# A Variation of Kraśkiewicz Insertion and Shifted Tableaux

**An Extended Abstract**

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**Summary**

We present a variation of Kraśkiewicz insertion, adapting it to permutations of $S_n$. It maps a permutation to a pair of shifted tableaux of the same shape. It is shown that it is equivalent to Haiman's shifted mixed insertion but it exhibits some properties that are different from those of shifted mixed insertion. For example, it has a natural notion of a reading word.

## 1 Introduction

We assume that the reader is familiar with the Robinson-Schensted insertion algorithm and jeu de taquin. A good reference is [9]. The insertion algorithm maps a permutation to a pair of standard Young tableaux of the same shape.

Edelman and Greene developed a variation of Robinson-Schensted algorithm that enables them to count the number of reduced words of a permutation [3]. This insertion maps each reduced word to a pair of tableaux. The symmetric functions related to these tableaux turn out to be stable Schubert polynomials [1, 4, 10].

There is an analogue of Edelman-Greene's insertion and a related theory of stable Schubert polynomials in the context of $B_n$, the group of signed permutations. The algorithm was developed by Kraśkiewicz [6] to enumerate the reduced words of a given signed permutation. The shifted tableaux that he used are different from those in the other algorithms. We propose a variation of Kraśkiewicz insertion algorithm that maps a permutation to a pair of shifted tableaux, one of which is of the same type as those used by Kraśkiewicz and the other is a standard shifted Young tableau.

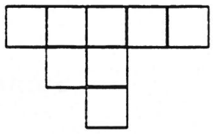
| | permutations | reduced words |
|---|---|---|
| unshifted | Robinson-Schensted | Edelman-Greene |
| shifted | proposed | Kraśkiewicz |

There are already insertion algorithms that map elements of $S_n$ to pairs of shifted tableaux, for example, Worley-Sagan insertion [11, 8] and Haiman's shifted mixed insertion[5]. It turns out that the new variation we are proposing is closely related to Haiman's shifted mixed insertion.

## 2 Notation

**Definition 2.1** *We define a* shifted shape $T$ *to be an arrangement of* $n$ *boxes, filled with numbers, into* $l$ *rows of strictly decreasing lengths* $\lambda_1, \lambda_2, \ldots, \lambda_l$. *Each row is indented 1 box to the right of the row above. The sequence* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *is called the* shape *of* $T$ *and is denoted by* $\mathrm{sh}(T)$. *The number of boxes in* $T$ *is called the* size *of* $T$.

**Example:** The figure shown below has size 8 and shape (5,2,1).



**Definition 2.2** *Let* $T$ *be a shifted tableau filled with integers from 1 to* $n$ *and such that*

1. *the entries in each row increases from left to right, and*
2. *the entries in each column increases from top to bottom.*

*Then* $T$ *is called a* standard shifted Young tableau. *Denote the set of such tableaux of size* $n$ *by* $\mathcal{Q}_n$.

**Example:**

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 7 |
| | 3 | 6 | | |
| | | 8 | | |

The set of permutations is denoted by $S_n$. We treat a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ as a sequence of distinct numbers from 1 to $n$. The *reverse* of $\pi$, denoted by $\pi^r$, is the permutation $\pi_n\pi_{n-1}\ldots\pi_2\pi_1$.

Very often we are interested in certain subsequences of a permutation that satisfy certain properties. For example, in the theory of Robinson-Schensted algorithm, the length of the longest increasing subsequence is reflected in the length of the first row of the pair of Young tableaux. Here, we are interested in unimodal subsequence.

**Definition 2.3** *Let* $\pi = \pi_1\pi_2\cdots\pi_n$ *be a sequence of distinct numbers. If there exists* $k$, $k \geq 1$ *such that*

$$\pi_1 > \pi_2 > \cdots > \pi_k < \pi_{k+1} < \cdots < \pi_n,$$

*then* $\pi$ *is called a* unimodal *sequence. We define* $\pi\!\downarrow = \pi_1\pi_2\ldots\pi_k$ *to be the* decreasing part *of* $\pi$ *and* $\pi\!\uparrow = \pi_{k+1}\pi_{k+2}\ldots\pi_n$ *to be the* increasing part *of* $\pi$.

