GENERALIZED LYASHKO–LOOIJENGA MAP, RAMIFIED COVERINGS OF THE SPHERE, AND ENUMERATION OF EDGE-LABELED k-TREES

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ABSTRACT. Let M be a Riemann surface of genus g, $f\colon M \to \mathbb{C}$ be a meromorphic function of degree n with l poles of orders $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$, $\sum_{i=1}^{l} n_i = n$. The generalized Lyashko–Looijenga map takes f to the polynomial Q whose roots are critical values of f. This mapping is a covering (up to the action of $\operatorname{Aut} M$) on the complement of the discriminant, that is, when all critical values of f are simple and distinct. Our aim is to find the multiplicity μ_ν^g, $\nu = (n_1, \dots, n_l)$, of this covering.

0. INTRODUCTION

The classic Lyashko–Looijenga map [A, L] takes a polynomial $P = x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ to the polynomial Q whose roots are critical values of P. This map is a covering on the complement of the swallowtail (which is the discriminant in the target space), and its multiplicity is equal to n^{n-3}. One observes easily that this number coincides with the number of edge-labeled trees on n vertices (see [AFPR]), and [L] presents a construction, which explains this coincidence. In this note we extend this construction to a general case of a meromorphic function on a Riemann surface and find explicit expressions for the number of edge-labeled graphs in several simple cases.

Let M be a Riemann surface of genus g, $f\colon M \to \mathbb{C}$ be a meromorphic function of degree n with l poles of orders $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$, $\sum_{i=1}^{l} n_i = n$. The generalized Lyashko–Looijenga map takes f to the polynomial Q whose roots are critical values of f. As in the case of polynomials, this mapping turns out to be a covering on the complement of the discriminant, that is, when all critical values of f are simple and distinct (more precisely, it is a covering up to the action of $\operatorname{Aut} M$). Our aim is to find the multiplicity μ_ν^g, $\nu = (n_1, \dots, n_l)$, of this covering.

In order to take into account the action of $\operatorname{Aut} M$ in the cases $g = 0$ and $g = 1$, one should introduce additional constraints. For $g = 0$ the group $\operatorname{Aut} M$ is 3-dimensional, and thus it suffices to fix one pole at the infinity, the leading coefficients of the numerator and the denominator at the unity, and the sum of the roots of the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX numerator at the origin. For $g = 1$ the group is 1-dimensional, thus it suffices to fix one of the poles. For $g \geqslant 2$ the group $\operatorname{Aut} M$ is discrete ([FK, p.242]); for $g = 2$ (a hyperelliptic surface) it is isomorphic to $\mathbb{Z}_2$, and for a general Riemann surface it may be assumed to be trivial.

Observe that our problem is intimately related to the Hurwitz problem of counting all ramified coverings of the sphere by a Riemann surface of genus g with a given set of ramification orders. References [M1, M2] contain a solution to the Hurwitz problem; however, the expression for the number of coverings presented there is extremely difficult to use and substantial efforts are needed to derive more suitable formulas for particular cases [M2].

Here we use the classic approach suggested by Hurwitz [H] for the case of holomorphic functions and rediscovered recently in the conformal field theory [D, GT].

The problem, along with several conjectures concerning the values of μ^g_ν for certain g and ν, was communicated to the authors by V. Arnold in summer, 1995. Later we had several stimulating discussions with him on various aspects of the problem. We cannot overestimate the role of T. Ekedahl, who explained to us the essence of the classic approach, and taught us several useful facts in the representation theory of the symmetric group. We are also grateful to S. Natanzon, who pointed out the references [M1, M2].

1. Edge–labeled k-trees

Let $G = (V, E)$ be a multigraph without loops, $|V| = n$, $|E| = m$. To each edge $e \in E$ we assign a mapping $\pi_e : V \to V$ that transposes the ends of e. Assume now that the edges of G are labeled by the numbers $1, \dots, m$. We then define a mapping $\pi_G : V \to V$ as the product of the transpositions π_e in the increasing order of labels. To represent π_G as an element of the symmetric group S_n, one have to choose a numbering of the elements of V. Evidently, all the permutations obtained in such a way for different numberings of the same graph belong to the same conjugacy class, and thus have the same cycle type. This cycle type is said to be the *cycle type* of the edge–labeled graph G. In the same way we define the *cycle partition* of G. Finally, if $\nu = (n_1, \dots, n_l)$ is the cycle partition of an edge–labeled graph G, then l is the *cycle length* of G.

We say that G is a *k-tree* if it is connected and its cyclomatic number equals k, that is, $k = m - n + 1$. The following proposition can be proved by induction on k.

