Oscillating Tableaux, $S_m \times S_n$-modules, and Robinson-Schensted-Knuth correspondence

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1 Introduction

The Robinson-Schensted-Knuth correspondence (RSK, see [8] and Corollary 2.5 below) is a bijection between pairs of semi-standard Young tableaux of the same shape with fixed weights and matrices with nonnegative integer entries with prescribed column and row sums. This correspondence plays an important role in the representation theory of the symmetric group and general linear groups, and in the theory of symmetric functions.

It is possible (see [2, 3, 4, 5, 10]) to construct an analogue of the RSK for oscillating tableaux, i.e., sequences of Young diagrams $\alpha = (\alpha_{(0)}, \ldots, \alpha_{(k)})$ such that each $\alpha_{(i)}$ and $\alpha_{(i+1)}$ differ by a horizontal strip.

We present a new approach to the RSK correspondence for oscillating tableaux. First, we show that the number of oscillating tableaux of a given weight and shape is equal to the multiplicity of the corresponding irreducible representation in a certain naturally defined $S_m \times S_n$-module. This allows us to recover the enumerative results from [4, 10, 11, 12] (see Section 4). In Section 5, we extend this construction to oscillating supertableaux. In Section 6, we discuss commutation relations for the operators which add or delete horizontal or vertical strips (cf. [5, 6]) and give a generalization of these relations.

In Section 7, we introduce a piecewise-linear analogue of RSK for oscillating tableaux in the spirit of [1]. We construct a continuous piecewise-linear map which establishes a bijection between two convex polyhedra. The restriction of this map to integer points gives the RSK correspondence for oscillating tableaux.

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2 Oscillating tableaux

First, recall several basic definitions from combinatorics of Young diagrams (see [9]).

Let $\lambda = (\lambda_1 \geq \ldots \lambda_l > 0)$ be a *partition* of an integer $n = |\lambda| = \sum \lambda_i$. With the partition λ one can associate its *Young* or *Ferrer diagram* which is the set of pairs $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \lambda_i$, $i = 1, 2, \ldots, l$. The poset of all Young diagrams ordered by inclusion is called the *Young lattice*. The *Young graph* is the Hasse diagram of this poset, in other words, we connect two diagrams if their difference consists of one cell. Let "$\supset$" be the partial order on $\mathcal{P}$ by inclusion of Young diagrams. For $\lambda \supset \mu$, a *skew Young diagram* λ/μ is the set-theoretic difference of the Young diagrams corresponding to λ and μ. A *semi-standard Young tableau* (also called *column-strict tableau*) of *shape* λ/μ is a map from λ/μ to nonnegative integers strongly increasing along the columns and weakly increasing along the rows of λ/μ. For example, a semi-standard tableau of shape λ/μ, $\lambda = (6, 4, 4, 1)$, $\mu = (3, 2)$ is given below.

			1	2	3
		1	2		
1	2	2	4		
3					

The weight of a tableau is the sequence $\beta = (\beta_1, \ldots, \beta_k)$ such that each i appears in the tableau exactly β_i times. The weight of the tableau in our example is equal to $\beta = (3, 4, 2, 1)$. A tableau is called *standard* if it has weight $\beta = (1, \ldots, 1)$, i.e., each number appears once.

A *horizontal* (resp., *vertical*) *m-strip* is a skew diagram λ/μ consisting of m cells such that each row (resp., column) contains at most one cell of λ/μ.

We can view a semi-standard tableau as a sequence of partitions $\lambda = \alpha_{(0)} \supset \alpha_{(1)} \supset \cdots \supset \alpha_{(k)} = \mu$ such that $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-strip for $i = 1, \ldots, k$. In other words, a semi-standard tableau is a path in a certain graph $\mathcal{Y}$. The vertices of $\mathcal{Y}$ are Young diagrams and diagrams λ and μ are connected by an edge in $\mathcal{Y}$ if λ/μ or μ/λ is a horizontal m-strip for some $m \geq 0$. We call $\mathcal{Y}$ the *extended Young graph* because it is obtained from the Young graph by adding edges connecting non-adjacent levels. It is clear that Young tableaux correspond to decreasing paths in the graph $\mathcal{Y}$. An oscillating tableau is an arbitrary path in $\mathcal{Y}$.

