FACTORIZATIONS, TREES, AND CACTI

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ABSTRACT. The number of ways of factoring an n-cycle in S_n into a minimal number of smaller cycles is counted combinatorially, both under the condition that all such factorizations are distinct and under the condition that factorizations which differ only up to commutation of disjoint cycles are considered equivalent in which case we enumerate equivalence classes. In the first case, a bijection is given with a class of cacti (graphs for which every edge is contained in exactly one cycle) which are in turn counted using a Prüfer encoding. In the latter case, a bijection is given with a class of rooted plane trees.

I. Introduction.

Factorizations of elements of the symmetric group as a product of other elements turn up in a number of combinatorial circumstances. When the factorization is minimal (in a sense to be defined precisely in section II), it is called *short*. Every short factorization of an element σ of S_n decomposes naturally into short factorizations of each cycle in the disjoint cycle form of σ. Hence in some sense it suffices to consider only short factorizations of n-cycles in S_n.

There are two natural ways to count factorizations in S_n: either count all factorizations as being distinct, or consider factorizations which differ only up to commutation of disjoint elements as equivalent, and count equivalence classes of factorizations rather than actual factorizations. We will refer to the former as an *ordered* factorization and to the latter as an *inequivalent* factorization.

Section II presents some definitions and technical lemmas on the structure of short factorizations which will be necessary as we proceed.

In Section III we deal with ordered factorizations. The work of Dénes [De] implies that there are equally many ordered short factorizations of an n-cycle into transpositions as there are labeled rooted trees on $n-1$ vertices. Moszkowski [Mo] and Goulden and Pepper [GP] later gave direct bijections between these sets.

Our main result of this section will be the following, generalizing the result of Dénes:

Theorem 1.1. *The set of ordered short factorizations of all n-cycles in S_n is bijective with the set of doubly-labeled oriented cacti through a bijection which preserves cycle lengths.*

The cacti produced can be enumerated by Lagrange inversion; however we give a second bijection which simplifies enumeration, generalizing the Prüfer code commonly used to count labeled trees. Specifically, we will prove that the number of

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ordered short factorizations of $(1\,2\cdots n)$ into k terms of which exactly a_i are i-cycles for each i is

$$\frac{n^{k-1}k!}{\prod_{i\geq 2} a_i!},$$

which is easily seen to specialize to give the result of Dénes when $k=n-1$, $a_2=k$, and all other $a_i=0$. While Goulden and Jackson [GJ2] give a more general result, our method avoids some of the difficult computations they required.

Section IV answers the same questions as section III for inequivalent rather than ordered factorizations. Eidswick [Ei] and Longyear [Lo] were independently able to establish analytically that there are $b_{n-1}=\frac{1}{n-1}\binom{3(n-1)}{n-2}$ inequivalent short factorizations of an n-cycle into transpositions.

If we let $R=\sum_{n\geq 1} b_n t^n$, then Longyear's proof of this establishes the functional equation $R=t(1+R)^3$, which we immediately recognize as the generating function for non-trivial rooted plane trees each of whose vertices has degree 3 or 0, with respect to the number of non-leaf vertices. By a *rooted plane tree* we mean a tree with a distinguished vertex (the *root vertex*) embedded in the plane. If we direct each edge of a tree away from its root vertex, the *degree* of a vertex is defined to be the number of edges directed away from that vertex, and the set of *descendants* of this vertex is the set of vertices reached by these out-directed edges. Vertices with degree 0 are called *leaf vertices*; vertices all of whose descendants are leaf vertices are called *preleafs*.

The correspondence between inequivalent short factorizations and rooted plane trees above is more than just a coincidence: Goulden and Jackson [GJ1] proved that the number of inequivalent short factorizations of a $(km+1)$-cycle into $(k+1)$-cycles is $\frac{1}{m}\binom{(2k+1)m}{m-1}$. As they point out, this number also counts rooted plane trees on m non-leaf vertices each of which has degree $2k+1$. Our main result of this section will be the following generalization:

Theorem 1.2. *Inequivalent short factorizations of* $(1\,2\,3\cdots n)$ *into* k *cycles with* a_i *i-cycles for each* i *are bijective with rooted plane trees with* k *non-leaf vertices all of odd degree,* a_i *of degree* $2i-1$.

