MULTIDIMENSIONAL MATRIX INVERSIONS
(Extended abstract)

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ABSTRACT. We compute the inverse of a specific infinite r-dimensional matrix, thus unifying multidimensional matrix inversions recently found by Milne, Lilly, and Bhatnagar. Our inversion is an r-dimensional extension of a matrix inversion previously found by Krattenthaler. We also compute the inverse of another infinite r-dimensional matrix. As applications of our matrix inversions, we derive new summation formulas for multidimensional basic hypergeometric series.

1. INTRODUCTION

Matrix inversions are very important in many fields of combinatorics and special functions. When dealing with combinatorial sums, application of matrix inversion may help to simplify problems, or yield new identities. Andrews [1] discovered that the Bailey transform [3], which is a very powerful tool in the theory of (basic) hypergeometric series, corresponds to the inversion of two infinite lower-triangular matrices. Gessel and Stanton [13] used a bibasic extension of that matrix inversion to derive a number of basic hypergeometric summations and transformations, and identities of Rogers-Ramanujan type. Even earlier, Carlitz [9] had found an even more general matrix inversion though without giving any applications.

Gasper and Rahman [10], [25], [11], [12, sec. 3.6] used another bibasic matrix inversion together with an indefinite bibasic sum to derive numerous beautiful hypergeometric summation and transformation formulas.

The most general (1-dimensional) matrix inversion, however, which contained all the inversions aforementioned, was found by Krattenthaler [16] who applied his inversion to derive a number of hypergeometric summation formulas. The inverse matrices he gave are basically $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers), where

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j - c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.1}$$

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and

$$g_{kl} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{j=l+1}^{k} (a_j - c_k)}{\prod_{j=l}^{k-1} (c_j - c_k)}. \tag{1.2}$$

In fact, the special case $a_j = aq^{-j}$, $c_k = q^k$ is equivalent to the matrix inversion of Andrews, and the case $a_j = ap^{-j}$, $c_k = q^k$ is equivalent to Gessel and Stanton's. Specializing $c_k = q^k$ we obtain Carlitz's matrix inversion, and $a_j = (bp^{-j}/a) + ap^j$, $c_k = q^{-k} + bq^k$ yields the inversion of Gasper and Rahman.

Multidimensional matrix inversions were found by Milne, Lilly and Bhatnagar. The A_r (or equivalently $U(r+1)$) and C_r inversions (corresponding to the root systems A_r and C_r, respectively) of Milne and Lilly [21, Theorem 3.3], [22], [17], [18], which are higher-dimensional generalizations of Andrews' Bailey transform matrices, were used to derive A_r and C_r extensions [21], [23] of many of the classical hypergeometric summation and transformation formulas. Bhatnagar and Milne [4, Theorem 5.7], [6, Theorem 3.48] were even able to find an A_r extension of Gasper and Rahman's bibasic hypergeometric matrix inversion. They used a special case of their matrix inversion, an A_r extension of Carlitz's inversion, to derive A_r identities of Abel-type. But none of these multidimensional matrix inversions contained Krattenthaler's inversion as a special case.

One of the main results of this paper is a multidimensional extension of Krattenthaler's matrix inverse (see Theorem 2.1). This multidimensional matrix inversion unifies all the matrix inversions mentioned so far as it contains them all as special cases. Besides, we present another interesting multidimensional matrix inversion (see Theorem 3.1) which is of different type.

The main motivation for finding a multidimensional extension of Krattenthaler's matrix inverse came from prospective applications to basic hypergeometric series. These applications are the contents of section 4. We combine a special case of Theorem 2.1 and a C_r ${}_8\phi_7$ summation theorem of Milne and Lilly [23] to derive a D_r ${}_8\phi_7$ summation theorem, which has been derived independently by Bhatnagar [5] using a different method. We also derive A_r and D_r extensions of a quadratic hypergeometric summation formula of Gessel and Stanton [13]. Finally, we derive a D_r extension of a cubic summation formula of Gasper and Rahman [11].

We are sure that our multidimensional matrix inversions are very useful in the theory of basic hypergeometric series of type A_r, C_r, and D_r, respectively, and will lead to the discovery of many more new identities. This claim is heavily supported by the fact that identities derived in this paper already lead to new C_r and D_r extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation [2]. This is ongoing research undertaken jointly with Bhatnagar [7].

Two determinant evaluations, which are elegant generalizations of the classical and "symplectic" Vandermonde determinants, turn out to be crucial for our computations in sections 2 and 3. We decided to give them in a separate appendix.

Full details and proofs of all the results in this paper are included in our preprint [27]. This work is part of the author's thesis, being written under the supervision of C. Krattenthaler. The author feels especially indebted to his supervisor who patiently has provided a lot of help and ideas.

Finally, we explain what we mean by "multidimensional matrix inversion". Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (as before, $\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular r-dimensional matrix; i.e. $f_{\mathbf{nk}} = 0$ unless $\mathbf{n} \geq \mathbf{k}$, by which we mean $n_i \geq k_i$ for all $i = 1, \ldots, r$.

The matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{\mathbf{n}\geq\mathbf{k}\geq\mathbf{l}} f_{\mathbf{nk}}g_{\mathbf{kl}} = \delta_{\mathbf{nl}}$$

for all $\mathbf{n},\mathbf{l}\in\mathbb{Z}^r$, where $\delta_{\mathbf{nl}}$ is the usual Kronecker delta.

2. A multidimensional matrix inversion

For convenience, we introduce the notation $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$.

Theorem 2.1. *Let $(a_t)_{t\in\mathbb{Z}}$, $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1,\dots,r$ be arbitrary sequences, b arbitrary, such that none of the denominators in (2.2) or (2.3) vanish. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}\left(a_t - b/\prod_{j=1}^r c_j(k_j)\right)\prod_{i=1}^r\prod_{t=|\mathbf{k}|}^{|\mathbf{n}|-1}\left(a_t - c_i(k_i)\right)}{\prod_{i=1}^r\prod_{t_i=k_i+1}^{n_i}\left(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j)\right)\prod_{i,j=1}^r\prod_{t_i=k_i+1}^{n_i}\left(c_i(t_i) - c_j(k_j)\right)} \tag{2.2}$$

and

$$g_{\mathbf{kl}} = \prod_{1\leq i<j\leq r}\frac{(c_i(l_i) - c_j(l_j))}{(c_i(k_i) - c_j(k_j))} \times \frac{(b - a_{|\mathbf{l}|}\prod_{j=1}^r c_j(l_j))}{(b - a_{|\mathbf{k}|}\prod_{j=1}^r c_j(k_j))}\prod_{i=1}^r\frac{(a_{|\mathbf{l}|} - c_i(l_i))}{(a_{|\mathbf{k}|} - c_i(k_i))}$$
$$\times \frac{\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}\left(a_t - b/\prod_{j=1}^r c_j(k_j)\right)\prod_{i=1}^r\prod_{t=|\mathbf{l}|+1}^{|\mathbf{k}|}\left(a_t - c_i(k_i)\right)}{\prod_{i=1}^r\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i) - b/\prod_{j=1}^r c_j(k_j)\right)\prod_{i,j=1}^r\prod_{t_i=l_i}^{k_i-1}\left(c_i(t_i) - c_j(k_j)\right)}. \tag{2.3}$$

Remark 2.4. The special case $a_t = 0$, $c_j(k_j) = x_j^{-1}q^{-k_j}$ is equivalent to the A_r Bailey transform of Milne and Lilly [21], [22], the specialization $a_t = 0$, $c_j(k_j) = x_j^{-1}q^{-k_j} + x_jq^{k_j}$, $b = 0$ is equivalent to their C_r Bailey transform [22], [17], [18]. The limiting case $a_t = baq^{-t}$, $c_j(k_j) = x_j^{-1}q^{-k_j}$, then $b\to 0$, is equivalent to a second A_r Bailey transform of Milne [21, Theorem 8.26]. The specialization $c_j(k_j) = x_j^{-1}q^{-k_j}$ is equivalent to the A_r matrix inverse of Bhatnagar and Milne [4, Theorem 5.7], [6, Theorem 3.48]. Moreover, the $r = 1$ case is a restatement of Krattenthaler's matrix inversion (eqs. (1.1) and (1.2)). Due to the fact that Theorem 2.1 covers all known A_r matrix inversions (to the author's knowledge), we view Theorem 2.1 as an A_r matrix inversion theorem (also see Remark 3.4).

Another important special case of Theorem 2.1 is a new multidimensional bibasic hypergeometric matrix inversion, stated seperately as Theorem 4.10 in section 4, which we utilize in our applications.

Sketch of proof of Theorem 2.1. We use the operator method of [15] extended with a multidimensional corollary (see [27, Corollary 2.4]). Proceeding this way we arrive at a particular system of operator equations which we solve by application of Lemma A.1. □

3. Another multidimensional matrix inversion

Theorem 3.1. *Let $(c_i(t_i))_{t_i\in\mathbb{Z}}$, $i = 1,\dots,r$, be arbitrary sequences, b arbitrary, such that none of the denominators in (3.2) or (3.3) vanish. Then $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are inverses of each other, where*

$$f_{\mathbf{nk}} = \prod_{i=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=k_i+1}^{n_i} (c_i(t_i) - c_j(k_j))} \tag{3.2}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = \prod_{1\le i<j\le r} &\left(\frac{(c_i(l_i) - c_j(l_j))}{(c_i(k_i) - c_j(k_j))} \frac{(1 - c_i(l_i)c_j(l_j))}{(1 - c_i(k_i)c_j(k_j))} \right) \\ &\times \prod_{i=1}^{r} \frac{(1 - c_i(l_i)^2)}{(1 - c_i(k_i)^2)} \prod_{i=1}^{r} \frac{c_i(l_i)}{c_i(k_i)} \\ &\times \prod_{i=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i} (1 - bc_i(t_i)/\prod_{j=1}^{r} c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - b/\prod_{j=1}^{r} c_j(k_j))} \prod_{i,j=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i} (1 - c_i(t_i)c_j(k_j))}{\prod_{t_i=l_i}^{k_i-1} (c_i(t_i) - c_j(k_j))}. \end{aligned} \tag{3.3}$$

Remark 3.4. The special case $c_j(k_j) = x_j^{-1}q^{-k_j}$ is a C_r generalization of Bressoud's matrix inversion formula [8], as pointed out in [18, second remark after Theorem 2.11]. Setting, in addition, $b = 0$ yields a C_r Bailey transform which is equivalent to the one derived in [18]. Therefore, we view Theorem 3.1 as a C_r matrix inversion theorem.

