HOMOTOPY IN Q-POLYNOMIAL DISTANCE-REGULAR GRAPHS

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1. Introduction.

Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$. In [7], Terwilliger showed that if Γ is the antipodal quotient of a distance-regular graph with diameter $D \geq 7$, then the dual eigenvalues of Γ satisfy a certain equation. We say that Γ is a *pseudoquotient* whenever this equation is satisfied. In our main result, speaking a bit vaguely for the moment, we show that if Γ is not a pseudoquotient, then each cycle in Γ can be "decomposed" into cycles of length at most six. We state this result precisely using homotopy.

The outline of this abstract is as follows. In Sections 2–4, we present material on homotopy. In Sections 5-6, we examine Q-polynomial distance-regular graphs. Specifically, in Section 5 we show that if Γ is a Q-polynomial distance-regular graph with diameter and valency at least three, then the intersection number p^3_{12} is at least two; consequently, the girth is at most six. In Section 6 we say what it means for Γ to be a pseudoquotient. Finally, in Section 7 we present our main theorem.[2]

By a **graph** we mean a pair $\Gamma = (X, R)$, where X is a finite non-empty set (the **vertices**) and R is a set of distinct two-element subsets of X (the **edges**). Observe that Γ is undirected without loops or multiples edges. Fix a graph $\Gamma = (X, R)$. Let x and y be vertices in X and let l be a nonnegative integer. By a **path** in Γ of length l from x to y we mean a sequence

$$p \ := \ (x = x_0, x_1, \ldots, x_l = y) \qquad (x_i \in X, \ 0 \leq i \leq l)$$

such that

$$\{x_{i-1}, \, x_i\} \in R \qquad (1 \leq i \leq l).$$

We call x the **initial vertex** of p and y the **terminal vertex** of p. Given p as above, we define p^{-1} to be the sequence

$$p^{-1} \ := \ (y = x_l, x_{l-1}, \ldots, x_0 = x).$$

Observe that p^{-1} is a path in Γ.

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[2]In the interests of space, we have omitted all of the proofs. A complete version of this paper, with proofs intact, is available from the author.

Let p be a path in Γ. We say that p is **closed** if the initial vertex and terminal vertex of p are the same. If p is closed, then we call the initial vertex the **base vertex** of p. For each $x \in X$, let $\psi(x)$ denote the set of all closed paths with base vertex x.

2. The Homotopy Relation.

Let $\Gamma = (X, R)$ be a graph, and pick any $x \in X$. In this section, we consider a binary relation $\sim$ on $\psi(x)$ called the homotopy relation (Definition 2.2). We also define what it means for a path in $\psi(x)$ to be reduced (Definition 2.5). We then show that each element of $\pi(x)$ has exactly one reduced representative (Theorem 2.6).

Definition 2.1. Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. Pick any $p \in \psi(x)$, and write

$$p = (x = x_0, x_1, \ldots, x_l = x).$$

An element $q \in \psi(x)$ is said to **extend** p if there exists an integer i $(0 \leq i \leq l)$ and a vertex $y \in X$ such that

$$q = (x = x_0, x_1, \ldots, x_{i-1}, x_i, y, x_i, x_{i+1}, \ldots, x_l = x).$$

Observe that if q extends p, then the length of q is two greater than the length of p.

Definition 2.2. Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. We define the binary relation $\sim$ on $\psi(x)$ as follows: for all p, $q \in \psi(x)$, write $p \sim q$ whenever there exists a nonnegative integer n and paths $p = p_0, p_1, \ldots, p_n = q \in \psi(x)$ such that p_i extends p_{i-1} for all i $(1 \leq i \leq n)$. We call this relation **homotopy**, and we say that p and q are **homotopic** if $p \sim q$. Observe that $\sim$ is an equivalence relation.

Definition 2.3. Let $\Gamma = (X, R)$ be a graph, and pick $x \in X$. Let $\pi(x)$ denote the set of equivalence classes of $\psi(x)$ under homotopy. For every $p \in \psi(x)$, let $[p]$ denote the element of $\pi(x)$ that contains p.