Definition 2.1 *Let λ, μ be partitions and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$. An oscillating tableau α of shape (λ, μ) and weight β is a sequence of partitions $(\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(k)})$, $\alpha_{(0)} = \lambda, \alpha_{(k)} = \mu$, such that for all $i = 1, 2, \ldots, k$ the following conditions hold:*

1. *If $\beta_i \geq 0$ then $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-strip,*
2. *If $\beta_i < 0$ then $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-strip.*

By $OT(\lambda, \mu, \beta)$ we will denote the set of all oscillating tableaux of shape (λ, μ) and weight β. If $\beta_i = \pm 1$ for all i then an oscillating tableau of weight β is called *standard.* Clearly, standard oscillating tableaux correspond to paths in the Young graph.

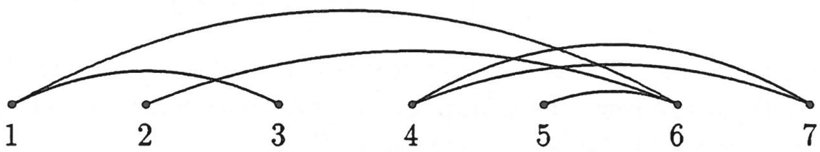
An analogous definition was given in [10, Definition 4.4.1]. Standard oscillating tableaux were earlier considered in [12, Definition 8.1].

Definition 2.2 *Let $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k$ be a sequence such that $\sum_i \beta_i = 0$. An intransitive graph of type β is an oriented graph Γ on the vertices $1, 2, \ldots, k$ (multiple edges allowed) such that:*

1. *If (i, j) is an edge of Γ then $i < j$.*
2. *If $\beta_i \geq 0$ then in-degree of i is β_i and out-degree of i is 0.*
3. *If $\beta_i \leq 0$ then out-degree of i is $-\beta_i$ and in-degree of i is 0.*

Denote by $G(\beta)$ the set of all intransitive graphs of type β.

For example, a graph from $G(-2, -1, 1, -2, -1, 3, 2)$ is shown below.



Theorem 2.3 *Let $\beta \in \mathbb{Z}^k$, $\sum_i \beta_i = 0$. Then the number of oscillating tableaux of shape $(\hat{0}, \hat{0})$ and weight β is equal to the number of intransitive graphs of type β*

$$|OT(\hat{0}, \hat{0}, \beta)| = |G(\beta)|.$$

Here $\hat{0}$ denotes a unique partition of 0.

For example, it is not difficult to check that $|OT(-2, -1, 1, -2, -1, 3, 2)| = |G(-2, -1, 1, -2, -1, 3, 2)| = 12$.

This theorem in slightly different notation was proven by T. W. Roby [10, Theorem 4.4.3] who generalized S. Fomin's results [3, 4, 5]. Roby constructed a bijection between the two sets in Theorem 2.3. The following special case was earlier found in [12, Lemma 8.3].

Corollary 2.4 *The number of paths in the Young graph from $\hat{0}$ to $\hat{0}$ of length $2k$ is equal to $(2k-1)!! = (2k-1)(2k-3)\dots 1$.*

For weight $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ such that $\beta_1, \dots, \beta_m \le 0$, $\beta_{m+1}, \dots, \beta_k \ge 0$, Roby's result yields the classical Robinson-Schensted-Knuth correspondence [8].

Corollary 2.5 *Let $\beta \in \mathbb{Z}_+^m$ and $\delta \in \mathbb{Z}_+^n$. Then the number of pairs (P, Q) of Young tableaux of the same shape and with weights β and δ respectively is equal to the number of $m \times n$-matrices $A = (a_{ij})$ such that*

1. *All entries a_{ij} are nonnegative integers;*
2. *$\sum_j a_{ij} = \beta_i$ for $i = 1, 2, \dots, m$;*
3. *$\sum_i a_{ij} = \delta_j$ for $j = 1, 2, \dots, n$.*

3 $S_m \times S_n$-module $M(m, n, \beta)$

In this section we study a permutational representation of $S_m \times S_n$ on the linear space generated by intransitive graphs. Multiplicities of irreducible components in this representation are given by the numbers of oscillating tableaux.

Let $m, n \in \mathbb{Z}_+$, $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{Z}^k$ such that $m - n = \sum_i \beta_i$; $N = m + k + n$; and let $G(m, n, \beta)$ be the set of intransitive graphs of type $\delta = (\delta_1, \delta_2, \dots, \delta_N)$, where

$$\delta_i = \begin{cases} -1 & \text{for } i = 1, \dots, m, \\ \beta_{i-m} & \text{for } i = m+1, \dots, m+k, \\ 1 & \text{for } i = m+k+1, \dots, N. \end{cases}$$

The direct product of two symmetric groups $S_m \times S_n$ acts on the graphs $\Gamma \in G(m, n, \beta)$ as follows: the group S_m permutes the first m vertices in Γ and the group S_n permutes the last n vertices in Γ.

