RISC Software for Symbolic Computation in Combinatorics

István Nemes
RISC - Linz, J. Kepler University
A–4040 Linz, Austria
email: Istvan.Nemes@risc.uni-linz.ac.at

1 Introduction

In this talk we report on five program packages for the computer algebra system *Mathematica*, which were developed at RISC-Linz by a working group in computational combinatorics headed by Peter Paule. We illustrate the packages **RComp**, **zb_alg**, **qZeil**, **GeneratingFunctions** and **Karr** by presenting typical applications.

RComp was developed by the author in a joint work with M. Petkovšek [10]. It provides tools for computing with sequences, which satisfy a linear recurrence relation with constant coefficients.

GeneratingFunctions was developed by Ch. Mallinger [9]. One can consider this package as an extension of **RComp**, providing functions for computing with sequences that satisfy a linear recurrence relation with polynomial coefficients, furthermore for manipulating generating functions of such sequences. In its functionality the package corresponds to the **gfun** package in MAPLE [14].

P. Paule and M. Schorn developed **zb_alg**, which contains an efficient implementation of Zeilberger's algorithm for finding recurrences for hypergeometric sums [11]. The package also includes a reliable implementation of Gosper's algorithm.

There is an algorithmic analogue to **zb_alg** for handling q-hypergeometric sums; to do that in an efficient way is a nontrivial task. In the computer algebra system MAPLE Zeilberger [13], Koornwinder [8] provided such an implementation. A. Riese's implementation in *Mathematica* **qZeil** [12] presents a more user friendly implementation, which also accepts input of more general type. Analogously to **zb_alg** this package contains a q-version of Gosper's algorithm.

Karr's algorithm [7] for summation in finite terms is being implemented by K. Eichhorn [2]. This implementation wil be the first that treats Karr's algorithm in full detail. To emphasize the relevance of this work we note that the algorithm contains indefinite hypergeometric (Gosper) and q-summation as a special case.

2 Example Sessions

2.1 RComp

Let us consider the following encoding of the Fibonacci numbers which originally is due to I. Schur:

$$\sum_k (-1)^k \binom{n}{\lfloor (n-5k)/2 \rfloor}.$$

In order to find a recurrence relation for the sum, we eliminate the floor function by splitting the summand according to odd and even n and k, respectively. For the resulting summands Zeilberger's algorithm applies and **zb_alg** finds the corresponding recurrence relations for the subsums with the same (constant) coefficients $\{7, -13, 4\}$ and initial terms {1,2,6}, {0,0,-1}, {1,3,10}, {0,0,-2}, respectively. The termwise addition and interlace functions of **RComp** return in one stroke the recurrence relation for the original sum:

```
In[5]:= Interlace[ Rec[{7,-13,4},{1,2,6}]  + Rec[{7,-13,4},{0,0,-1}],
                   Rec[{7,-13,4},{1,3,10}] + Rec[{7,-13,4},{0,0,-2}]]

Out[5]= Rec[{1, 1}, {1, 1}],
```

which is indeed a defining relation for the Fibonacci numbers.

2.2 GeneratingFunctions

We demonstrate the functionality of **GeneratingFunctions** by computing a closed form for

$$s_n := \sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} (-1)^k / 2^k.$$

We apply Euler's transformation like it was used in [3], noticing that the ordinary generating function of $\langle s_n \rangle_{n \geq 0}$ is the Euler transform of $f(x) := \sum_{n \geq 0} u_n x^n$, where $u_n = \binom{2n}{n}/2^n$. First, we derive from the recurrence equation for u_n a differential equation for f:

```
In[1]:= <<GeneratingFunctions.m

Out[1]= GeneratingFunctions version 0.4 (01. 12. 1995) loaded.

In[2]:= RE2DE[{(n+1)u[n+1]-(2n+1)u[n] == 0,u[0]==1},u[n],f[x]]

Out[2]= {-f[x] + (1 - 2 x) f'[x] == 0, f[0] == 1}.
```

To get Euler's transform of f we compute the substitution $x \mapsto x/(x-1)$ and then multiply the result by $1/(1-x)$:

```
In[3]:= AlgebraicCompose[ -f[x] + (1 - 2 x) f'[x], f == x/(x-1),f[x]]

                     2
Out[3]= f[x] + (1 - x ) f'[x] == 0


In[4]:= DECauchy[ f[x]+(1-x^2) f'[x]==0,  f[x]==1/(1-x), f[x] ]

                          2
Out[4]= x f[x] + (-1 + x ) f'[x] == 0.
```