Sketch of proof of Theorem 3.1. We proceed as in the proof of Theorem 2.1 but utilize Lemma A.3 instead of Lemma A.1. □

4. Applications to A_r and D_r basic hypergeometric series

Probably, the most important application of matrix inversion is the derivation of hypergeometric series identities. There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1], [13], [26]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}} a_{\mathbf{k}} = b_{\mathbf{n}} \tag{4.1}$$

if and only if

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}} b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{4.2}$$

We expect that applications of our matrix inversions in Theorems 2.1 and 3.1 will lead to many new identities for multidimensional (basic) hypergeometric series. As an illustration, we use special cases of our Theorem 2.1 to derive A_r and D_r extensions of a terminating quadratic summation of Gessel and Stanton [13], D_r extensions of Jackson's ${}_8\phi_7$ summation [14], and a D_r extension of a cubic summation of Gasper and Rahman [11].

We recall the standard definition of the rising q-factorial (cf. [12]). Define

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j),$$

and for any integer k,

$$(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j) = \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{4.3}$$

Theorem 4.4 (An A_r quadratic sum). *Let $x_1,\ldots,x_r$, a, b, and d be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers, let $r\geq 1$, and suppose that none of the denominators in (4.5) vanish. Then*

$$\sum_{\substack{0\leq k_i\leq n_i\\ i=1,\ldots,r}}\left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i,j=1}^{r}\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}\right.$$
$$\times\prod_{i=1}^{r}\frac{(dx_i;q^2)_{k_i}\,(a^2x_iq^{1+2|\mathbf{n}|}/d;q^2)_{k_i}}{(ax_iq^2/b;q^2)_{k_i}\,(abx_iq;q^2)_{k_i}}\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}}$$
$$\left.\times\frac{(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{(aq/d;q)_{|\mathbf{k}|}\,(dq^{-2|\mathbf{n}|}/a;q)_{|\mathbf{k}|}}\,q^{-|\mathbf{k}|+2\sum_{i=1}^{r}ik_i}\right)$$
$$=\frac{(aq^2/bd;q^2)_{|\mathbf{n}|}\,(abq/d;q^2)_{|\mathbf{n}|}}{(aq/d;q)_{2|\mathbf{n}|}}\prod_{i=1}^{r}\frac{(ax_iq;q)_{2n_i}}{(ax_iq^2/b;q^2)_{n_i}\,(abx_iq;q^2)_{n_i}}. \tag{4.5}$$

Remark 4.6. This quadratic summation formula is an A_r extension of

$$\sum_{k=0}^{n}\frac{1-aq^{3k}}{1-a}\,\frac{(a;q)_k(b;q)_k(q/b;q)_k}{(q^2;q^2)_k(aq^2/b;q^2)_k(abq;q^2)_k}\,\frac{(d;q^2)_k(a^2q^{1+2n}/d;q^2)_k(q^{-2n};q^2)_k}{(aq/d;q)_k(dq^{-2n}/a;q)_k(aq^{2n+1};q)_k}\,q^k$$
$$=\frac{(aq;q)_{2n}}{(aq/d;q)_{2n}}\,\frac{(abq/d;q^2)_n(aq^2/bd;q^2)_n}{(aq^2/b;q^2)_n(abq;q^2)_n}, \tag{4.7}$$

due to Gessel and Stanton [13, eq. (1.4), $q\to q^2$], to which it reduces for $r=1$. Many identities like (4.7), involving bases of different powers of q, are known. Hypergeometric series with several bases were extensively studied by Gasper and Rahman [10], [25], [11], [12, sec. 3.8].

