Graphs, Syzygies and Multivariate Splines

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Abstract

The module of splines on a polyhedral complex can be viewed as the syzygy module of a graph with linear forms assigned to the edges. We describe a process for decomposing the graph in order to compute the homological dimension and the Hilbert series of these modules.

Let Δ be a polyhedral subdivision of a region in R^d. Billera initiated the algebraic study of the ring of r-differentiable piecewise polynomials on Δ, denoted $C^r(\Delta)$. $C^r(\Delta)$ is also a module over $S= R[x_1, \ldots, x_d]$, the polynomial ring in d variables over S, via pointwise multiplication. The elements of $C^r(\Delta)$ are known as *multivariate splines.*

In previous work we were concerned with finding combinatorial and topological conditions on Δ for $C^r(\Delta)$ to be a free module. We are interested here in determining the homological dimension and the Hilbert series of $C^r(\Delta)$. This study has applications to the problem of finding bases for certain finite dimensional R-subspaces of $C^r(\Delta)$, called spline spaces.

Our primary object of study is $syz(\Delta, r)$, the syzygy module of the dual graph of Δ. $C^r(\Delta)$ and $syz(\Delta, r)$ are projectively equivalent, and the Hilbert series of one can easily be expressed in terms of the other. Given any graph G, with an assignment $L = \{\ell_e\}$ of linear forms to edges of G, we can define $syz(G, L^r)$ as follows. Fix an orientation of the m edges of G and let $\ell_{-e} = -\ell_e$. Let $L^r = \{\ell_e^{r+1}\}$ and let $\mathcal{C}$ denote the set of cycles of G. We now define

$$syz(G, L^r) = \{(\alpha_1, \ldots, \alpha_m) \in S^m : \text{for all } C \in \mathcal{C}, \sum_{e \in C} \alpha_e \ell_e{}^{r+1} = 0\}.$$

Our main result is that when (G, L) decomposes into cycles the homological dimension and the Hilbert series of $syz(G, L^r)$ are completely determined by the one cycle case. This is equivalent to the problem in commutative algebra of finding a minimal graded free resolution of an ideal generated by powers of linear forms. When $r = 0$, this resolution is completely known, but for higher r, the problem is unsolved in general. We describe conditions on G, L and r for such a decomposition to occur, and give a method for computing the Hilbert series based on this decomposition.

Proposition 0.1 *$syz(G, L^r)$ is a torsion free S module of rank $n - 1$, where n is the number of vertices of G.*

Definition 0.2 *The rank of a cycle C in G is the dimension of the linear span of the ℓ_e's.*

Let $hd(M)$ denote the homological dimension of the module M. In [8], we prove the following.

Theorem 0.3 *If G_Δ contains only one cycle C, then $hd(syz(G, L^r)) = rk(C) - 2$.*

Note that when the graph is a cycle, the homological dimension of $syz(G, L^r)$ is the same for every r. This is false in general. In fact, freeness is usually lost when going from $r = 0$ to $r = 1$, and the homological dimension is likely to increase as r increases. For computations and related results see [8].

1 A Description of $C^r(\Delta)$ and $syz(\Delta, r)$

Let Δ be a pure polyhedral complex in R^d, i.e., a convex polyhedral subdivision of a region in R^d. For details, see [6].

Definition 1.1 *The* ***dual graph*** *of* Δ, G_Δ, *is defined as follows. The vertices of* G_Δ *correspond to d-polytopes of* Δ, *and two vertices share an edge whenever the corresponding polytopes meet in a* $d-1$ *dimensional face.*

If two d-polytopes meet in a face of dimension $d-1$, we say they are **adjacent**.

Definition 1.2 Δ *is* ***strongly connected*** *if* G_Δ *is connected, and is called hereditary if for every face* σ *of* Δ, $G(st(\sigma))$ *is connected.*

Definition 1.3 *For a positive integer r and a d-complex* Δ, $C^r(\Delta)$ *is the set of r- differentiable functions* $F : \Delta \rightarrow R$ *such that for every d-face* σ, F *restricted to* σ *is given by a polynomial in* S.