**Definition 2.4** *Let $P$ be a shifted tableau filled with distinct entries. Denote the entries in the ith row of $P$ by $P_i$. If for each $i$, $P_i$ is a unimodal subsequence of maximum length in $P_l P_{l-1} \ldots P_{i+1} P_i$ then $P$ is called a* unimodal tableau.

*If the entries in $P$ are from* 1 *to $n$, then $P$ is called a* standard unimodal tableau. *We denote the set of standard unimodal tableaux of size $n$ by $\mathcal{P}_n$.*

**Example:**

| | | | | |
|---|---|---|---|---|
| 8 | 7 | 6 | 2 | 3 |
| | 5 | 4 | | |
| | | 1 | | |

# 3 Kraśkiewicz Insertion

Even though the Kraśkiewicz insertion algorithm works on reduced words of signed permutations, we can use it on permutations of $S_n$ as well. This is made possible by the fact that each permutation, when written as a sequence of distinct numbers, corresponds to a reduced word of some signed permutation.

We will abuse nomenclature and call the proposed insertion algorithm Kraśkiewicz insertion algorithm as well. So the Kraśkiewicz insertion maps a permutation $\pi$ to a pair of tableaux $(P, Q)$. We call $P$ and $Q$ the insertion tableau and the recording tableau of $\pi$ respectively. The insertion tableau is obtained by constructing a sequence of unimodal tableaux each obtained by the steps described below.

$$\emptyset = P^{(0)},\ P^{(1)},\ P^{(2)}, \ldots,\ P^{(n)} = P$$

Each $P^{(i)}$ is obtained from $P^{(i-1)}$ and $\pi_i$.

**Input:** $\pi_i$ and $P^{(i-1)}$. **Output:** $P^{(i)}$.

**Step 1:** Let $a = \pi_i$ and $R$ be the first row of $P^{(i-1)}$.
**Step 2:** Insert $a$ into $R$ as follows:

- Case 0: $R = \emptyset$. If the empty row is the $k$th row, we write $a$ indented $k-1$ boxes away from the left margin. This new tableau is $P^{(i)}$. Stop.
- Case 1: $Ra$ is unimodal. Append $a$ to $R$ and let $P^{(i)}$ be this new tableau and stop.
- Case 2: $Ra$ is not unimodal. Let $b$ be the smallest number in $R\uparrow$ bigger than $a$. Put $a$ in $b$'s position. Let $c$ be the biggest number in $R\downarrow$ smaller than $b$. Put $b$ in $c$'s place.

**Step 3:** Repeat Step 2 with $a = c$ and $R$ equals to the next row.

The recording tableau $Q$ records the changes in shapes of $P^{(i)}$. The box in $Q$ that corresponds to the box where the insertion procedure ends in $P^{(i)}$ is filled by $i$.

This makes $Q$ a standard shifted Young tableau of the same shape as $P$. We denote the insertion map by

$$\pi \xrightarrow{K} (P, Q).$$

**Example:** Let $\pi = 3142$. We list the sequence of unimodal tableaux $P^{(i)}$ obtained during the insertion process.

$$P^{(1)} = \boxed{3} \qquad P^{(2)} = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline\end{array} \qquad P^{(3)} = \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline\end{array}$$

$$P = \begin{array}{ccc} \boxed{4} & \boxed{1} & \boxed{2} \\ & \boxed{3} & \end{array} \qquad Q = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ & \boxed{4} & \end{array}$$

**Theorem 3.1** *Kraśkiewicz insertion maps permutations of $S_n$ to $\{(P, Q) \in \mathcal{P}_n \times \mathcal{Q}_n : \mathrm{sh}(P) = \mathrm{sh}(Q)\}$. Moreover, it is a bijection.*

Associated with this insertion algorithm is a set of relations called the B-Knuth relations.