Definition 2.4. Let $\Gamma = (X, R)$ be a graph. Fix $x \in X$, and pick $u \in \pi(x)$. We say that $p \in \psi(x)$ is a **representative** of u if $u = [p]$.

Definition 2.5. Let $\Gamma = (X, R)$ be a graph. Fix $x \in X$, and pick $p \in \psi(x)$. We say that p is **reduced** if p does not extend q for all $q \in \psi(x)$.

Theorem 2.6. *Let $\Gamma = (X, R)$ be a graph. Fix $x \in X$, and pick any $u \in \pi(x)$. Then u has exactly one reduced representative. Furthermore, this is the unique representative of u of minimal length. We denote this representative by $\tilde{u}$.*

3. The Fundamental Group of a Graph.

Let $\Gamma = (X, R)$ be a graph, and pick $x \in X$. In this section, we show that concatenation in $\psi(x)$ induces a group structure on $\pi(x)$ (Theorem 3.3).

Definition 3.1. Let $\Gamma = (X, R)$ be a graph. Let p and q be any paths in Γ such that the terminal vertex of p is the same as the initial vertex of q, and write

$$\begin{aligned} p &= (x_0, x_1, \ldots, x_{l-1}, x_l), \\ q &= (x_l = y_0, y_1, \ldots, y_m). \end{aligned}$$

By the **concatenation** of p and q we mean the sequence

$$pq := (x_0, x_1, \ldots, x_{l-1}, x_l = y_0, y_1, \ldots, y_m).$$

Observe that pq is a path in Γ.

Note: Whenever we write pq for paths p and q in Γ, it will be assumed that the terminal vertex of p is the same as the initial vertex of q.

Definition 3.2. Let $\Gamma = (X, R)$ be a graph. Fix $x \in X$, and pick any u, $v \in \pi(x)$.

(i) We define uv to be element $[pq] \in \pi(x)$, where p is any representative of u and q is any representative of v.

(ii) We define u^{-1} to be the element $[p^{-1}] \in \pi(x)$, where p is any representative of u.

(iii) We define e to be the element $[(x)] \in \pi(x)$.

Theorem 3.3. *Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. With reference to Definition 3.2, the following hold for all u, v, $w \in \pi(x)$:*

(i) $(uv)w = u(vw)$,

(ii) $ue = u = eu$,

(iii) $uu^{-1} = e = u^{-1}u$.

In particular, concatenation on $\psi(x)$ induces a group structure on $\pi(x)$. We call this group the **fundamental group with respect to x.**

Note: The fundamental group is sometimes referred to as the first homotopy group. It is usually written as $\pi(\Gamma, x)$ or $\pi_1(\Gamma, x)$, but we have chosen to drop Γ from the notation in this abstract since there is no ambiguity about the identity of Γ.

4. The Subgroups $\pi(x, i)$.

Let $\Gamma = (X, R)$ be a graph and pick any $x \in X$. In this section we define the essential length of an element of $\pi(x)$ (Definition 4.3), and we use this concept to define a collection of subgroups $\pi(x, i)$ of $\pi(x)$ (Definition 4.4).

Definition 4.1. Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. Pick any path $p \in \psi(x)$, and write

$$p \quad = \quad (x = x_0, x_1, \ldots, x_l = x).$$

We say that p is **cyclically reduced** if $l = 0$ or if p is reduced with $x_1 \neq x_{l-1}$.

Lemma 4.2. *Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. Let p be any reduced element of $\psi(x)$. Then there exists a unique cyclically reduced closed path q and a unique path r such that*

$$p \quad = \quad rqr^{-1}.$$

Definition 4.3. Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. Pick any $u \in \pi(x)$ and write $\tilde{u} = pqp^{-1}$, where q is cyclically reduced. By the **essential length** of u, we mean the length of q.