This will be proved through a direct bijection, reflecting the combinatorics of the functional equation implied by Longyear's work (and its generalization to the case where we include larger cycles than just transpositions). It should be noted that a completely different bijection can be constructed based instead on Eidswick's paper. His work amounts to proving the functional equation $R(t(1-t)^2)=t-t^2$, which somewhat surprisingly also counts these trees. Only the first of these two bijective proofs will be presented here.

Note that the number of rooted plane trees of the type specified in this theorem is known (see Erdélyi and Etherington [EE]); there are exactly

$$\frac{(\sum_{i\geq 1}(2i-1)a_i)!}{(1+\sum_{i\geq 2}(2i-2)a_i)!\prod_{i\geq 1} a_i!}$$

such trees, hence exactly this number of inequivalent short factorizations of a cycle into prescribed cycle types.

II. Preliminaries.

We call a factorization of $(1\,2\cdots n)$ (or any other cycle of length n) into a_i i-cycles ($i = 1, 2, \ldots$) *short* if $\sum_{i\geq 2}(i-1)a_i = n-1$. For example, $(1\,2\,3\,4\,5) = (2\,3)(4\,5)(1\,3\,5)$ is a short factorization of $(1\,2\,3\,4\,5)$ into two 2-cycles and one 3-cycle. To prevent confusion, we shall use the term *cycle* to denote a cycle (in the usual sense of the word) for which we desire a factorization; irreducible terms in this factorization will be called *factors*.

Two factorizations are considered *equivalent* if one may be obtained from the other through a sequence of steps, where each step consists of switching the order of two adjacent non-overlapping (i.e. on mutually disjoint sets of elements) factors. For example, the factorization above is equivalent to the factorization $(1\,2\,3\,4\,5) = (4\,5)(2\,3)(1\,3\,5)$.

If c is a cycle, we use the notation $[c]$ to denote an arbitrary short factorization of c. Thus the first example above is of the form $[(2\,3)][(1\,3\,4\,5)]$ while the second is not (although it is certainly equivalent to a factorization of that form).

The following result is well known (see for example [Ei]):

Lemma 2.1. *If $c_1, \ldots, c_r$ are disjoint cycles of lengths $n_1, \ldots, n_r$, respectively, then the product $c_1 \cdots c_r$ can be written as a product of no fewer than $(\sum_{i=1}^{r} n_i) - r$ transpositions. Moreover, any factorization of $c_1 \cdots c_r$ into exactly this many transpositions is equivalent to one of the form $[c_1][c_2]\cdots[c_r]$.*

The second part of this lemma is the most powerful. In essence, it states that we need only be concerned about short factorizations of a full cycle since factorizations of any other element ρ of S_n decompose naturally, up to equivalence, into the product of short factorizations of the cycles in the disjoint cycle form of ρ.

For our purposes, we require the following generalization of this lemma (to the case where factors other than transpositions may be used):

Lemma 2.2. *If $c_1, \ldots, c_r$ are disjoint cycles of lengths $n_1, \ldots, n_r$ respectively, then whenever the product $c_1c_2\cdots c_r$ is written as the product of a_i i-cycles for each i, we must have $(\sum_{i=1}^{r} n_i) - r \leq \sum_{i\geq 2}(i-1)a_i$. Moreover, any factorization of $c_1c_2\cdots c_r$ for which equality holds above is equivalent to one of the form $[c_1][c_2]\cdots[c_r]$.*

Proof. Given a factor $(a_1\,a_2\cdots a_l)$ where $a_1 < a_i$ whenever $2 \leq i \leq l$, we say its *canonical decomposition into transpositions* is $(a_1\,a_2)(a_2\,a_3)\cdots(a_{l-1}\,a_l)$ (which is well-defined since a_1 is the minimal element in our factor). We extend this definition by concatenation to a product of factors.