Let $M(m, n, \beta)$ be the vector space over $\mathbb{C}$ with basis $\{v_\Gamma\}$, $\Gamma \in G(m, n, \beta)$. There is a natural action of the product for two symmetric groups $S_m \times S_n$ on $M(m, n, \beta)$ given by $w : v_\Gamma \mapsto v_{w^{-1}\Gamma}$, for $w \in S_m \times S_n$. Thus we have a structure of $S_m \times S_n$-module on $M(m, n, \beta)$.

Denote by π_λ the irreducible S_m-module associated with a partition $\lambda \vdash m$ (see [7, 9]). Every irreducible representation of the group $S_m \times S_n$ is of the form $\pi_\lambda \otimes \pi_\mu$, where $|\lambda| = m$ and $|\mu| = n$.

Theorem 3.1 *The multiplicity of $\pi_\lambda \otimes \pi_\mu$ in $M(m, n, \beta)$ is equal to the number of oscillating tableaux with shape (λ, μ) and weight β, i.e.,*

$$M(m, n, \beta) \simeq \bigoplus |OT(\lambda, \mu, \beta)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the direct sum is over all partitions $\lambda \vdash m$ and $\mu \vdash n$.

Consider several examples:
1. Clearly, Theorem 2.3 is a special case of Theorem 3.1 for $m = n = 0$.
2. Let $m = n$ and let $\beta = \emptyset$ be the empty sequence. Then graphs from $G(n, n, \emptyset)$ can be identified with permutations in S_n. In this case $M(n, n, \emptyset)$ is the the group algebra $\mathbb{C}[S_n]$ viewed as an $S_n \times S_n$-module on which one copy of S_n acts by left multiplication and the other copy of S_n acts by right multiplication. Theorem 3.1 gives the following well-known identity.

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} \pi_\lambda \otimes \pi_\lambda. \tag{1}$$

3. Let $n = 0$ and $\beta_i \geq 0$ for all $i = 1, 2, \ldots, k$. Then a graph $\Gamma \in G(m, 0, \beta)$ can be identified with the word $w = w_1 w_2 \ldots w_m$ such that the vertex j is connected with $w_j + m$ in Γ, $j = 1, \ldots, m$. For $i = 1, \ldots, k$, the word w has β_i i's. The symmetric group S_m acts on such words w by permuting the letters w_i. The representation $M_\beta = M(m, 0, \beta)$ is a well-known *monomial representation*, see [7], i.e., $M_\beta = \mathrm{Ind}_{S_{\beta_1} \times \ldots \times S_{\beta_k}}^{S_m} \mathbf{1}$ (induced from a parabolic subgroup in S_m). By Theorem 3.1 we get

$$M_\beta = M(m, 0, \beta) = \bigoplus_{\lambda \vdash m} \mathrm{Tab}(\lambda, \beta) \cdot \pi_\lambda, \tag{2}$$

where $\mathrm{Tab}(\lambda, \beta)$ is the number of semi-standard tableaux of shape λ and weight β. This is the classical *Young's rule* for decomposition of the monomial representation M_β, see [7, 9].

In order to prove Theorem 3.1 one can check it first for β of length 1 (using Pieri's rule) and then deduce the general statement by induction on the length of β.

4 Combinatorial theorem

A sequence $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{Z}^k$ is called *normal* if $\tau_1, \tau_2, \ldots, \tau_i > 0$; $\tau_{i+1} = \ldots = \tau_l = 0$; $\tau_{l+1}, \ldots, \tau_k < 0$ for some $0 \leq i \leq i \leq l$. For a sequence $\beta \in \mathbb{Z}^k$, let $\mathrm{nor}(\beta)$ be the normal sequence obtained from β by shifting all positive entries of β to the beginning and all negative entries to the end. For example, $\mathrm{nor}(0, -3, 1, -1, 0, -2, 0, 1, 3) = (1, 1, 3, 0, 0, 0, -3, -1, -2)$.

For $\beta, \delta \in \mathbb{Z}^k$, the notation $\delta \prec \beta$ means that δ_i is between 0 and β_i for all $i = 1, \ldots, k$, i.e., $0 \leq \delta_i \leq \beta_i$ or $0 \geq \delta_i \geq \beta_i$.

It is not difficult to deduce the following result from Theorem 3.1.