Proof of Theorem 4.4. If we substitute $c_i(t_i)\mapsto q^{-2t_i}/x_i$, $i=1,\ldots,r$, $a_t\mapsto aq^t$, and $b\mapsto a/bx_1\cdots x_n$ in Theorem 2.1 (this special case can be also obtained from the inversion [6, Theorem 3.48] of Bhatnagar and Milne) we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}}=\prod_{i=1}^{r}\left(\frac{1-q^{1+2k_i+2|\mathbf{k}|}a^2x_i/d}{1-qa^2x_i/d}\right)\prod_{1\leq i<j\leq r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{i,j=1}^{r}\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}}\prod_{i=1}^{r}\frac{(ax_iq^{|\mathbf{n}|};q)_{2k_i}}{(a^2x_iq^{3+2n_i}/d;q^2)_{|\mathbf{k}|}}\cdot\frac{q^{2\sum_{i=1}^{r}ik_i}}{(aq^{2-|\mathbf{n}|}/d;q)_{2|\mathbf{k}|}}$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^{r} \left(\frac{1 - ax_i q^{2l_i + |\mathbf{l}|}}{1 - ax_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{2l_i - 2l_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{r} \frac{(q^{-2k_j} x_i/x_j; q^2)_{l_i}}{(q^2 x_i/x_j; q^2)_{l_i}}$$
$$\times \prod_{i=1}^{r} \frac{(ax_i; q)_{|\mathbf{l}|}}{(ax_i q^{1+2k_i}; q)_{|\mathbf{l}|}} \prod_{i=1}^{r} \frac{(a^2 x_i q^{1+2|\mathbf{k}|}/d; q^2)_{l_i}}{(a^2 x_i q^3/d; q^2)_{l_i}} \prod_{i=1}^{r} \frac{(a^2 x_i q/d; q^2)_{|\mathbf{k}|}}{(ax_i q; q)_{2k_i}}$$
$$\times \frac{(1 - q^{1+|\mathbf{l}|} a/d)}{(1 - qa/d)} \frac{(d/aq; q)_{|\mathbf{l}|} (qa/d; q)_{2|\mathbf{k}|}}{(q^{2-2|\mathbf{k}|} a/d; q)_{|\mathbf{l}|}} q^{-|\mathbf{l}| + 2\sum_{i=1}^{r} i l_i}.$$

Now (4.1) holds for

$$a_{\mathbf{k}} = (baq/d; q^2)_{|\mathbf{k}|} (aq^2/bd; q^2)_{|\mathbf{k}|} \prod_{i=1}^{r} \frac{(a^2 x_i q/d; q^2)_{|\mathbf{k}|}}{(ax_i q^2/b; q^2)_{k_i} (abx_i q; q)_{k_i}}$$

and

$$b_{\mathbf{n}} = \frac{(q^{2-|\mathbf{n}|}/b; q^2)_{|\mathbf{n}|} (bq^{1-|\mathbf{n}|}; q^2)_{|\mathbf{n}|}}{(aq^{3-|\mathbf{n}|}/d; q^2)_{|\mathbf{n}|} (dq^{-|\mathbf{n}|}/a; q^2)_{|\mathbf{n}|}} \prod_{i=1}^{r} \frac{(a^2 x_i q^3/d; q^2)_{n_i} (dx_i/a; q^2)_{n_i}}{(ax_i q^2/b; q^2)_{n_i} (abx_i q; q^2)_{n_i}}$$

by means of an A_r extension of Jackson's ${}_8\phi_7$-sum, taken from [20, Theorem 6.14] (or in more convenient notation [24, Theorem A12]). This implies the inverse relation (4.2) which is easily transformed into (4.5). □

It is not hard to see from a polynomial identity argument that Theorem 4.4 implies the following summation theorem.

Theorem 4.8 (An A_r quadratic sum). *Let $x_1, \dots, x_r$, $c_1, \dots, c_r$, a, and d be indeterminate, let N be a nonnegative integer, let $r \ge 1$, and suppose that none of the denominators in (4.9) vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r \ge 0 \\ 0 \le |\mathbf{k}| \le N}} \Bigg(\prod_{i=1}^{r} \left(\frac{1 - ax_i q^{2k_i + |\mathbf{k}|}}{1 - ax_i} \right) \prod_{1 \le i < j \le r} \left(\frac{1 - q^{2k_i - 2k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j; q^2)_{k_i}}{(q^2 x_i/x_j; q^2)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{(dx_i; q^2)_{k_i} (a^2 x_i q/d \prod_{j=1}^{r} c_j; q^2)_{k_i}}{(ax_i q^{2+N}; q^2)_{k_i} (ax_i q^{1-N}; q^2)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i; q)_{|\mathbf{k}|}}{(ax_i q/c_i; q)_{|\mathbf{k}|}}$$
$$\times \frac{(q^{-N}; q)_{|\mathbf{k}|} (q^{1+N}; q)_{|\mathbf{k}|}}{(aq/d; q)_{|\mathbf{k}|} (d \prod_{j=1}^{r} c_j/a; q)_{|\mathbf{k}|}} q^{-|\mathbf{k}| + 2\sum_{i=1}^{r} i k_i} \Bigg)$$
$$= \begin{cases} \dfrac{(dq/a; q^2)_M (aq^2/d \prod_{j=1}^{r} c_j; q^2)_M}{(aq^2/d; q^2)_M (dq \prod_{j=1}^{r} c_j/a; q^2)_M} \displaystyle\prod_{i=1}^{r} \frac{(ax_i q^2; q^2)_M (c_i q/ax_i; q^2)_M}{(q/ax_i; q^2)_M (ax_i q^2/c_i; q^2)_M} & (N = 2M), \\[2ex] \dfrac{(d/a; q^2)_M (aq/d \prod_{j=1}^{r} c_j; q^2)_M}{(aq/d; q^2)_M (d \prod_{j=1}^{r} c_j/a; q^2)_M} \displaystyle\prod_{i=1}^{r} \frac{(ax_i q; q^2)_M (c_i/ax_i; q^2)_M}{(1/ax_i; q^2)_M (ax_i q/c_i; q^2)_M} & (N = 2M - 1). \end{cases} \tag{4.9}$$

Proof. First we write the right sides of (4.9) as quotients of infinite products using (4.3). Then by the $b = q^{-N}$ case of Theorem 4.4 it follows that the identity (4.9) holds for $c_j = q^{-2n_j}$, $j = 1, \dots, r$. By clearing out denominators in (4.9), we get a polynomial equation in c_1, which is true for q^{-2n_1}, $n_1 = 0, 1, \dots$. Thus we obtain an identity in c_1. By carrying out this process for $c_2, c_3, \dots, c_r$ also, we obtain Theorem 4.8. □

By another specialization of Theorem 2.1 we obtain an interesting bibasic hypergeometric matrix inversion. We use this inversion to derive D_r basic hypergeometric summation formulas. For explanations why we associate D_r with these formulas the reader is referred to [5].