Suppose $c_1c_2c_3\cdots c_r = (a_{1,1}\,a_{1,2}\cdots a_{1,l_1})(a_{2,1}\cdots a_{2,l_2})\cdots(a_{k,1}\cdots a_{k,l_k})$ is a factorization of $c_1c_2\cdots c_r$ into factors, a_i of which are i-cycles for all i. Replacing each factor by its canonical decomposition into transpositions, we can apply Lemma 2.1:

$$(\sum_{i=1}^{r} n_i) - r \leq \sum_{i=1}^{k}(l_i - 1) = \sum_{i\geq 2}(i-1)a_i.$$

Now, if equality holds here then by Lemma 2.1 the transpositions of the canonical factorization into transpositions may be rearranged to give $c_1, c_2, \ldots, c_r$ in that order.

Since the cycles $c_1, c_2, \ldots, c_r$ are disjoint, transpositions which arose from the canonical decomposition of one factor all correspond to a single cycle c_j. Then within the set of transpositions giving c_j, we may again rearrange so that the transpositions giving the original factor are adjacent in the same order as before (since they were initially adjacent). Replacing these transpositions by the factor from which they arose, and repeating this process for all of the original factors, we have constructed an equivalent rearrangement of the initial factors of the decomposition of the form $[c_1][c_2]\cdots[c_r]$. ■

Our next lemma gives a useful description of any factors which may be commuted to either end of a short factorization of $(1\,2\cdots n)$. Notice that this result depends critically on the shortness of our factorization; without this assumption, the lemma is false.

Lemma 2.3. *Suppose* $(1\,2\,\cdots n) = cc_1c_2\cdots c_k$ *or* $(1\,2\,\cdots n) = c_1c_2\cdots c_kc$ *are short factorizations. Then we always have* $c = (t_1\,t_2\cdots t_l)$ *for some* l *and* $1 \le t_1 < t_2 < \cdots < t_l \le n$.

Proof. We will consider only the case where $(1\,2\,\cdots n) = cc_1c_2\cdots c_k$, as the other case follows similarly.

Let $c = (t_1\,t_2\cdots t_l)$. We may certainly assume $t_1 < t_i$ for all $i > 1$. Suppose there is some $j < l$ for which $t_j > t_{j+1}$, where without loss of generality we may choose the smallest possible j; we will reach a contradiction of the fact that the factorization was short.

Under our assumption, for some m we must have $t_1 < t_2 < \cdots < t_{m-1} < t_{j+1} < t_m < \cdots < t_j$. Note that $j > 1$ since t_1 was the minimal element of the cycle. Then we have:

$$\begin{aligned}
&(t_{j+1}\,t_{j-2}\cdots t_{l-1}\,t_l)c_1c_2\cdots c_k\\
&\qquad = (t_{j+1}\,t_j\cdots t_2\,t_1)(1\,2\cdots n)\\
&\qquad = (1\,2\cdots t_1-1\,t_{j+1}\,t_{j+1}+1\cdots t_m-1\,t_{m-1}t_{m-1}+1\cdots t_{j+1}-1\,t_j\cdots n)\\
&\qquad\qquad (t_1\,t_1+1\cdots t_2-1)\cdots(t_{m-2}\,t_{m-2}+1\cdots t_{m-1}-1)\\
&\qquad\qquad (t_m\,t_m+1t_{m+1}-1)\cdots(t_{j-1}\,t_{j-1}+1\cdots t_j-1).
\end{aligned}$$

Now, the right hand side of this last expression cannot be written as a product of fewer than $n-j+1$ transpositions. If $(t_1\,t_2\cdots t_{l-1}\,t_l)c_1c_2\cdots c_k$ is short, then $(t_{j+1}\,t_{j+2}\cdots t_l)c_1c_2\cdots c_k$ is also short. However the latter expression requires only $n-j-1$ transpositions, so our factorization could not have been short. ■

III. Ordered Factorizations and Cacti.

Denés was able to enumerate ordered short factorizations (as opposed to inequivalent short factorizations) of an n-cycle in S_n into transpositions by producing a bijection between ordered short factorizations of all n-cycles in S_n into transpositions and labeled, edge-labeled trees on n vertices. In this section we extend his result to deal with arbitrary ordered short factorizations into cycles (not necessarily just transpositions).