Theorem 4.1 *Let λ, μ be partitions, $\beta \in \mathbb{Z}^k$. Then*

$$|OT(\lambda, \mu, \beta)| = \sum |G(\delta)| \cdot |OT(\lambda, \mu, \mathrm{nor}(\beta - \delta))|,$$

where the sum is over all $\delta \in \mathbb{Z}^k$ such that $\sum_i \delta_i = 0$ and $\delta \prec \beta$.

An analogous result but in different notation was obtained in [10, Theorem 4.4.6]. Clearly, Theorem 2.3 is a special case of Theorem 4.1 for $\lambda = \mu = \hat{0}$.

It is possible (see [10]) to construct a bijection $\Phi_{\lambda\mu\beta}$ between the two sets in Theorem 4.1. This construction is based on certain local operations (see Section 6).

We complete this section with an example. Let $\lambda = (3)$, $\mu = (2,1)$ and $\beta = (-2,-1,3)$. Then we have $|OT(\lambda,\mu,\beta)| = 6$, $|OT(\lambda,\mu,(3,-2,-1))| = 1$, $|OT(\lambda,\mu,(2,-1,-1))| = 2$, $|OT(\lambda,\mu,(2,0,-2))| = 1$, $|OT(\lambda,\mu,(1,0,-1))| = 1$, and $|OT(\lambda,\mu,(0,0,0))| = 0$. Theorem 4.1 implies that $6 = 1\cdot1+1\cdot2+1\cdot1+1\cdot1+1\cdot1+1\cdot0$.

5 Superanalogue

In this section we outline "superanalogues" of the definitions and theorems from Sections 2–4.

Let $\beta \in \mathbb{Z}^k$, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_k) \in \{1,-1\}^k$. By β^ε we will denote the sequence $b = (b_1,b_2,\ldots,b_k)$ in the alphabet $\{m, \overline{m} \mid m \in \mathbb{Z}\}$ such that $b_i = \beta_i$ if $\varepsilon_i = 1$ and $b_i = \overline{\beta}_i$ if $\varepsilon_i = -1$.

Definition 5.1 *Let λ,μ be partitions. An oscillating supertableau of shape (λ,μ) and weight $b = \beta^\varepsilon$ is a sequence of partitions $(\alpha_{(0)},\alpha_{(1)},\ldots,\alpha_{(k)})$, $\alpha_{(0)} = \lambda, \alpha_{(k)} = \mu$, such that for all $i = 1,2,\ldots,k$ the following conditions hold.*

1. *If $\varepsilon_i = 1$ then (a) for $\beta_i \geq 0$ we have $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-strip;*

   (b) for $\beta_i < 0$ we have $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-strip;

2. *If $\varepsilon_i = -1$ then (a) for $\beta_i \geq 0$ we have $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a vertical β_i-strip;*

   (b) for $\beta_i < 0$ we have $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a vertical $(-\beta_i)$-strip.

The set of all oscillating supertableaux of shape (λ,μ) and weight $b = \beta^\varepsilon$ is denoted by $OST(\lambda,\mu,b)$.

Definition 5.2 *Let $\delta \in \mathbb{Z}^k$ and $\epsilon = (\epsilon_1,\epsilon_2,\ldots,\epsilon_k) \in \{1,-1\}^k$. An intransitive graph of type $d = \delta^\epsilon$ is an oriented graph Γ on the set of vertices $\{1,2,\ldots,k\}$ satisfying the conditions 1–3 of Definition 2.2 and also the condition:*

4. If $\epsilon_i \neq \epsilon_j$ then Γ contains at most one edge (i,j).

Let $SG(\delta^\epsilon)$ be the set of all such graphs.

Let $m,n \in \mathbb{Z}_+$, $\beta = (\beta_1,\ldots,\beta_k) \in \mathbb{Z}^k$, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_k) \in \{1,-1\}^k$, $b = \beta^\varepsilon$, and $\psi,\omega \in \{1,-1\}$. Let $SG(m^\psi, n^\omega, \beta^\varepsilon)$ be the set of intransitive graphs of type

$d = (d_1, d_2, \ldots, d_N)$, where $N = m + k + n$ and

$$d_i = \begin{cases} -1 & \text{if } \psi = 1,\ i = 1, \ldots, m, \\ -\overline{1} & \text{if } \psi = -1,\ i = 1, \ldots, m, \\ b_{i-m} & \text{for } i = m+1, \ldots, m+k, \\ 1 & \text{if } \omega = 1,\ i = m+k+1, \ldots, N, \\ \overline{1} & \text{if } \omega = -1,\ i = m+k+1, \ldots, N. \end{cases}$$

Consider the formal variables $x_{ij}, 1 \leq i < j \leq k$ with relations $x_{ij} x_{lm} = (-1)^{\sigma_{ij}\sigma_{lm}} x_{lm} x_{ij}$, where

$$\sigma_{ij} = \begin{cases} 0 & \epsilon_i = \epsilon_j, \\ 1 & \epsilon_i \neq \epsilon_j. \end{cases}$$

In other words, the x_{ij} with $\sigma_{ij} = 0$ are even variables and the x_{ij} with $\sigma_{ij} = 1$ are odd variables.

