $(p-1)$-spheres. □

## 8. Shellability of $\Pi_{n,k}(l)$, computation of the Möbius function of $\Pi_\lambda$

Let us denote $\mu_\lambda = \mu(\Pi_\lambda)$. To compute $\mu_\lambda$ we will need an auxiliary class of lattices. The following definition is adopted from Björner and Welker, [BWe].

**Definition 8.1.** *For $n \geq 2$, $k \geq 2$, and $l \geq 0$, let $\Pi_{n,k}(l)$ be the family of all partitions $\pi$ of the set $\{1, 2, \ldots, n\}$ such that each block $B$ of $\pi$ satisfies at least one of the following requirements:*

(i) $|B| = 1$,
(ii) $k \leq |B| \leq n$,
(iii) $B \cap \{1, 2, \ldots, l\} \neq \emptyset$.

Ordered by refinement, $\Pi_{n,k}(l)$ is a lattice. Observe that $\Pi_{n,k}(0) = \Pi_{n,k}$ and $\Pi_{n,2}(l) = \Pi_n$. We will denote $\mu(\Pi_{n,k}(l)) = \mu_{n,k}(l)$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, let $b(\lambda)$ denote the number of blocks in $\lambda$, i.e. $b(\lambda) = p$ and let $s(\lambda) = \{\max j \mid \lambda_j \neq 1\} = \{\# j \mid \lambda_j \neq 1\}$.

**Proposition 8.2.** *Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition. Take $x \in \Pi_\lambda$, $\pi \neq \hat{0}$ and write $x$ in the block form: $x = (B_1, \ldots, B_l, B_{l+1}, \ldots, B_m)$, such that $|B_1| \geq \cdots \geq |B_m|$, $|B_l| > 1$ and $|B_{l+1}| = \cdots = |B_m| = 1$. Then*

$$[x, \hat{1}] \simeq \Pi_{m, \lambda(s(\lambda))}(l)$$

**Proof.** Denote $k = \lambda(s(\lambda))$. Let us define a map $f : [x, \hat{1}] \to \Pi_m$. If $B = B_{i_1} \cup \cdots \cup B_{i_t}$ then let $f^*(B) = \{i_1, \ldots, i_t\}$. Take $y \in [x, \hat{1}]$, $y = (Y_1, \ldots, Y_q)$ and define $f(y) = (f^*(Y_1) \ldots, f^*(Y_q)) \in \Pi_m$. Every block of $y$ is either a singleton, a union of at least $k$ singletons or a union of at least one of the blocks $B_1, \ldots, B_l$ and an arbitrary number of singletons.

These cases correspond exactly to (*i*), (*ii*) and (*iii*) in Definition 8.1, so $f$ defines a bijection $f : [x, \hat{1}] \to \Pi_{m,k}(l)$. On the other hand, both $[x, \hat{1}]$ and $\Pi_{m,k}(l)$ are ordered under refinement, so it is easy too see that $f$ is actually a poset map. Hence $[x, \hat{1}] \simeq \Pi_{m,k}(l)$. □

**Proposition 8.3.** *$\Pi_{n,k}(l)$ is shellable for all $n \geq 2$, $k \geq 2$, and $l \geq 0$.*

**Proof.** Consider $\Pi_{m,k}$ for $m = n + lk - l$. It has been proved in the Theorem 6.1, [BWa94], that $\Pi_{m,k}(l)$ is EL-shellable for any $m \geq k \geq 2$. Let $x \in \Pi_{m,k}$ be some partition of shape $(\underbrace{\overbrace{k, k, \ldots, k}^{l}, 1, 1 \ldots, 1}_{n})$, then according to the Proposition 8.2 we get $[x, \hat{1}] \simeq \Pi_{n,k}(l)$. Since every interval of an EL-shellable poset is EL-shellable (see Lemma 5.6 in [BWa94]), we conclude that $\Pi_{n,k}(l)$ is EL-shellable, hence simply shellable. □

**Proposition 8.4.**

$$\mu_\lambda = - \sum_{\lambda \text{ refines } \delta} n(\delta) \cdot \mu(\delta), \tag{8.1}$$

*where $n(\delta)$ is equal to number of partitions of $[n]$ of shape $\delta$ and $\mu(\delta) = \mu_{b(\delta),k}(s(\delta))$.*

**Proof.** Immediate from Proposition 8.2 and definition of Möbius function. □

Next we would like to specialize the formula 8.1 to the case when the partition $\lambda$ satisfies the condition (NES). First of all, let us notice that if $\delta = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$ then it is well known that

$$n(\delta) = \frac{n!}{\alpha_1! \ldots \alpha_n!\ 1!^{\alpha_1} \ldots n!^{\alpha_n}} \tag{8.2}$$

Let $\tilde{\beta}^i_{n,k}(l) = \mathrm{rank}\tilde{H}_i(\Pi_{n,k}(l))$, i.e. $\tilde{\beta}^i_{n,k}(l)$ is the $i$th reduced Betti number of $\Pi_{n,k}(l)$. It has been proved in [BWe] that these numbers satisfy the following recursive formula:

$$\tilde{\beta}^i_{n,k}(l) = \binom{n-l-1}{k-1} \cdot \tilde{\beta}^{i-1}_{n-k+1,k}(l) + l \cdot \tilde{\beta}^{i-1}_{n-1,k}(l). \tag{8.3}$$

Summing up over all $i$ with a sign $(-1)^i$, we obtain

$$\mu_{n,k}(l) + \binom{n-l-1}{k-1} \cdot \mu_{n-k+1,k}(l) + l \cdot \mu_{n-1,k}(l) = 0. \tag{8.4}$$

In particular, if $\lambda$ satisfies (NES), then

$$b(\delta) - s(\delta) \leq b(\lambda) - s(\lambda) < k.$$

But if $n - l < k$, then it is easy to derive from formulae 8.3 and 8.4 that

$$\mu_{n,k}(l) = (-1)^{n-3} \cdot \tilde{\beta}^{n-3}_{n,k} = (-1)^{n-3} \cdot l^{n-l-1} \cdot l!\,.$$

**Corollary 8.5.** *If the partition $\lambda$ satisifes condition (NES), then*

$$\mu(\Pi_\lambda) = - \sum_{\lambda \text{ refines } \delta} n(\delta) \cdot (-1)^{b(\delta)-3} \cdot s(\delta)! \cdot s(\delta)^{b(\delta)-s(\delta)-1}, \tag{8.5}$$

*where $\delta = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$ and $n(\delta) = \dfrac{n!}{\alpha_1! \ldots \alpha_n!\ 1!^{\alpha_1} \ldots n!^{\alpha_n}}$.*

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