**Definition 3.2** *Let $\pi, \sigma$ be permutations of $S_n$. Suppose $\pi = \pi_1 \dots \pi_{k-1} wxyz \pi_{k+4} \dots \pi_n$ and $\sigma = \pi_1 \dots \pi_{k-1} w'x'y'z' \pi_{k+4} \dots \pi_n$. If $wxyz \sim w'x'y'z'$ or $zyxw \sim z'y'x'w'$ is one of the forms shown below where $a < b < c < d$, we say that $\pi$ is* elementary B-Knuth related *to $\sigma$.*

$$\begin{array}{rcll} abdc & \sim & adbc & (1) \\ acdb & \sim & acbd & (2) \\ adcb & \sim & dacb & (3) \\ badc & \sim & bdac & (4) \end{array}$$

*Let $\pi, \sigma \in S_n$. If there exists a sequence of permutations $\pi = \pi^{(1)}, \pi^{(2)}, \dots, \pi^{(k)} = \sigma$ such that each $\pi^{(i)}$ is elementary B-Knuth related to $\pi^{(i+1)}$, then we say that $\pi$ and $\sigma$ are* B-Knuth related. *We denote this by $\pi \sim \sigma$.*

It is well known that 2 permutations give rise to the same insertion tableau under Robinson-Schensted algorithm if and only if they are Knuth related. We have the analogue here as well.

**Theorem 3.3** *Let $\pi, \sigma \in S_n$. Denote the insertion tableaux of $\pi$ and $\sigma$ under Kraśkiewicz insertion by $P_\pi$ and $P_\sigma$ respectively. Then*

$$P_\pi = P_\sigma \text{ iff } \pi \sim \sigma.$$

This insertion algorithm has several beautiful properties. We describe some of these here. The notion of a *reading word* of a standard unimodal tableau is obvious.

**Definition 3.4** *Let $\pi_P$ denote the sequence of entries of a standard unimodal tableau $P$ when read from left to right starting from the bottom row of $P$. This sequence is called the reading word of $P$.*

This is an analogue of the reading word in the theory of Robinson-Schensted algorithm. It can be shown that when we apply Kraśkiewicz insertion on $\pi_P$, we get back $P$ as the insertion tableau.

The recording tableau of Kraśkiewicz insertion also has some nice properties. We can describe how the recording tableau changes when we

1. delete $\pi_1$
2. reverse $\pi$.

The first case involves what is known as the *delta operator* $\Delta$. We follow the notation in [9, Section 3.11].

**Definition 3.5** *Let $Q$ be a standard shifted Young tableau. Define $\Delta(Q)$ to be the resulting tableau after applying the following operations:*

1. *Remove the entry* 1 *from $Q$.*
2. *Apply jeu de taquin into this box.*
3. *Deduct* 1 *from each of the remaining boxes.*

This is essentially the same as [9, Definition 3.11.1] but here, we are applying $\Delta$ to a shifted Young tableau. In the notation of [5], $\Delta(Q)$ is the tableau which is obtained by subtracting 1 from every box in $Q(1 \to \infty)$.

**Theorem 3.6** *Let $\pi \in S_n$ and suppose*

$$\begin{aligned} \pi_1\pi_2\cdots\pi_n &\xrightarrow{K} (P,Q), \\ \pi_2\cdots\pi_n &\xrightarrow{K} (R,S). \end{aligned}$$

*Then*

$$S = \Delta(Q).$$

The second case involves the process called evacuation.

**Definition 3.7** *Let $Q$ be a standard shifted Young tableau with $|Q| = m$. We define the* evacuation *of $Q$, denoted by* $\mathrm{ev}(Q)$, *to be a shifted Young tableau of the same shape as $Q$ such that each box $(i,j)$ has entry $m-k+1$ iff $\mathrm{sh}(\Delta^k(Q))$ and $\mathrm{sh}(\Delta^{k-1}(Q))$ differ in box $(i,j)$.*

Again, this is essentially the same as [9, Definition 3.11.1] but we are defining it on shifted Young tableaux instead. An alternate definition can be found in [5, Section 8]. Observe that if $Q$ is of size $m$, then

$$\mathrm{ev}(\Delta(Q)) = \mathrm{ev}(Q)|_{m-1}.$$

**Theorem 3.8** *Let $\pi \in S_n$ and*

$$\pi \xrightarrow{K} (P, Q),$$
$$\pi^{r} \xrightarrow{K} (R, S).$$

*Then,*

$$S = \mathrm{ev}(Q).$$

The experienced reader will recognize that these correspond exactly to the situations in the theory of Robinson-Schensted algorithm.