Let m_Γ denote the product of x_{ij}'s over all edges (i,j) of a graph Γ; and let $M(m^\psi, n^\omega, \beta^\varepsilon)$ be the linear span of m_Γ for $\Gamma \in SG(m^\psi, n^\omega, \beta^\varepsilon)$.

The group $S_m \times S_n$ acts on this space, cf. Section 3. The symmetric group S_m permutes the first index i of variables x_{ij} with $i = 1, 2, \ldots, m$ and S_n permutes the second index j of variables x_{ij} with $j = m+k+1, \ldots, m+k+n$.

For a partition $\lambda \in \mathcal{P}$ and $\psi \in \{1, -1\}$, let $\lambda^\psi = \lambda$ if $\psi = 1$ and $\lambda^\psi = \lambda'$ (the conjugate partition) if $\psi = -1$. Now we can present a superanalogue of Theorem 3.1.

Theorem 5.3

$$M(m^\psi, n^\omega, \beta^\varepsilon) \simeq \bigoplus |OST(\lambda^\psi, \mu^\omega, \beta^\varepsilon)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the direct sum is over all partitions $\lambda \vdash m$ and $\mu \vdash n$.

The following example is an odd analogue of (1). Let $\beta^\varepsilon = \emptyset$ be the empty sequence, $m = n$, and $\psi = -\omega = 1$. Then $\mathrm{Alt}_n := M(n, \overline{n}, \emptyset)$ is the representation of $S_n \times S_n$ on the group algebra $\mathbb{C}[S_n]$ such that for $(\sigma, \pi) \in S_n \times S_n$ and $f \in \mathbb{C}[S_n]$, $(\sigma, \pi) \cdot f = \mathrm{sgn}(\sigma\pi^{-1})\, \sigma f \pi^{-1}$. By Theorem 5.3, we have

$$\mathrm{Alt}_n = \sum_{\lambda \vdash n} \pi_\lambda \otimes \pi_{\lambda'}.$$

It is not difficult to deduce this formula directly from definition of Alt_n.

We can give a "superanalogue" of Theorem 4.1. Let $b = (b_1, b_2, \ldots, b_k) = \beta^\varepsilon$; and let $\mathrm{nor}(b)$ denote the word obtained from the word $b = (b_1, b_2, \ldots, b_k)$ by shifting all negative entries to the beginning and all positive entries to the end. For example, $\mathrm{nor}(0, \overline{3}, -1, \overline{1}, 0, 2, \overline{0}, -\overline{1}, -3) = (-1, -\overline{1}, -3, 0, 0, \overline{0}, \overline{3}, \overline{1}, 2)$.

Theorem 5.4 *Let $\lambda, \mu \in \mathcal{P}$ be partitions, $\beta \in \mathbb{Z}^k$, $\varepsilon \in \{1, -1\}^k$. Then*

$$|OST(\lambda, \mu, \beta^\varepsilon)| = \sum_{\delta \prec \beta} |SG(\delta^\varepsilon)| \cdot |OST(\lambda, \mu, \mathrm{nor}((\beta - \delta)^\varepsilon))|.$$

One way to prove this theorem is to deduce it from Theorem 5.3. It is also possible to construct a bijection $\Phi^{super}_{\lambda\mu b}$ between the two set from Theorem 5.4 using the operations ψ_3 and ψ_4 from Section 6 below.

For $\lambda = \mu = \hat{0}$ Theorem 5.4 implies the following result.