Theorem 4.10. *Let*

$$f_{\mathbf{n}\mathbf{k}} = \frac{\prod_{i=1}^{r}\left[(ap^{|\mathbf{k}|}q^{k_i}x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\,(ap^{|\mathbf{k}|}q^{-k_i}/x_i;p)_{|\mathbf{n}|-|\mathbf{k}|}\right]}{\prod_{i,j=1}^{r}\left[(q^{1+k_i-k_j}x_i/x_j;q)_{n_i-k_i}\,(q^{1+k_i+k_j}x_ix_j;q)_{n_i-k_i}\right]} \tag{4.11}$$

and

$$\begin{aligned} g_{\mathbf{k}\mathbf{l}} = (-1)^{|\mathbf{k}|-|\mathbf{l}|}q^{\binom{|\mathbf{k}|-|\mathbf{l}|}{2}} &\prod_{1\le i<j\le r}\frac{(1-x_ix_jq^{l_i+l_j})}{(1-x_ix_jq^{k_i+k_j})}\\ &\times\prod_{i=1}^{r}\frac{(1-ap^{|\mathbf{l}|}q^{l_i}x_i)(1-ap^{|\mathbf{l}|}q^{-l_i}/x_i)}{(1-ap^{|\mathbf{k}|}q^{k_i}x_i)(1-ap^{|\mathbf{k}|}q^{-k_i}/x_i)}\\ &\times\frac{\prod_{i=1}^{r}\left[(ap^{1+|\mathbf{l}|}q^{k_i}x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\,(ap^{1+|\mathbf{l}|}q^{-k_i}/x_i;p)_{|\mathbf{k}|-|\mathbf{l}|}\right]}{\prod_{i,j=1}^{r}\left[(q^{1+l_i-l_j}x_i/x_j;q)_{k_i-l_i}\,(q^{1+l_i+k_j}x_ix_j;q)_{k_i-l_i}\right]}. \end{aligned} \tag{4.12}$$

Then $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are infinite lower-triangular r-dimensional matrices being inverses of each other.

Proof. In Theorem 2.1 we set $b = 0$, $a_t = ap^t + p^{-t}/a$, and $c_i(t_i) = x_iq^{t_i} + q^{-t_i}/x_i$ for $i = 1,\dots,r$. After some elementary manipulations we obtain the inverse pair (4.11) and (4.12). □

Remark 4.13. The inversion in Theorem 4.10 is a D_r extension of Gasper and Rahman's bibasic matrix inversion [12, (3.6.19) and (3.6.20)], to which it reduces for $r = 1$.

Theorem 4.14 (A D_r Jackson's sum). *Let $x_1,\dots,x_r$, a, b, and c be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in (4.15) vanish. Then*

$$\begin{aligned} \sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}}\Bigg(&\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\le i<j\le r}(x_ix_j;q)^{-1}_{k_i+k_j}\\ &\times\prod_{i,j=1}^{r}\frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(x_ix_jq^{n_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-k_i}}{(aq^{1+n_i}x_i;q)_{|\mathbf{k}|}\,(aq^{1-n_i}/x_i;q)_{|\mathbf{k}|}}\\ &\times\frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}\,(a^2q/bc;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}\left[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(bcx_i/a;q)_{k_i}\right]}\,q^{\sum_{i=1}^{r}ik_i}\Bigg)\\ &=\prod_{i=1}^{r}\frac{(ax_iq;q)_{n_i}\,(ax_iq/bc;q)_{n_i}\,(bx_i/a;q)_{n_i}\,(cx_i/a;q)_{n_i}}{(x_i/a;q)_{n_i}\,(bcx_i/a;q)_{n_i}\,(ax_iq/b;q)_{n_i}\,(ax_iq/c;q)_{n_i}}. \end{aligned} \tag{4.15}$$

Remark 4.16. For $r = 1$ Theorem 4.14 reduces to Jackson's very-well-poised ${}_8\phi_7$ summation formula [14], [12, (II.22)].