**Example:** Let $\pi = 18547263 \in S_8$. Applying Kraśkiewicz insertion to $\pi$ gives

$$\left( \begin{array}{|c|c|c|c|c|} \hline 8 & 7 & 6 & 2 & 3 \\ \hline & 5 & 4 & & \\ \cline{2-3} & & 1 & & \\ \cline{3-3} \end{array} , \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 \\ \hline & 3 & 6 & & \\ \cline{2-3} & & 8 & & \\ \cline{3-3} \end{array} \right)$$

and to 8547263 gives

$$\left( \begin{array}{|c|c|c|c|c|} \hline 8 & 7 & 6 & 2 & 3 \\ \hline & 5 & 4 & & \\ \cline{2-3} \end{array} , \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline & 5 & 7 & & \\ \cline{2-3} \end{array} \right)$$

The reader can check that the second recording tableau can be obtained by applying $\Delta$ to the first recording tableau. Applying Kraśkiewicz insertion to $\pi^{r} = 36274581$ gives

$$\left( \begin{array}{|c|c|c|c|c|} \hline 7 & 4 & 1 & 5 & 8 \\ \hline & 6 & 3 & & \\ \cline{2-3} & & 2 & & \\ \cline{3-3} \end{array} , \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 7 \\ \hline & 3 & 5 & & \\ \cline{2-3} & & 8 & & \\ \cline{3-3} \end{array} \right)$$

It can be verified that third recording tableau can be obtained from the first by evacuation.

# 4 Shifted Mixed Insertion

We describe briefly Haiman's shifted mixed insertion. The reader should refer to [5] for the details. The shifted mixed insertion algorithm maps a permutation $\pi$ of $S_n$ into a pair of tableaux $(T, Q)$. We call $T$ the insertion tableau and $Q$ the recording tableau. The insertion tableau looks like a standard shifted Young tableau except that we allow numbers which are not in the first box of any row to have a bar above them. If the number has a bar above it, we call it a *barred* number. Let us denote the set of such tableau of size $n$ by $\mathcal{T}_n$ and call the diagonal formed by the first boxes of all rows the first diagonal.

As in the description of Kraśkiewicz insertion, we construct a sequence of tableaux

$$\emptyset = T^{(0)}, T^{(1)}, \ldots, T^{(n-1)}, T^{(n)} = T$$

where $T^{(i)}$ is obtained by inserting $\pi_i$ into $T^{(i-1)}$. The recording tableau $Q$ records the changes in shapes of $T^{(i)}$. So $Q$ is a standard shifted Young tableau of the same shape as $T$.

For details, please see [5, Definition 6.7]. We denote it as

$$\pi \xrightarrow{H} (P, Q).$$

**Input:** $\pi_i$ and $T^{(i-1)}$. **Output:** $T^{(i)}$.

**Step 1:** Let $a = \pi_i$ and $R$ be the first row of $T^{(i-1)}$.

**Step 2:** Let $b$ be the smallest number in $R$ which is bigger than $a$. Put $a$ in $b$'s place. If no such $b$ exists, append $a$ to $R$ and end.

**Step 3:**

- Case 1: $b$ is from a box in the first diagonal of $T^{(i-1)}$. Repeat Step 2 with $a = \bar{b}$ and $R$ equals to the next column.
- Case 2: $b$ is not from the first diagonal and $b$ is unbarred. Repeat Step 2 with $a = b$ and $R$ equals to the next row.
- Case 3: $b$ is not from the first diagonal and $b$ is barred. Repeat Step 2 with $a = b$ and $R$ equals to the next column.