An *oriented labeled cactus* is a connected directed graph on n vertices which are labeled $1, 2, \ldots, n$ such that each edge is contained in exactly one directed cycle (it follows easily from the definition that there can be at most one edge from vertex a to vertex b, but there certainly could additionally be an edge from b to a). An *oriented doubly- labeled cactus* is an oriented labeled cactus with k directed cycles which are labeled $1, 2, \ldots, k$. An example of an oriented doubly- labeled cactus is given in Figure 3.1 (cycle labels are inside the corresponding cycles, vertex labels are by the corresponding vertex):

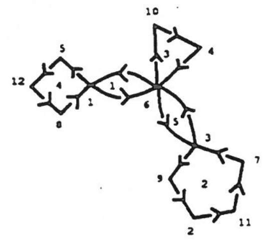

FIGURE 3.1. An oriented, doubly-labeled cactus

We will show that short factorizations of full cycles in S_n of prescribed cycle structure and oriented doubly-labeled cacti with prescribed cycle lengths are bijective. The latter will be counted using an extension of the usual Prüfer coding for labeled trees. We begin by establishing the connection between ordered factorizations and oriented cacti.

Theorem 3.1. *The set of ordered short factorizations of all n-cycles in S_n is bijective with the set of oriented doubly-labeled cacti on n vertices. Moreover this bijection preserves cycle structure: a short factorization containing exactly a_i i-cycles produces a cactus with exactly a_i i-cycles, and conversely.*

Proof.

Factorizations $\rightsquigarrow$ *Cacti.* Given a short factorization

$$(a_1\, a_2 \cdots a_n) = (a_{11}\, a_{12} \cdots a_{1l_1}) \cdots (a_{k1}\, a_{k2} \cdots a_{kl_k})$$

we correspond the directed graph with vertices $\{a_1, a_2, \ldots, a_n\}$ and directed cycles $a_{i1} \rightharpoondown a_{i2} \rightharpoondown \cdots \rightharpoondown a_{il_i} \rightharpoondown a_{i1}$ for $i = 1, 2, \ldots, k$. Label the cycle containing the a_{ij}'s by i. Figure 3.2 shows a factorization which produces the cactus given earlier:

(1 6)(2 11 7 3 9)(4 6 10)(1 5 12 8)(3 6)

= (1 5 12 8 6 9 2 11 7 3 10 4)

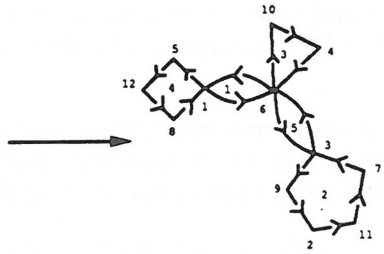

FIGURE 3.2. The cactus associated with a factorization

We claim that the digraph produced by this method is an oriented doubly-labeled cactus. Certainly the graph is connected and each edge is contained in at least one cycle. We must show each edge is on *exactly* one cycle.

The proof is by induction on k, the number of terms in our factorization. For $k = 1$, the result is clear. Suppose that the result holds for all short factorizations into fewer than k terms. Then if our factorization is as given above,

$$(a_1\, a_2 \cdots a_n)(a_{kl_k} \cdots a_{k2}\, a_{k1}) = (a_{11}\, a_{12} \cdots a_{1l_1}) \cdots (a_{k-11} \cdots a_{k-1l_{k-1}})$$

From Lemma 2.3, we may assume without loss of generality that $a_{k1} \prec a_{k2} \prec \cdots \prec a_{kl_k}$ in the order defined by $a_1 \prec a_2 \prec \cdots \prec a_n$, so the left hand side is seen to be a permutation consisting of l_k cycles in its disjoint cycle form. By Lemma 2.2, the terms on the right may be partitioned into l_k groups, each group multiplying to give the corresponding cycle in the disjoint cycle form on the left. By induction, each subgraph corresponding to a cycle in the disjoint cycle form is an oriented doubly-labeled cactus, so since the graph in question consists of these cacti (which are disjoint) and the cycle $a_{k1} - a_{k2} - \cdots \rightarrow a_{kl_k} \rightarrow a_{k1}$, each edge is on exactly one cycle, and hence the graph in question is indeed a cactus as claimed.