Proof of Theorem 4.14. Setting $p = q$ in Theorem 4.10 (i.e. here we consider a D_r extension of Bressoud's matrix inverse [8]) we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1\le i<j\le r} \frac{(1-q^{k_i-k_j}x_i/x_j)(1-x_ix_jq^{k_i+k_j})}{(1-x_i/x_j)(1-x_ix_j)} \prod_{i=1}^{r} \frac{(1-x_i^2q^{2k_i})}{(1-x_i^2)}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-n_j}x_i/x_j;q)_{k_i}\,(x_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(x_ix_jq^{1+n_j};q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_iq^{|\mathbf{n}|};q)_{k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{k_i}} \cdot q^{\sum_{i=1}^r ik_i}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r} \frac{(1-q^{l_i-l_j}x_i/x_j)(1-x_ix_jq^{l_i+l_j})}{(1-x_i/x_j)(1-x_ix_j)}$$
$$\times \prod_{i,j=1}^{r} \frac{(q^{-k_j}x_i/x_j;q)_{l_i}\,(x_ix_jq^{k_j};q)_{l_i}}{(qx_i/x_j;q)_{l_i}\,(x_ix_jq;q)_{l_i}} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|+l_i}x_i)(1-aq^{|\mathbf{l}|-l_i}/x_i)}{(1-ax_i)(1-a/x_i)}$$
$$\times \prod_{i=1}^{r} \frac{(x_i/a;q)_{k_i}}{(ax_iq;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-k_i}/x_i;q)_{|\mathbf{l}|}} \cdot q^{\sum_{i=1}^r il_i}.$$

Now (4.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(bx_i/a;q)_{k_i}\,(cx_i/a;q)_{k_i}\,(ax_iq/bc;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(bcx_i/a;q)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^{r} \frac{(x_i^2q;q)_{n_i}}{(bcx_i/a;q)_{n_i}\,(x_iq^{1-|\mathbf{n}|}/a;q)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq;q)_{n_i}}{(x_ix_jq^{1+n_j};q)_{n_i}}$$
$$\times \frac{(bcq^{-|\mathbf{n}|}/a^2;q)_{|\mathbf{n}|}\,(q^{1-|\mathbf{n}|}/b;q)_{|\mathbf{n}|}\,(c;q)_{|\mathbf{n}|}}{\prod_{i=1}^{r}[(ax_iq/b;q)_{n_i}\,(cq^{-n_i}/ax_i;q)_{n_i}]}$$

by Milne and Lilly's C_r ${}_8\phi_7$ summation [23, Theorem 6.13]. This implies the inverse relation (4.2) which is easily transformed into (4.15). □

By using a polynomial argument we get

Theorem 4.17 (A D_r Jackson's sum). *Let $x_1,\dots,x_r$, $c_1,\dots,c_r$, a, and b be indeterminate. let N be a nonnegative integer. let $r \ge 1$, and suppose that none of the denominators in* (4.18) *vanish. Then*

$$\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} \left(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r}(x_ix_j;q)^{-1}_{k_i+k_j}\right.$$

$$\times \prod_{i,j=1}^{r} \frac{(c_j x_i/x_j;q)_{k_i}\,(x_i x_j/c_j;q)_{k_i}}{(q x_i/x_j;q)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-k_i}}{(ax_i q/c_i;q)_{|\mathbf{k}|}\,(ac_i q/x_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(b;q)_{|\mathbf{k}|}\,(a^2q^{1+N}/b;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(ax_iq/b;q)_{k_i}\,(bx_iq^{-N}/a;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}]}\, q^{\sum_{i=1}^r ik_i}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(ax_iq;q)_N\,(aq/x_i;q)_N\,(ax_iq/bc_i;q)_N\,(ac_iq/bx_i;q)_N}{(aq/bx_i;q)_N\,(ax_iq/b;q)_N\,(ac_iq/x_i;q)_N\,(ax_iq/c_i;q)_N}. \tag{4.18}$$

Limiting cases of Theorem 4.14 or Theorem 4.17 include various D_r summations. By reversing the multisum in Theorem 4.14 we obtain another D_r Jackson's sum which was independently derived by G. Bhatnagar [5] using a different method. D_r extensions of many of the classical basic hypergeometric summation theorems are given in [5]. Further consequences of the new D_r ${}_8\phi_7$ summations, such as C_r and D_r extensions of Bailey's very-well-poised ${}_{10}\phi_9$ transformation formula [2], [12, (III.28)] will be given in [7].

The remainder of this section is devoted to D_r quadratic and cubic summation formulas.

Theorem 4.19 (A D_r quadratic sum). *Let $x_1,\ldots,x_r$, a, and b be indeterminate, let $n_1,\ldots,n_r$ be nonnegative integers, let $r \geq 1$, and suppose that none of the denominators in (4.20) vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\ldots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{2k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{2k_i-2k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r}(x_ix_j;q^2)^{-1}_{k_i+k_j}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2n_j};q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}} \prod_{i=1}^{r}\frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-2k_i}}{(ax_iq^{1+2n_i};q)_{|\mathbf{k}|}\,(aq^{1-2n_i}/x_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(a^2q;q^2)_{|\mathbf{k}|}\,(b;q)_{|\mathbf{k}|}\,(q/b;q)_{|\mathbf{k}|}}{\prod_{i=1}^{r}[(abx_iq;q^2)_{k_i}\,(ax_iq^2/b;q^2)_{k_i}]}\,(-a)^{|\mathbf{k}|}\prod_{i=1}^{r}x_i^{-k_i}\cdot q^{2e_2(\mathbf{k})-\binom{|\mathbf{k}|+1}{2}+2\sum_{i=1}^r ik_i}\Bigg)$$

$$= \prod_{i=1}^{r}\frac{(ax_iq;q)_{2n_i}\,(x_iq/ab;q^2)_{n_i}\,(bx_i/a;q^2)_{n_i}}{(x_i/a;q)_{2n_i}\,(abx_iq;q^2)_{n_i}\,(ax_iq^2/b;q^2)_{n_i}}, \tag{4.20}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\ldots,k_r\}$.