Corollary 5.5 *Let $\beta \in \mathbb{Z}^k$ and $\varepsilon \in \{1,-1\}^k$. Then the number of oscillating tableaux of shape $(\hat{0},\hat{0})$ and weight $b = \beta^\varepsilon$ is equal to the number of intransitive graphs of type b*

$$|OST(\hat{0},\hat{0},b)| = |SG(b)|.$$

Corollary 5.6 *Let $\beta \in \mathbb{Z}_+^m$ and $\delta \in \mathbb{Z}_+^n$. Then the number of pairs of tableaux (P,Q) with conjugated shapes and with weights β and δ respectively is equal to the number of $n\times m$-matrices satisfying the conditions 1–3 of Corollary 2.5 with all entries equal to 0 or 1.*

Knuth [8] also constructed a variant of RSK which gives a bijection between the set of $m\times n$-matrices and the set of pairs of tableaux (P,Q) from Corollary 5.6. In this case the bijection $\Phi^{super}_{\lambda\mu b}$ coincides with Knuth's correspondence.

6 Local operators

Let $n \in \mathbb{Z}_+$. Consider the operators $I(n), I(\overline{n}), D(n), D(\overline{n})$ in the space of formal linear combinations of partitions such that $I(n)$ (respectively, $I(\overline{n})$) deletes a horizontal (respectively, vertical) n-strip and $D(n)$ (respectively, $D(\overline{n})$) adds a horizontal (respectively, vertical) n-strip. These operators were considered by I. Gessel in [6].

Let $b \in \{n, \overline{n} \mid n \in \mathbb{Z}\}$; and let $A(b)$ denote the operator $I(b)$ if $b \geq 0$ or the operator $D(-b)$ if $b \leq 0$. Then

$$A(b_k)A(b_{k-1})\cdots A(b_1)(\lambda) = \sum_\mu |OST(\lambda, b, \mu)|\,\mu.$$

In the following theorem $[x,y]$ denotes the usual commutator $xy - yx$.

Theorem 6.1 *Let $m, n \in \mathbb{Z}_+$. The following commutation relations hold:*

1. $[I(m), I(n)] = [I(\overline{m}), I(\overline{n})] = [D(m), D(n)] = [D(\overline{m}), D(\overline{n})] = 0.$
2. $[I(m), I(\overline{n})] = [D(m), D(\overline{n})] = 0.$
3. $[I(m+1), D(n+1)] = I(m)D(n)$, $[I(\overline{m+1}), D(\overline{n+1})] = I(\overline{m})D(\overline{n}).$
4. $[I(m+1), D(\overline{n+1})] = D(\overline{n})I(m)$, $[I(\overline{m+1}), D(n+1)] = D(n)I(\overline{m}).$

Clearly, this theorem follows from

Proposition 6.2 *Let $m, n \geq 1$. There exist bijections between the following sets*

1. $\psi_1 : YT(\lambda/\nu, (m,n)) \rightarrow YT(\lambda/\nu, (n,m))$,

2. $\psi_2 : ST(\lambda/\nu, (m,\overline{n})) \rightarrow ST(\lambda/\nu, (\overline{n},m))$,

3. $\psi_3 : OT(\lambda, \nu, (-m,n)) \rightarrow \coprod_{0\leq k\leq \min(m,n)} OT(\lambda, \nu, (n-k, -m+k))$,

4. $\psi_4 : OST(\lambda, \nu, (-m,\overline{n})) \rightarrow \coprod_{k=0,1} OST(\lambda, \nu, (\overline{n-k}, -m+k))$.

Here $YT(\lambda/\nu, \beta)$ and $ST(\lambda/\nu, \beta)$ denote the sets of Young tableaux and supertableaux, resp., of weight β

It is not difficult to construct these four bijections. As an example, we will show how to construct the bijection ψ_3 (see [6]).

Let $\alpha = (\lambda, \mu, \nu) \in OT(\lambda, \mu, (-m,n))$, $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, and $\nu = (\nu_1, \nu_2, \ldots)$. In the following diagram an arrow $x \rightarrow y$ denotes the inequality $x \geq y$.

$$\begin{array}{cccccccccccccccccc}
 & & \lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \cdots & \lambda_i & & & & \lambda_{i+1} & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & \searrow & & \nearrow & & \\
\mu_1 & & & & \mu_2 & & & & \mu_3 & & & \cdots & & & \mu_{i+1} & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & \nearrow & & \searrow & & \\
 & & \nu_1 & & & & \nu_2 & & & & \nu_3 & \cdots & \nu_i & & & & \nu_{i+1} & \cdots
\end{array}$$