Theorem 1. *Let G be an edge–labeled k-tree, then the cycle length of G can assume an arbitrary value in the set* $\{j : 1 \leqslant j \leqslant k + 1,\ j \neq k \mod 2\}$.

Let N^k_ν denote the number of edge–labeled k-trees on $n = \sum_{i=1}^l n_i$ vertices whose cycle partition is $\nu = (n_1, \dots, n_l)$. The Riemann–Hurwitz formula and a straightforward generalization of the Lyashko–Looijenga construction yield the following proposition.

Theorem 2.

$$\mu^g_\nu = N^{l+2g-1}_\nu, \quad n \geqslant 3.$$

2. Algebraic computations

Let Z denote the center of the group algebra of the symmetric group. It is well known that Z, as an algebra, is generated by the conjugacy classes of S_n (or, more exactly, by the sums of all the elements of a conjugacy class). Therefore, for each $z \in Z$ one can define $\delta(z)$ as the coefficient of the unity in the decomposition of z in a weighted sum of conjugacy classes.

Let now $\tilde{N}_\nu^k$ denote the number of edge–labeled graphs (not necessarily connected) on $n = \sum_{i=1}^l n_i$ vertices and $n+k-1$ edges whose cycle partition is $\nu = (n_1, \dots, n_l)$. The following proposition is an easy consequence of the above definitions.

Proposition 3.

$$\tilde{N}_\nu^k = \frac{1}{n!}\delta(z_2^{n+k-1} z_\nu), \tag{1}$$

where z_2 is the class of transpositions and z_ν is the class with the cycle partition ν.

To evaluate the right hand side of (1) we use certain results in the representation theory of the symmetric group. As follows from the main theorem of this theory,

$$\delta(z) = \frac{1}{n!}\sum_{\rho \vdash n} (f^\rho)^2 \psi_\rho(z),$$

where the sum is taken over all Young diagrams of length n, f^ρ is the multiplicity of the representation labeled by ρ, ψ_ρ is the central character corresponding to this representation. Applying the Frobenius theorem ([Ma, p.64]), we get the following result.

Proposition 4.

$$\delta(z_2^m z_\nu) = \frac{1}{n!}\sum_{\rho \vdash n} f^\rho (h(\rho') - h(\rho))^m |C_\nu| \chi_\rho^\nu, \tag{2}$$

where $h(\rho) = \sum (i-1)\rho_i$, ρ' is the diagram conjugate to ρ, C_ν is the conjugacy class with the cycle partition ν, χ_ρ^ν is the value of the character of the representation labeled by ρ on the class C_ν.

A typical expression one encounters while trying to evaluate the right hand side of (2) is

$$\sigma(t, p, \alpha) = \sum_{m=0}^{t} \binom{t}{m} (-1)^m (\alpha - m)^p,$$

where $p, t \in \mathbb{N}$, $\alpha \in \mathbb{R}$. We introduce the generating function

$$\Sigma(t, \alpha; x) = \sum_{p=0}^{\infty} \sigma(t, p, \alpha)\frac{x^p}{p!}$$

and obtain the following proposition.

Proposition 5.

$$\Sigma(t,\alpha;x) = e^{\alpha x}(1-e^{-x})^t. \tag{3}$$

It is easy to see that $\Sigma(t,\alpha;x)$ has a zero of order t at the origin. Thus, introducing coefficients $\Delta_q^t(\alpha)$ by

$$\sum_{q=0}^{\infty} \Delta_q^t(\alpha)x^q = e^{\alpha x}\left(\frac{1-e^{-x}}{x}\right)^t,$$

we can rewrite (3) as

$$\sigma(t,p,\alpha) = p!\Delta_{p-t}^t(\alpha). \tag{4}$$

3. Enumeration

Now we are ready to start computing N_ν^k for several simple cases. The simplest situation occurs when $l = 1$ and $\nu = (n)$, which corresponds to meromorphic functions with one pole. In this case the permutation π_G is a cycle, and thus graph G is forced to be connected. Thus, $N_n^k = \tilde{N}_n^k$. As an immediate consequence of Theorem 1 we get $N_n^{2g+1} = 0$ for $g = 0,1,\dots$; so we are now interested only in N_n^{2g}, $g = 0,1,\dots$, which, by Theorem 2, are just μ_n^g. A close examination of (2) reveals that in this case we are dealing with the sum $\sigma(n-1,p,\frac{n-1}{2})$. This fact was actually proved in [J] by the same representation–theoretic methods as above (see also [G] for a direct combinatorial proof). Thus, the generating function of $\Delta_q^{n-1}(\frac{n-1}{2})$ is just

$$\left(\frac{\sinh\frac{x}{2}}{\frac{x}{2}}\right)^{n-1},$$

and we get the following result.