Finally, the differential equation of the transform is converted to a recurrence relation, from which a closed form can be easily read off:

```
In[5]:= DE2RE[x f[x] + (1 -  x^2 ) f'[x] == 0,f[x],s[n]]


Out[5]= (1 - n) s[n] + (2 + n) s[2 + n] == 0.
```

2.3 zb_alg

To illustrate the implementation **zb_alg** of Zeilberger's fast algorithm [13], we refer to [11] for a variety of interesting examples. Now, we consider Problem 10424 of the American Mathematical Monthly [4]. The solution found jointly with P. Paule demonstrates, that **zb_alg** handles the case of non standard boundaries, which leads to a non-homogeneous recurrence relation. The problem is to evaluate:

$$\mathrm{SUM}(n) := \sum_{0 \le k \le n/3} 2^k \frac{n}{n-k} \binom{n-k}{2k}.$$

Noticing that $\mathrm{SUM}(n)$ does not change when the summation is extended for $n/3 < k \le n-1$, one sees that Zeilberger's algorithm applies:

```
In[4]:= Zb[ 2^k n/(n-k) Binomial[n-k,2k],{k,0,n-1},n,3]
If '-1 + n' is a natural number, then:

Out[4]= {-2 SUM[n] + SUM[1 + n] - 2 SUM[2 + n] + SUM[3 + n] ==

       n
>   (2  (-3 + n) (-2 + n) (-1 + n) (-5 + 2 n) (-3 + 2 n) (-1 + 2 n)

                          2       3
>   (18 + 63 n + 53 n  + 10 n ) Binomial[10, 6 + 2 (-1 + n)]) / 113400}.
```

Observing that the right side vanishes for positive integer n we get an easily solvable recurrence relation for $\mathrm{SUM}(n)$.

2.4 qZeil

Andrews [1] gave a detailed account on conjectures recently raised by P. Borwein. All are related to partition theory, and the one stated most easily is:
Define polynomials $A_n(q)$,$B_n(q)$, and $C_n(q)$ by

$$\prod_{j=1}^{n}(1-q^{3j-2})(1-q^{3j-1}) = A_n(q^3) - q\,B_n(q^3) - q^2\,C_n(q^3),$$

then each of these polynomials has nonnegative coefficients.
Andrews presented the polynomials in question in terms of sums over Gaussian polynomials, for instance ([1], (3.4)):

$$A_n(q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(9k+1)/2} \begin{bmatrix} 2n \\ n+3k \end{bmatrix};$$

the other representations are of similar form. In addition, he derived three recurrences relating the polynomials in the following way, for instance ([1] (3.1)):

$$A_n(q) = (1+q^{2n-1})A_{n-1}(q) + q^n B_n(q) + q^n C_{n-1}(q);$$

the other representations are also of mixed type, but involving *negative* signs. Thus, one could raise the question whether a recurrence involving only polynomials of *one* type would add some further insight to the problem.

Using `qZeil`, the task of deriving a recurrence of the type specified above is pure routine:

```
In[1]:= <<qZeil.m

Out[1]= Axel Riese's q-Zeilberger implementation version 1.4
        loaded
```

```
In[2]:= Timing[
        qZeil[(-1)^k q^(9/2 k^2+1/2 k) qBinomial[2n,n+3k,q],
            {k, -Infinity, Infinity}, n, 3]]

Out[2]= {123.65 Second, SUM[n] ==

   3       -8 + 3 n         -7 + 3 n                          2
>  q  (1 - q         ) (1 - q         ) SUM[-3 + n] + (1 + q + q )

     4    2 n    2 + n    4    2 n    2 + n
   (-q  - q    + q     ) (q  + q    + q     ) SUM[-2 + n]
>  ------------------------------------------------------ +
                                7
                               q

              2         -3 + 2 n
>  (1 + q + q ) (1 + q         ) SUM[-1 + n]}.
```

We want to remark that M. Hirschhorn [6] independently came up with this recurrence by hand computation — spending nontrivial human effort.

2.5 Karr

Besides indefinite hypergeometric (Gosper) and q-hypergeometric summation, Karr's algorithm is capable to handle also non hypergeometric extensions of the field of rational functions. The last example provides a q-hypergeometric summation.

The following two problems are taken from [5] (Problems 6.53 and 6.67), where H_k denotes the k-th harmonic number.

- Find a closed form for $\sum_{k=0}^{m} \binom{n}{k}^{-1} (-1)^k H_k$, when $0 \le m \le n$.
- Find A closed form for $\sum_{k=1}^{n} k^2 H_{n+k}$.