Cacti $\rightsquigarrow$ *Factorizations.* Given an oriented, doubly-labeled cactus with k cycles, the ith of which is $a_{i1} \rightarrow a_{i_2} \rightarrow \cdots - a_{il_i} \rightarrow a_{i1}$, we correspond the ordered factorization

$$(a_{11}\, a_{12} \cdots a_{1l_1}) \cdots (a_{k1} \cdots a_{kl_k})$$

in S_n. In Figure 3.2, it's clear that this map is just the inverse of the map demonstrated there, however in general it's not clear that the factorization produced by this map is a short factorization of a full cycle in S_n. We now show this is indeed the case by induction; again, for $k = 1$ the result is clear.

Suppose the result holds for cacti with fewer than k cycles. Removing the cycle with highest label then gives a set of l_k smaller oriented doubly-labeled cacti (possibly trivial). By induction, each corresponds to a short factorization of a full cycle on its vertex set. Since these cacti are disjoint, the terms of the product giving this set of cacti may be rearranged in the order

$$(a_{11} \cdots a_{1l_1}) \cdots (a_{k-11} \cdots a_{k-1l_{k-1}})$$

Multiplying by $(a_{k1} \cdots a_{kl_k})$ on the right joins these disjoint cycles into one full cycle (since each smaller cycle has exactly one element in common with the new cycle). Hence the factorization does indeed correspond to a short factorization of a full cycle in S_n.

The maps given above are easily seen to be mutually inverse, so the sets they map are bijective. Moreover, since the maps preserve cycle structure, short factorizations into a given cycle structure are bijective with cacti of that same cycle structure. ■

We now enumerate labeled oriented cacti (from which it is easy to enumerate doubly-labeled oriented cacti and hence, from the theorem just proved, ordered short factorizations).

Theorem 3.2. *There are exactly*

$$\frac{n^{k-1}\cdot(n-1)!}{\prod_{i\geq 2} a_i!}$$

labeled oriented cacti on n vertices with a total of k directed cycles, a_i of length i for each $i = 2, 3, 4, \ldots$.

Proof. To a cactus on n vertices with exactly k cycles, a_i of length i, we associate a sequence $c_1, c_2, \ldots, c_{k-1}$ of length $k-1$ whose values are in $[n] = 1, 2, \ldots, n$ and a set of k ordered tuples such that there are exactly a_i $(i-1)$-tuples and the tuples, viewed as unordered sets, form a partition of $[n] - \{c_{k-1}\}$.

Our proof will bijectively associate labeled oriented cacti with these objects. Note that these are easy to count: there are n^{k-1} such sequences and $\frac{(n-1)!}{a_1!a_2!\cdots}$ such sets of tuples, so the result will follow once our encoding is verified.

Given an oriented labeled cactus on more than one cycle, an *extreme cycle* is a cycle which, when removed, leaves a smaller connected cactus (an extreme cycle is the analogue of a leaf vertex in a tree). Note that there is always at least one extreme cycle.

Suppose $a_0 \to a_1 \to a_2 \to \cdots \to a_l \to a_0$ is the extreme cycle, attached at a_0 to the rest of the cactus, for which a_1 is minimal. For example, in the initial cactus given in Figure 3.4, this cycle would be $1 \to 5 \to 12 \to 4 \to 1$. Then we add a_0 to the sequence, $(a_1, a_2, \ldots, a_l)$ to the set of tuples, and continue recursively with the rest of the cactus. Eventually the cactus will consist of only one cycle $b_0 \to b_1 \to \cdots \to b_m \to b_0$, where b_0 is the point of attachment of the last removed extreme cycle. Adding $(b_1, b_2, \ldots, b_m)$ to the set of tuples completes our encoding of the cactus. Note that the sequence formed does indeed have length $k-1$, and each vertex except b_0 appears in exactly one removed cycle, so the tuples have the required structure. Figure 3.4 demonstrates this process for a sample cactus.