Definition 4.4. Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. For every nonnegative integer i, let $\pi(x, i)$ denote the subgroup of $\pi(x)$ generated by the elements of essential length at most i.

We summarize some elementary results about these subgroups in the following lemma.

Lemma 4.5. *Let $\Gamma = (X, R)$ be a graph, and fix $x \in X$. Then*

(i) $\pi(x, i) \subseteq \pi(x, i + 1)$ *for every nonnegative integer i,*

(ii) $\pi(x, 0) = \pi(x, 1) = \pi(x, 2) = \{e\}$.

Recall that a graph $\Gamma = (X, R)$ is **connected** if for every $x, y \in X$ there exists a path from x to y. Let $\Gamma = (X, R)$ be a connected graph, and pick x, $y \in X$. By the **distance** $\partial(x, y)$, we mean the length of the shortest path in Γ from x to y. By the **diameter** of Γ we mean the maximal distance between any two vertices in X.

Theorem 4.6. *Let $\Gamma = (X, R)$ be a connected graph with diameter d. Fix any $x \in X$. Then $\pi(x, 2d + 1) = \pi(x)$.*

5. The intersection number $p^3_{12} \geq 2$ in any Q-Polynomial Distance-Regular Graph.

For the rest of the abstract, we restrict our attention to distance-regular graphs. In this section, we show that if a distance-regular graph Γ is Q-polynomial with diameter and valency at least three, then the intersection number p^3_{12} is at least two (Theorem 5.1); consequently, the girth is at most six (Corollary 5.3).

We shall begin this section by briefly reviewing the key definitions and basic results related to Q-polynomial distance-regular graphs. For general information about distance -regular graphs and the Q-polynomial property, see Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [2].

Let $\Gamma = (X, R)$ denote a connected graph of diameter $d \geq 1$. We say that Γ is **distance-regular** if for all integers h, i, j $(0 \leq h, i, j \leq d)$ and for all x, $y \in X$ with $\partial(x, y) = h$, the numbers

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|$$

depend only on h, i, j, and not on x or y. We call the p^h_{ij} the **intersection numbers** of Γ. Note that if Γ is distance-regular, then Γ is regular with valency $k := p^0_{11}$.

Let Γ be a distance-regular graph of diameter d. Let A_0, $A_1, \ldots, A_d$ denote the distance matrices for Γ. Then A_0, $A_1, \ldots, A_d$ form a basis for a commutative semi-simple -algebra M known as the **Bose-Mesner** algebra. The algebra M has a second basis E_0, $E_1, \ldots, E_d$ such that

$$\begin{aligned} E_0 + E_1 + \ldots + E_d &= I, \\ E_i E_j &= \delta_{ij} E_i && (0 \leq i, j \leq d), \\ E_0 &= \tfrac{1}{|X|} J, \\ E_i &= E_i^t && (0 \leq i \leq d), \end{aligned}$$

where I is the identity matrix and J is the all-1s matrix [2, Theorem 2.6.1]. We refer to E_0, $E_1, \ldots, E_d$ as the **primitive idempotents** of Γ.

By the **Krein parameters** of Γ (with respect to the above ordering E_0, $E_1, \ldots, E_d$ of the primitive idempotents), we mean the real scalars q^h_{ij} $(0 \leq h, i, j \leq d)$ such that

$$E_i \circ E_j \quad = \quad \frac{1}{|X|} \sum_{h=0}^{d} q^h_{ij} E_h \qquad (0 \leq i, j \leq d),$$

where $\circ$ denotes entry-wise matrix multiplication [2].

Suppose that E is a primitive idempotent of Γ. We say that E is a **Q-idempotent** if there exists an ordering E_0, $E = E_1, \ldots, E_d$ of the primitive idempotents of Γ such that the corresponding Krein parameters satisfy

$$q^i_{1j} = 0 \quad \text{if } |i-j| > 1 \quad (0 \le i,\, j \le d),$$
$$q^i_{1j} \ne 0 \quad \text{if } |i-j| = 1 \quad (0 \le i,\, j \le d).$$

We say that Γ is ***Q*-polynomial** if Γ has at least one Q-idempotent.