**Example:** Let $\pi = 3142$. Haiman's shifted mixed insertion maps it to:

$$\left( \begin{array}{|c|c|c|} \hline 1 & 2 & \bar{3} \\ \hline & 4 & \\ \cline{2-2} \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline & 4 & \\ \cline{2-2} \end{array} \right).$$

The shifted mixed insertion possesses a number of properties similar to those of the Kraśkiewicz insertion. Thus the next result is not totally unexpected.

**Theorem 4.1** *Let $\pi \in S_n$ and*

$$\begin{array}{ccc} \pi & \xrightarrow{K} & (P, Q) \\ \pi & \xrightarrow{H} & (T, T') \end{array}$$

*Then $Q = T'$. Furthermore, there exists a shape preserving bijection, $\Phi$ from $\mathcal{T}_n$ to $\mathcal{P}_n$ such that $\Phi(T) = P$.*

This bijection between $\mathcal{P}_n$ and $\mathcal{T}_n$ gives an easy formula for enumerating the number of unimodal tableaux.

**Corollary 4.2** *The number of standard unimodal tableaux of shape $\lambda$ is given by*

$$2^{n - \text{ \# of rows of } \lambda} \times \text{ the number of standard shifted Young tableaux of shape } \lambda$$

It is not obvious how to arrive at this formula without using this bijection.

In the same vein, we can make use of other properties of the shifted mixed insertion to derive new results for the Kraśkiewicz insertion and vice versa. For example,

**Theorem 4.3** *Let $\pi$ be mapped to a unimodal tableau $P$ with shape $(\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then the length of a longest unimodal subsequence in $\pi$ is $\lambda_1$.*

becomes

**Theorem 4.4** *Let $\pi \in S_n$ and suppose*

$$\pi \xrightarrow{H} (T, T')$$

*where $sh(T) = \lambda$. Then $\lambda_1$ is the length of the longest unimodal sequence in $\pi$.*

Recall the famous result [7] that given any sequence of distinct numbers of length $n$, it necessary contains an increasing or a decreasing subsequence of length at least $\lceil \sqrt{n} \rceil$. In [2], the analogue for the length of a unimodal subsequence was given. The bound is attributed to J. M. Steele, V. Chvátal and others. We give a different proof of this result.

**Theorem 4.5** ([2]) *Let $\pi$ be a sequence of distinct numbers of length $n$. Then it contains a unimodal subsequence of length at least $\lceil \frac{\sqrt{8n+1}-1}{2} \rceil$.*

**Proof:** We can assume $\pi$ is a permutation of $S_n$. and apply the Kraśkiewicz insertion algorithm to $\pi$. Let $P$ be the insertion tableau that is obtained and let $\text{sh}(P) = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. From Theorem 4.3, the longest unimodal subsequence of $\pi$ is equal to $\lambda_1$. Clearly, $P$ must sit inside the shifted shape $(\lambda_1, \lambda_1 - 1, \lambda_1 - 2, \ldots, 2, 1)$. This shows that $\frac{\lambda_1(\lambda_1+1)}{2} \geq n$. Solving this inequality gives the desired bound. ■

# 5 Comparison and Open Problems

Though the Kraśkiewicz insertion and the shifted mixed insertion are equivalent, we find that the respective insertion tableaux reveal somewhat different information. For example, the reading word for a standard unimodal tableaux is easily "readable" from it, whereas the shifted mixed insertion has not been shown to have a similar property. But this does not mean that Kraśkiewicz insertion is better. As we have seen, the formula for the number of standard unimodal tableaux of size $n$ is obtained via shifted mixed insertion. Hence, it would be preferable to treat these two insertion algorithms as different representations of the same theory.

Of course, this does not deter us from asking if there is an easy way to "read" the insertion tableau of shifted mixed insertions or whether we can count the number of standard unimodal tableau without resorting to shifted mixed insertion. There are many more such questions that can be looked into.

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