Remark 4.21. This quadratic summation formula is a D_r extension of Gessel and Stanton's summation [13, eq. (1.4)], displayed in (4.7), to which it reduces for $r=1$.

Proof of Theorem 4.19. Doing the replacements $q\to q^2$, $p\to q$ in Theorem 4.10 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = \prod_{1\le i<j\le r}\frac{(1-q^{2k_i-2k_j}x_i/x_j)(1-x_ix_jq^{2k_i+2k_j})}{(1-x_i/x_j)(1-x_ix_j)}\prod_{i=1}^{r}\frac{(1-x_i^2q^{4k_i})}{(1-x_i^2)}$$

$$\times \prod_{i,j=1}^{r}\frac{(q^{-2n_j}x_i/x_j;q^2)_{k_i}\,(x_ix_j;q^2)_{k_i}}{(q^2x_i/x_j;q^2)_{k_i}\,(x_ix_jq^{2+2n_j};q^2)_{k_i}}\prod_{i=1}^{r}\frac{(ax_iq^{|\mathbf{n}|};q)_{2k_i}}{(x_iq^{1-|\mathbf{n}|}/a;q)_{2k_i}}\cdot q^{2\sum_{i=1}^r ik_i}$$

and

$$g_{\mathbf{kl}} = \prod_{1\le i<j\le r} \frac{(1-q^{2l_i-2l_j}x_i/x_j)(1-x_ix_jq^{2l_i+2l_j})}{(1-x_i/x_j)(1-x_ix_j)}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-2k_j}x_i/x_j;q^2)_{l_i}\,(x_ix_jq^{2k_j};q^2)_{l_i}}{(q^2x_i/x_j;q^2)_{l_i}\,(x_ix_jq^2;q^2)_{l_i}} \prod_{i=1}^{r} \frac{(1-aq^{|\mathbf{l}|+2l_i}x_i)(1-aq^{|\mathbf{l}|-2l_i}/x_i)}{(1-ax_i)(1-a/x_i)}$$

$$\times \prod_{i=1}^{r} \frac{(x_i/a;q)_{2k_i}}{(ax_iq;q)_{2k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{l}|}\,(a/x_i;q)_{|\mathbf{l}|}}{(aq^{1+2k_i}x_i;q)_{|\mathbf{l}|}\,(aq^{1-2k_i}/x_i;q)_{|\mathbf{l}|}} \cdot q^{2\sum_{i=1}^{r} il_i}.$$

Now (4.1) holds for

$$a_{\mathbf{k}} = \prod_{i=1}^{r} \frac{(x_iq/ab;q^2)_{k_i}\,(bx_i/a;q^2)_{k_i}}{(abx_iq;q^2)_{k_i}\,(ax_iq^2/b;q^2)_{k_i}}$$

and

$$b_{\mathbf{n}} = \prod_{i=1}^{r} \frac{(x_i^2q^2;q^2)_{n_i}}{(abx_iq;q^2)_{n_i}\,(x_iq^{2-|\mathbf{n}|}/a;q^2)_{n_i}} \prod_{1\le i<j\le r} \frac{(x_ix_jq^2;q^2)_{n_i}}{(x_ix_jq^{2+2n_j};q^2)_{n_i}}$$

$$\times \frac{(bq^{1-|\mathbf{n}|};q^2)_{|\mathbf{n}|}\,(q^{2-|\mathbf{n}|}/b;q^2)_{|\mathbf{n}|}\,(a^2q;q^2)_{|\mathbf{n}|}}{\prod_{i=1}^{r}\left[(ax_iq^2/b;q^2)_{n_i}\,(aq^{1+|\mathbf{n}|-2n_i}/x_i;q^2)_{n_i}\right]}$$

by Milne and Lilly's C_r ${}_8\phi_7$ summation [23, Theorem 6.13]. This implies the inverse relation (4.2) which is easily transformed into (4.20). □

Remark 4.22. By reversing the multisum in (4.20) we may obtain another, differently looking, extension of Gessel and Stanton's quadratic summation formula (4.7).

By a polynomial argument we may obtain some more quadratic summation theorems from Theorem 4.19.

Finally, we derive some cubic summations.