Let $\Gamma = (X, R)$ denote any distance-regular graph of diameter d, and let E denote any primitive idempotent of Γ. There exist real scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ such that

$$E \quad = \quad \frac{1}{|X|} \sum_{h=0}^{d} \theta^*_h A_h. \tag{1}$$

If E is a Q-idempotent of Γ, then we say that the sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ is a ***Q*-sequence.**

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d \ge 1$. By the **standard module** for Γ we mean the vector space $V = {}^X$ of column vectors, whose coordinates are indexed by X. We equip V with the inner product

$$\langle u, v \rangle \quad = \quad u^t v \qquad (u,\, v \in V).$$

For each vertex $x \in X$, let $\hat{x}$ denote the vector in V with a one in the x coordinate and zeros elsewhere. Observe that $\{\hat{x} \,|\, x \in X\}$ is an orthonormal basis for V.

Theorem 5.1. *Let $\Gamma = (X, R)$ be a Q-polynomial distance-regvlar graph with diameter $d \ge 3$ and valency $k \ge 3$. Then the intersection number $p^3_{12} \ge 2$.*

Definition 5.2. Let $\Gamma = (X, R)$ be a distance-regular graph of valency at least two. By the **girth** of Γ, we mean the minimal integer $i > 0$ such that there exists a cyclically reduced path $p \in \psi(x)$ of length i, where x is any vertex in X.

Corollary 5.3. *Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph such that the valency is at least three. Then the girth of Γ is at most six.*

6. Pseudoquotients.

Let $\Gamma = (X, R)$ denote a Q-polynomial distance-regular graph of diameter $d \ge 3$. In this section, we examine a property that Γ must satisfy if it is the quotient of a distance-regular antipodal graph of diameter $D \ge 7$. We use this property to define what it means for Γ to be a pseudoquotient (Definition 6.6).

Lemma 6.1. *(Leonard [3]) Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph of diameter $d \ge 3$. Suppose that $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ is a Q-sequence. Then there exists a unique real number λ such that*

$$\theta^*_{i-2} - \theta^*_{i-1} = \lambda(\theta^*_{i-3} - \theta^*_i) \qquad (3 \le i \le d).$$

Moreover, $\lambda \neq 0$.

Corollary 6.2. *(Leonard [3], Bannai and Ito [1, Theorem 5.1, p. 263]) Let* $\Gamma = (X, R)$ *be a Q-polynomial distance-regular graph of diameter* $d \geq 3$. *Let* $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ *be a Q-sequence for* Γ. *Then exactly one of the following occurs:*

$$
\begin{array}{lll}
\textit{Case (i)} & \theta_i^* = \theta_0^* + h^*(1-q^i)(1-s^*q^{i+1})q^{-i} & (0 \leq i \leq d), \quad (2) \\
\textit{Case (ii)} & \theta_i^* = \theta_0^* + h^* i(1+i+s^*) & (0 \leq i \leq d), \quad (3) \\
\textit{Case (iii)} & \theta_i^* = \theta_0^* + s^* i & (0 \leq i \leq d), \quad (4) \\
\textit{Case (iv)} & \theta_i^* = \theta_0^* + h^*\left(s^* - 1 + (1 - s^* + 2i)(-1)^i\right) & (0 \leq i \leq d), \quad (5)
\end{array}
$$

where q, h^*, s^* *are appropriate complex numbers.*

Let $\Gamma' = (X', R')$ be a distance-regular graph of diameter D. Define a relation $\approx$ on X' as follows: for all x, $y \in X'$, write $x \approx y$ whenever $x = y$ or $\partial(x, y) = D$. The graph Γ' is said to be **antipodal** whenever $\approx$ is an equivalence relation.