Let $a_i = \min(\lambda_i, \nu_i)$ and $b_i = \max(\lambda_{i+1}, \nu_{i+1})$, $i = 1, 2 \ldots$. Set $\widetilde{\mu}_i = a_i + b_i - \mu_{i+1}$, $i = 1, 2, \ldots$ and $k = \mu_1 - \min(\lambda_1, \nu_1)$. Clearly, $0 \leq k \leq \min(n,m)$. Now $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ is a partition and $\widetilde{\alpha} = (\lambda, \widetilde{\mu}, \nu) \in OT(\lambda, \mu, (n-k, -m+k))$. Define $\psi_3 : \alpha \mapsto \widetilde{\alpha}$. Then ψ_3 gives a bijection between the sets $OT(\lambda, \mu, (-m,n))$ and $\coprod_k OT(\lambda, \mu, (n-k, -m+k))$, $0 \leq k \leq \min(m,n)$. Indeed, if we have a partition $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ and $0 \leq k \leq \min(m,n)$ then we can reconstruct μ setting $\mu_1 = k + \min(\lambda_1, \nu_1)$ and $\mu_{i+1} = a_i + b_i - \widetilde{\mu}_i$, $i = 1, 2, \ldots$.

Remark 6.3 We can assume that λ_i, μ_i, ν_i, m, n, and k are arbitrary nonnegative real numbers. Thus one can construct a continuous analogue of the bijection ψ_3.

The construction of the bijection $\Phi_{\lambda\mu\beta}$ mentioned in Section 4 is based on the operation ψ_3. Let $\alpha = (\alpha_{(0)}, \ldots, \alpha_{(k)})$ be an oscillating tableaux of weight $(\beta_1, \ldots, \beta_k)$. We need to apply ψ_3 repeatedly to "subtableaux" of the type $(\alpha_{(i-1)}, \alpha_{(i)}, \alpha_{(i+1)})$ (see [10]) such that $\beta_i \geq 0$ and $\beta_{i+1} \leq 0$ (at least one of the inequalities is strict) until we get a tableau α' of a normal shape β'. In the same way, using the operations ψ_3 and ψ_4, one can construct the bijection $\Phi^{super}_{\lambda\mu\beta}$ from Section 5.

In the end of this section we give a generalization of Theorem 6.1. Let Λ be the ring of symmetric function of infinitely many variables $x_1, x_2, \ldots$, see [9]. Schur functions $s_\lambda(x)$, $\lambda \in \mathcal{P}$ (see [9]) form a linear basis in Λ. Thus we can identify Λ with the space of formal linear combinations of partitions. Consider the non-degenerate symmetric bilinear form on Λ such that $s_\lambda(x)$ form an orthonormal basis with respect to this form. Consider the linear operator on Λ given by $S_{\lambda/\mu} : f \rightarrow s_{\lambda/\mu} \cdot f$. Let

$S^*_{\lambda/\mu}$ be the conjugate operator with respect to the bilinear form. Then the operators $D(n)$, $D(\overline{n})$, $I(n)$, and $I(\overline{n})$ coincide with S_n, S_{1^n}, S^*_n, and $S^*_{1^n}$, respectively (Pieri's rule). In general, we have the following commutation relations for the operators S_λ and S^*_ν.

Theorem 6.4

$$S^*_\nu S_\lambda = \sum_{\mu \subset \lambda \cap \nu} S_{\lambda/\mu} S^*_{\nu/\mu}.$$

7 Continuous analogue

In this section we sketch a continuous piecewise-linear analogue of the RSK correspondence for oscillating tableaux.

Using the bijection ψ_3 from the previous section (see Remark 6.3) it is possible to construct a continuous piecewise-linear volume-preserving map $\Phi : A \to B$ between two convex polyhedra. Rather than rigorously state the theorem we will give an example.

Consider an array $\{p_{ij}\}$ whose shape is a Young diagram, say

p_{11}	p_{12}	p_{13}	p_{14}
p_{21}	p_{22}	p_{23}	
p_{31}	p_{32}		

where all entries p_{ij} are nonnegative real numbers weakly decreasing from left to right and from top to bottom. Each diagonal in this array is a decreasing sequence of nonnegative real numbers, i.e., it is a "continuous partition". Thus we can view the array $\{p_{ij}\}$ as a "continuous oscillating tableau"

$$\alpha = ((p_{31}), (p_{21}, p_{32}), (p_{11}, p_{22}), (p_{12}, p_{23}), (p_{13}), (p_{14})).$$

Consider the polyhedron A which consists of all such arrays with fixed diagonal-sums: $p_{31} = \gamma_1$, $p_{21} + p_{32} = \gamma_2$, $p_{11} + p_{22} = \gamma_3$, $p_{12} + p_{23} = \gamma_3$, $p_{13} = \gamma_5$, $p_{14} = \gamma_6$.