Theorem 4.23 (A D_r cubic sum). *Let $x_1,\dots,x_r$, and a be indeterminate, let $n_1,\dots,n_r$ be nonnegative integers, let $r\ge 1$, and suppose that none of the denominators in (4.24) vanish. Then*

$$\sum_{\substack{0\le k_i\le n_i\\ i=1,\dots,r}} \Bigg(\prod_{i=1}^{r}\left(\frac{1-ax_iq^{3k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le r}\left(\frac{1-q^{3k_i-3k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le r} (x_ix_j;q^3)^{-1}_{k_i+k_j}$$

$$\times \prod_{i,j=1}^{r} \frac{(q^{-3n_j}x_i/x_j;q^3)_{k_i}\,(x_ix_jq^{3n_j};q^3)_{k_i}}{(q^3x_i/x_j;q^3)_{k_i}} \prod_{i=1}^{r} \frac{(ax_i;q)_{|\mathbf{k}|}\,(aq/x_i;q)_{|\mathbf{k}|-3k_i}}{(ax_iq^{1+3n_i};q)_{|\mathbf{k}|}\,(aq^{1-3n_i}/x_i;q)_{|\mathbf{k}|}}$$

$$\times \frac{(1/a^2;q)_{|\mathbf{k}|}\,(a^2q;q)_{2|\mathbf{k}|}}{\prod_{i=1}^{r}(a^3x_iq^3;q^3)_{k_i}}\, a^{2|\mathbf{k}|} \prod_{i=1}^{r} x_i^{-2k_i} \cdot q^{6e_2(\mathbf{k})-\binom{2|\mathbf{k}|+1}{2}+3\sum_{i=1}^{r} ik_i}\Bigg)$$

$$= \prod_{i=1}^{r} \frac{(ax_iq;q)_{3n_i}\,(x_i/a^3;q^3)_{n_i}}{(x_i/a;q)_{3n_i}\,(a^3x_iq^3;q^3)_{n_i}}, \qquad (4.24)$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\{k_1,\dots,k_r\}$.

Remark 4.25. This cubic summation formula is a D_r extension of

$$\sum_{k=0}^{m} \frac{1-aq^{4k}}{1-a} \frac{(a;q)_k\,(b;q)_k\,(q/b;q)_{2k}\,(a^2bq^{3n};q^3)_k\,(q^{-3n};q^3)_k}{(q^3;q^3)_k\,(aq^3/b;q^3)_k\,(ab;q)_{2k}\,(q^{1-3n}/ab;q)_k\,(aq^{3n+1};q)_k} q^k = \frac{(aq;q)_{3n}}{(ab;q)_{3n}} \frac{(ab^2;q^3)_n}{(aq^3/b;q^3)_n}, \tag{4.26}$$

due to Gasper and Rahman [11, eq. (4.1), $c \to 1$], to which it reduces for $r = 1$.

Sketch of proof of Theorem 4.23. We replace $q \to q^3$, $p \to q$ in Theorem 4.10 and proceed as in the proof of Theorem 4.19. □

Remark 4.27. By reversing the multisum in (4.24) we may obtain another, differently looking, D_r extension of Gasper and Rahman's cubic summation formula (4.26).

Finally, by a polynomial argument, we may derive some more D_r cubic summation theorems from Theorem 4.23.

Appendix A.

Here we provide two determinant lemmas which we needed in the proofs of our Theorems 2.1 and 3.1. Our lemmas are interesting generalizations of the classical Vandermonde determinant evaluation

$$\det_{1\le i,j\le r} (x_i^{r-j}) = \prod_{1\le i<j\le r} (x_i - x_j),$$

and the "symplectic" Vandermonde determinant evaluation

$$\det_{1\le i,j\le r} (x_i^{r-j} - x_i^{r+j}) = \prod_{1\le i<j\le r} (x_i - x_j) \prod_{1\le i\le j\le r} (1 - x_i x_j),$$

respectively.

Lemma A.1. *Let $x_1, \dots, x_r$, $y_1, \dots, y_r$, a, and c be indeterminate. Then*

$$\det_{1\le i,j\le r} \left(x_i^{r+1-j} - a^{r+1-j} \frac{(x_i - c/\prod_{s=1}^r y_s)}{(a - c/\prod_{s=1}^r y_s)} \prod_{s=1}^{r} \frac{(x_i - y_s)}{(a - y_s)} \right) = \frac{(a - c/\prod_{j=1}^r x_j)}{(a - c/\prod_{j=1}^r y_j)} \prod_{i=1}^{r} \frac{(a - x_i)}{(a - y_i)} \prod_{i=1}^{r} x_i \prod_{1\le i<j\le r} (x_i - x_j). \tag{A.2}$$

Lemma A.3. *Let $x_1, \dots, x_r$, $y_1, \dots, y_r$, and c be indeterminate. Then*

$$\det_{1\le i,j\le r} \left(x_i^{r+1-j} - x_i^j \frac{(x_i - c/\prod_{s=1}^r y_s)}{(1 - x_i c/\prod_{s=1}^r y_s)} \prod_{s=1}^{r} \frac{(x_i - y_s)}{(1 - x_i y_s)} \right) = \prod_{i=1}^{r} \frac{(1 - y_i c/\prod_{j=1}^r y_j)}{(1 - x_i c/\prod_{j=1}^r y_j)} \prod_{i=1}^{r} (1 - x_i^2) \prod_{i=1}^{r} x_i \times \prod_{i,j=1}^{r} (1 - x_i y_j)^{-1} \prod_{1\le i<j\le r} [(x_i - x_j)(1 - x_i x_j)(1 - y_i y_j)]. \tag{A.4}$$

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