Suppose that Γ' is an antipodal distance-regular graph of diameter D, and let $\approx$ be as above. By the **quotient** of Γ', we mean the graph $\Gamma = (X, R)$ where

$$
\begin{aligned}
X &= \text{the set of equivalence classes of } \approx, \\
R &= \Big\{\{u, v\} \mid u,\ v \in X,\ \exists x \in u,\ \exists y \in v \text{ such that } \{x, y\} \in R'\Big\}.
\end{aligned}
$$

(For more information on antipodal distance-regular graphs, see Brouwer, Cohen, and Neumaier [2]).

Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph of diameter at least three. The following theorem gives a restriction that every Q-sequence of Γ satisfies if Γ is the quotient of an antipodal distance-regular graph.

Theorem 6.3. *(Terwilliger [7]) Let* $\Gamma = (X, R)$ *be a Q-polynomial distance-regular graph of diameter* $d \geq 3$. *Suppose that* Γ *is the quotient of an antipodal distance-regular graph of diameter* $D \geq 7$. *If* $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ *is a Q-sequence of* Γ, *then*

$$\theta_{i-2}^* - \theta_{i-1}^* = \lambda(\theta_{i-3}^* - \theta_i^*) \qquad (3 \leq i \leq D),$$

where λ *is as in Lemma 6.1, and where* $\theta_{d+1}^*, \theta_{d+2}^*, \ldots, \theta_D^*$ *are defined by*

$$\theta_i^* \quad := \quad \theta_{D-i}^* \qquad (d+1 \leq i \leq D).$$

The following lemma shows some conditions that are equivalent to the condition that appears in Theorem 6.3.

Lemma 6.4. *Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph with diameter $d \geq 3$. Let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ be a Q-sequence of Γ and let λ be as in Lemma 6.1. Then for all integers $D \in \{2d, 2d+1\}$, the following three conditions are equivalent:*

(i)

$$\theta_{i-2}^* - \theta_{i-1}^* = \lambda(\theta_{i-3}^* - \theta_i^*) \qquad (3 \leq i \leq D),$$

where $\theta_{d+1}^*, \theta_{d+2}^*, \ldots, \theta_D^*$ are defined by

$$\theta_i^* := \theta_{D-i}^* \qquad (d+1 \leq i \leq D).$$

(ii)

$$\theta_{d-1}^* - \theta_d^* = \lambda(\theta_{d-2}^* - \theta_{D-d-1}^*).$$

(iii) Referring to lines (2)–(5) in Corollary 6.2,

$$\begin{aligned} &\text{Case (i) occurs with } s^* = q^{-D-1}, \\ &\text{Case (ii) occurs with } s^* = -D-1, \\ \text{or}\quad &\text{Case (iv) occurs with } s^* = D+1, \text{ and } D \text{ is odd.} \end{aligned}$$

Lemma 6.5. *Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph with diameter $d \geq 3$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ be a Q-sequence of Γ. Suppose that conditions (i)–(iii) hold in Lemma 6.4 for some $D \in \{2d, 2d+1\}$. Then D is unique. In this case, we say that $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ is* ***D-symmetric***.

Definition 6.6. Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph of diameter $d \geq 3$. We say that Γ is a **pseudoquotient** if there exists $D \in \{2d, 2d+1\}$, with $D \geq 7$, such that every Q-sequence in D-symmetric. In this case we call D the **covering diameter** of Γ.

7. The Fundamental Group of a Q-polynomial Distance-Regular Graph.

We now present our main result.

Theorem 7.1. *Let $\Gamma = (X, R)$ be a Q-polynomial distance-regular graph of diameter $d \geq 3$ and valency $k \geq 3$. Fix any $x \in X$. Then the following hold.*

(i) $\pi(x, 6) \neq \{e\}$.

(ii) Suppose $\pi(x, 6) \neq \pi(x)$. Then Γ is a pseudoquotient. Furthermore,

$$\pi(x, 6) = \pi(x, D-1) \quad \pi(x, D) = \pi(x)$$

where D is the covering diameter of Γ.

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