Theorem 6.

$$\mu_n^g = N_n^{2g} = n^{n+2g-2}\frac{(n+2g-1)!}{n!}\Delta_{2g}^{n-1}\left(\frac{n-1}{2}\right), \quad n \geqslant 3.$$

In particular, from Theorem 6 we get

Corollary 7. *Let $n \geqslant 3$, then*

$$\mu_n^0 = n^{n-3}, \qquad \text{(Lyashko–Looijenga)}$$

$$\mu_n^1 = \frac{n^n(n^2-1)}{24},$$

$$\mu_n^2 = \frac{n^{n+2}(n^2-1)(n+3)(n+2)(5n-7)}{5760}.$$

Let us now consider the case when the cycle length of π_G equals 2, which corresponds to meromorphic functions having two poles. In this case G is either connected, or contains exactly two connected components. Let $\nu = (n-r,r)$, $r \leqslant n/2$. From Theorem 1 we immediately get $N_{n-r,r}^{2g} = 0$, $g = 0,1,\dots$. For the case of an odd cyclomatic number one gets the following result.

Proposition 8.

$$\tilde{N}^{2g+1}_{n-r,r} = N^{2g+1}_{n-r,r} + \sum_{s=0}^{g+1} \binom{n+2g}{r+2s-1} N^{2s}_{r} N^{2g+2-2s}_{n-r}, \qquad 0 < r < n-r.$$

Taking into account Theorem 6, we get the following

Theorem 9.

$$\mu^{g}_{n-r,r} = N^{2g+1}_{n-r,r} = \tilde{N}^{2g+1}_{n-r,r} - \binom{n}{r} \frac{(n+2g)!}{n!} r^{r-2}(n-r)^{n-r+2g} \times$$
$$\sum_{s=0}^{g+1} \left(\frac{r}{n-r}\right)^{2s} \Delta^{r-1}_{2s}\left(\frac{r-1}{2}\right) \Delta^{n-r-1}_{2g+2-2s}\left(\frac{n-r-1}{2}\right), \qquad 0 < r < n-r.$$

In the case $r = n-r$ the sum in the right hand side of the above expression should be taken over s varying from 0 to $\lfloor \frac{g+1}{2} \rfloor$.

In particular, for $g = 0$ we are dealing with a linear combination of the sums $\sigma(t,p,\alpha)$ for $\alpha = (n-7)/2$, $(n-5)/2$, $(n-1)/2$, $(n+1)/2$ and $p \leqslant n+5$. We thus get from Propositions 3–5 and Theorem 9 the following

Corollary 10.

$$\mu^{0}_{n-r,r} = N^{1}_{n-r,r} = \binom{n}{r} \frac{r^{r}(n-r)^{n-r}}{n}, \qquad 0 < r < n-r,$$
$$\mu^{0}_{r,r} = N^{1}_{r,r} = \binom{2r}{r} \frac{r^{2r-1}}{4}.$$

Observe that an expression for the total number of edge–labeled 1-trees, that is, for $\sum_{r=1}^{n-1} N^{1}_{n-r,r}$, was obtained earlier in [AFPR]. Comparing the two results we get the following identity.

Corollary 11.

$$\sum_{k=0}^{n-2} \frac{n^k}{k!} = \sum_{k=1}^{n-1} \frac{k^k}{k!} \frac{(n-k)^{n-k}}{(n-k)!}.$$

According to Proposition 8, in order to find N^{k}_{ν} for graphs of cycle length 2, we need to know these numbers for graphs of cycle length 1. In the same way, N^{k}_{ν} for the case $l(\nu) = 3$ can be expressed via the same numbers for smaller l. Since the calculations become more and more involved, we state here only one result.

Proposition 12.

$$\mu^{0}_{n-2,1,1} = N^{2}_{n-2,1,1} = \frac{(n-2)^{n-2} n (n^2-1)}{2}, \qquad n > 3.$$

Added in the final version: An anonymous referee kindly pointed out two important references of which we were unaware previously. Reference [CT] contains a simple explicit expression for $\mu^{0}_{1^n}$, and a recent preprint [GJ] offers a general formula for μ^{0}_{ν} for an arbitrary partition ν, which generalizes both the result of [CT] and several results of the present paper (Proposition 12, Corollary 10, and the first part of Corollary 7).

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