For this work, we will only consider hereditary complexes. In this case, there is an easy test to determine whether a piecewise polynomial function is in $C^r(\Delta)$. Choose an ordering $\sigma_1, \ldots, \sigma_n$ of the d-faces of Δ. With respect to this ordering, a piecewise polynomial function can be represented as an n-tuple of polynomials, $(f_1, \ldots, f_n)$, where is f_i is the restriction of F to the face σ_i. If σ_i is adjacent to σ_j, then $\sigma_i \cap \sigma_j$ has dimension $d-1$ and $I(\sigma_i \cap \sigma_j)$, the ideal of polynomials which vanish on $\sigma_i \cap \sigma_j$, is generated by an affine form, denoted ℓ_{ij}. Note that ℓ_{ij} is unique up to constant multiple. Another way to think of this is as follows: The affine span of $\sigma_i \cap \sigma_j$ is a hyperplane in R^d, and ℓ_{ij} is an affine form whose kernel is that hyperplane. The following proposition is proved in [3] in a more general case.

Proposition 1.4 *If Δ is hereditary and $F = (f_1, \ldots, f_n)$ is a piecewise polynomial function on Δ, then F is in $C^r(\Delta)$ if and only if whenever σ_i is adjacent to σ_j, ℓ_{ij}^{r+1} divides $f_i - f_j$.*

Choose an ordering of the vertices of G_Δ. This induces an orientation on the edges of G_Δ. If $e = ij$ is the directed edge from i to j in G_Δ, let $\ell_e = \ell_{ij}$ and let $\ell_{-e} = \ell_{ji} = -\ell_{ij}$. Let $e_1, \ldots, e_m$ be an ordering of the positively oriented edges in G_Δ, with corresponding affine forms $\ell_1, \ldots, \ell_m$. Let $\mathcal{C}$ denote the set of cycles G_Δ.

Definition 1.5 *$syz(\Delta, r) = syz(G_\Delta, L^r)$, where $L = \ell_{ij}$.*

The following theorem, proved in [8], connects the modules $syz(\Delta, r)$ and $C^r(\Delta)$.

Theorem 1.6 *([8]) If Δ is hereditary, then $C^r(\Delta) \cong syz(\Delta, r) \oplus S$. Moreover, when Δ is central this is a graded isomorphism with a degree shift in $syz(\Delta, r)$ of $r + 1$.*

See [3] and [9] for Hilbert series computations and their connection to the dimension problem for splines spaces.

2 Decompositions of (G, L^r)

Definition 2.1 *Fix a basis $\mathcal{B}$ for the cycle space of G. If e is contained in only one C in $\mathcal{B}$, we say e is a **free** edge.*

Let C be a cycle in G.

Definition 2.2 *If the span$\{\ell_e^{r+1} : e \in C,\ e$ is free$\}$ = span$\{\ell_e^{r+1} : e \in C\}$, then G **decomposes** into $G - \{e : e \in C\} \oplus C$, with respect to L^r.*

Theorem 2.3 *If G decomposes into $A \oplus B$, then*

$$syz(G) = syz(A) \oplus syz(B)$$

Proof: We use the fact that $syz(G)$ is the module of syzgies of a matrix M with rows indexed by the cycles in a a given basis for $\mathcal{C}$ and columns indexed by the edges of G. The (i,j)th entry of M is 0 if edge j is not contained in cycle C_i and $\pm\ell_j^{r+1}$ depending on the orientation of edge j in cycle C_i. If e is a free edge, ℓ_e only appears in one row of M, so we can use its column to perform column operations on M. This won't change the the image of M, i.e., the module generated by the columns of M, and by the Hilbert Syzygy Theorem, it won't change the isomorphism class of $syz(G)$. If the linear forms on free edges span the same space as all the linear forms in say row i, then by column operations, we can replace row i with 0's except for the (i,j)th entries where j is a free edge of C_i. Thus, the image of M = span $\{$column $j : j$ is a free edge of $C_i\}$ $\oplus$ span $\{$column $j : j$ is not a free edge of $C_i\}$. The theorem now follows from the Hilbert Syzygy Theorem. □

Corollary 2.4 *If $G = C_1 \oplus \ldots \oplus C_k$ is a decomposition with respect to L^r into cycles, then $hd(syz(G)) = max_i\{rk(C_i)\} - 2$*

Proof: Repeated applications of Theorem 4.2 and 2.3. □

We can now compute the Hilbert Series of $syz(\Delta, r)$in terms of the series of the decomposition. If $r = 0$ and G decomposes into cycles, then the Hilbert series is completely determined by the ranks and the lengths of the cycles. For $r \geq 1$, this is false in general.

We describe classes of polyhedral complexes whose dual graphs decompose into cycles, and give examples of the resulting Hilbert series.

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