Concerning the second problem, the authors of [5] explicitly state the desire to automate the derivation of formulas of such type. Eichhorn's implementation of Karr's algorithm does this job as follows:

```
In[1]:= <<k.m
 -- Karr Summation Package V 0.6.2 loaded. --

In[2]:= Karr[ 1/Binomial[n,k] (-1)^k H[k], {k,0,m}]

              1 + n
Out[2]= -(--------) +
                  2
          (2 + n)
```

```
            m
       (-1)  (1 - m + n + 2 H[m] + 2 m H[m] + n H[m] + m n H[m])
>      ---------------------------------------------------------
                                2
                         (2 + n)  Binomial[n, m]
```

```
In[3]:= Karr[(k-n)^2 Sum[1/j,{j,1,k}],{k,n+1,2n}]
```

```
                                                      2            2
         -1 - n    n (1 + n)     2             (1 + n)     n (1 + n)
Out[3]= ------ + --------- + n (1 + n) - -------- - ----------+
          36         2                          12           2

          3                                         2
   (1 + n)     1 + n              2             (1 + n)             2
> -------- - (----- + n (1 + n) + n  (1 + n) - -------- - n (1 + n)  +
     9          6                                   2

          3
   (1 + n)         1
> --------)    (----- + H[n]) +
      3          1 + n

             2       3                          2             3
   2 n - 24 n  - 32 n  + 12 n H[2 n] + 36 n  H[2 n] + 24 n  H[2 n]
> -----------------------------------------------------------------.
                                   36
```

This output easily simplifies to the solution given in [5].

We conclude by the following simple q-hypergeometric summation:

```
In[4]:= Karr[ qBinomial[m+k,k] q^k,k,{{k,1,1},{q^k,q,0},
               {qBinomial[m+k,k], (1-t[2] q^(m+1))/(1-t[2] q),0}}].
```

The tupel $\{k,1,1\}$ represents the difference field extension of type $\sigma t[0] = 1 \cdot t[0] + 1$, the tupel $\{q^k, q, 0\}$ the extension of type $\sigma t[1] = q \cdot t[1] + 0$, the last tupel the extension of type $\sigma t[2] = (1 - t[2]\, q^{(m+1)})/(1 - t[2]q) \cdot t[2] + 0$.

The system returns

```
                 k
        (-1 + q ) qBinomial[k + m, k]
Out[4]= -----------------------------,
                     1 + m
                -1 + q
```

which is the expected answer.

References

[1] G.E. Andrews: On a Conjecture of Peter Borwein. Preprint 1994. To appear in *J. Symbolic Computation.*

[2] K. Eichhorn: *A Mathematica Implementation of Karr's Algorithm for Summation in Finite Terms.* Dipl. Thesis in preparation, RISC-Linz, J. Kepler University Linz.

[3] P. Flajolet, B. Salvy: Computer Algebra Libraries for Combinatorial Structures. To appear in *J. Symbolic Comput.*

[4] I. Gessel: Problem 10424. *Am. Math. Monthly* 102:70, 1995.

[5] R. L. Graham, D. E. Knuth and O. Patashnik: *Concrete Mathematics - A foundation for computer science.* 2nd ed. Addison-Wesley, Reading, Massachusetts, 1994.

[6] M. Hirschhorn: E-mail correspondence with P. Paule, February, 1995.

[7] M. Karr: Summation in Finite Terms. *Journal of the ACM.* 28:305–350, 1981.

[8] T. Koornwinder: On Zeilberger's Algorithm and it's q-analogue. *J. of Comput. and Appl. Math.* 48:91-111, 1993.

[9] Ch. Mallinger: *A Mathematica Package for Manipulating Holonomic Functions.* Dipl. Thesis in preparation, RISC-Linz, J. Kepler University Linz.

[10] I. Nemes, M. Petkovšek: `RComp`: A *Mathematica* Package for Computing with Recursive Sequences. To appear in *J. Symbolic Computation.*

[11] P. Paule, M. Schorn: A Mathematica version of Zeilberger's Algorithm for Proving Binomial Coefficient Identities. To appear in *J. Symbolic Computation.*

[12] P. Paule, A. Riese: A Mathematica q-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping. To appear in *Fields Proceedings of the Workshop "Specila Functions, q-Series and Related Topics* (June 1995. Toronto).

[13] M. Petkovšek, H.S. Wilf, and D. Zeilberger: *"A=B".* A. K. Peters, 1996.

[14] B. Salvy, P. Zimmermann: GFUN: a Maple Package for the Manipulation of Generating and Holonomic Functions in one Variable. *ACM Transactions of Mathematical Software* 20:163–177, 1994.