Consider another array $\{q_{ij}\}$ of the same shape where all entries q_{ij} are nonnegative real numbers. (We drop the monotonicity requirements for the entries.) Let B be the polyhedron of all such arrays with fixed column and row sums $q_{11}+q_{21}+q_{31} = \beta_1$, $q_{12} + q_{22} + q_{32} = \beta_2$, $q_{13} + q_{23} = \beta_3$, $q_{14} = \beta_4$, $q_{31} + q_{32} = \delta_1$, $q_{21} + q_{22} + q_{23} = \delta_2$, $q_{11} + q_{12} + q_{13} + q_{14} = \delta_3$.

Suppose that the parameters $\{\gamma_i\}$ and $\{\beta_j, \delta_k\}$ satisfy the following relations: $\gamma_1 = \beta_1$, $\gamma_2 = \gamma_1 + \beta_2$, $\gamma_3 = \gamma_2 - \delta_1$, $\gamma_4 = \gamma_3 + \beta_3$, $\gamma_5 = \gamma_4 - \delta_2$, $\gamma_6 = \gamma_5 + \beta_4$.

Repeatedly applying the operation ψ_3 from the previous section, one can construct a continuous piecewise linear volume-preserving bijection Φ between the polyhedra A and B. In particular,

(a) $\mathrm{Vol}(A) = \mathrm{Vol}(B)$,
(b) The number of integer points in A is equal to the number of integer points in B.

As we mentioned before the elements of A are "continuous oscillating tableaux". Analogously, the elements of B are "continuous intransitive graphs". If we restrict ourself to the case when all p_{ij} and q_{ij} are integer, we can recover the RSK correspondence for oscillating tableaux (see Theorem 2.3).

In the end we consider a simple example. Suppose

$$\Phi : \quad \begin{array}{|c|c|} \hline p_{11} & p_{12} \\ \hline p_{21} & p_{22} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|} \hline q_{11} & q_{12} \\ \hline q_{21} & q_{22} \\ \hline \end{array} \; ,$$

where $p_{11} \geq p_{12} \geq 0$, $p_{21} \geq p_{22} \geq 0$, $p_{11} \geq p_{21} \geq 0$, $p_{12} \geq p_{22} \geq 0$, and $q_{ij} \geq 0$.

Then the map Φ is given by $q_{11} = p_{11} - \min(p_{12}, p_{21})$, $q_{12} = p_{12}$, $q_{21} = p_{21}$, and $q_{22} = \min(p_{12}, p_{21}) - p_{22}$ (cf. construction of ψ_3 in Section 6).

References

[1] A. D. Berenstein, A. N. Kirillov, Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux, *Algebra i Analiz* **7** (1995), N. 1. 92–152.

[2] S. Dulucq, B. E. Sagan, La correspondance de Robinson-Schensted pour les tableaux oscillants gauches, *Disc. Math.* **139** (1995), 129–142.

[3] S. V. Fomin, Generalized Robinson-Schensted-Knuth correspondence [in Russian], *Zapiski Nauchn. Sem. LOMI* **155** (1985), 156–175.

[4] S. Fomin, Duality of graded graphs; Schensted algorithms for dual graded graphs *J. Alg. Comb.* **3** (1994), 357–404; **4** (1995), 5–45.

[5] S. Fomin, Schur operators and Knuth correspondence, to appear in *J. Comb. Theory, Ser. A.*

[6] I. Gessel, Counting paths in Young's lattice, *J. Stat. Plan.* **34** (1993), 125–134.

[7] G. James, A. Kerber, The representation theory of the symmetric groups. *Encyclopedia of Math. and its Appl.*, 16, Addison-Wesley, Reading, Mass. 1981.

[8] D. E. Knuth, Permutations, matrices and generalized Young tableaux, *Pacific J. Math.* **34** (1970), N. 3, 709–727.

[9] I. G. Macdonald, *Symmetric Functions and Hall Polynomials,* Oxford Univ. Press, 1995, 2nd ed.

[10] T. W. Roby, Applications and extensions of Fomin's generalization of the Robinson-Schensted-Knuth correspondence to differential posets, Ph. D. thesis, M.I.T., 1991.

[11] B. E. Sagan, R. P. Stanley, Robinson-Schensted algorithms for skew tableaux, *J. Comb. Theory, Ser. A* **55** (1990), 161–193.

[12] S. Sundaram, On the combinatorics of representations of $Sp(2n, \mathbb{C})$, Ph. D. thesis, M.I.T., 1986.