To reverse this process, scan for the lexicographically least tuple $(a_1, a_2, \ldots, a_l)$ which contains none of the entries in the sequence $c_1, c_2, \ldots, c_{k-1}$ (existence of such a tuple is guaranteed by the pigeon-hole principle). This corresponds to the cycle $c_1 \to a_1 \to a_2 \to \cdots \to a_k \to c_1$. Remove c_1 from the sequence and $(a_1, a_2, \ldots, a_k)$ from the set of tuples, and continue recursively. Eventually, only one tuple $(b_1, b_2, \ldots, b_m)$ will remain; this corresponds to the cycle $c_{k-1} \to b_1 \to b_2 \to \cdots \to b_m \to c_{k-1}$.

The resulting graph is a cactus since, considering the created cycles in reverse order, at each stage we adjoin a cycle to a vertex of a smaller cactus. This map is also clearly the inverse of our cactus decomposition map. Hence the sets are bijective, and so the number of labeled oriented cacti is as claimed. ■

As mentioned earlier, this encoding method is closely related to the usual Prüfer code used to count labeled trees. In fact, in the case where our cactus consists only of cycles of length 2, a labeled cactus is isomorphic to a labeled tree (associating a pair of directed edges in opposite directions from the cactus with an edge in the tree). The coding given above is then just a sequence of length $n-2$ where each value is chosen from $[n]$, since the set of tuples in this case is determined by the sequence. Indeed, the sequence associated to our labeled cactus in this case is exactly the same as given by the usual Prüfer correspondence.

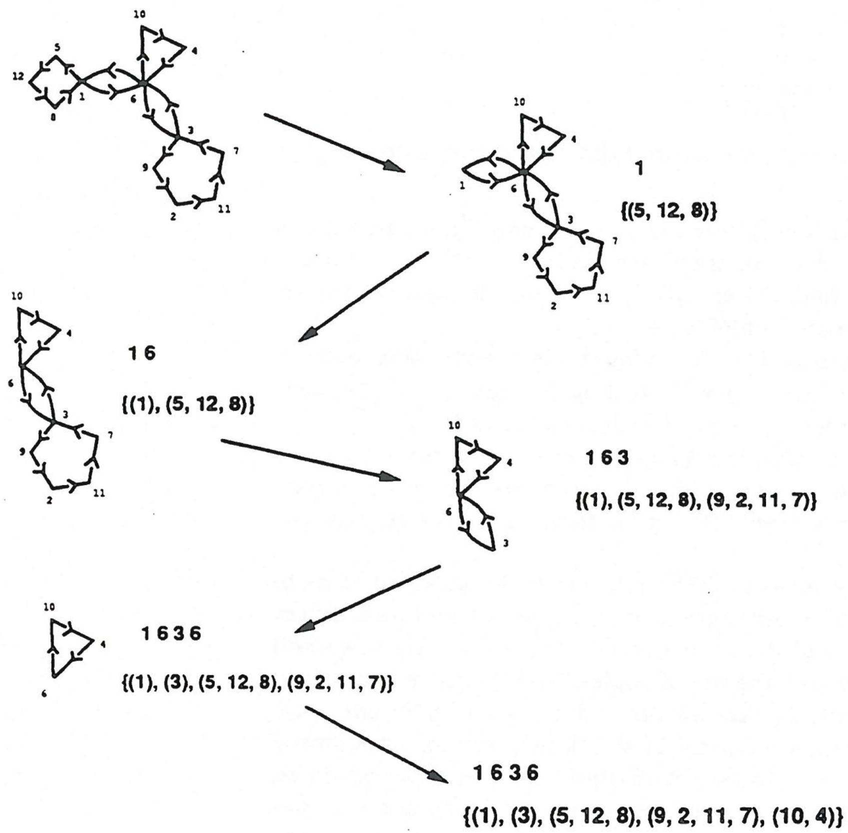

FIGURE 3.4. The recursive encoding procedure

We note also that this result can easily be proved by writing down a recursive definition for the generating function of a labeled, oriented, rooted cactus, applying Lagrange inversion, and dividing by n (to undo the effect of having chosen a root for our cactus).

Corollary 3.3. *There are exactly*

$$\frac{n^{k-1}k!}{\prod_{i\geq 2} a_i!}$$

short ordered factorizations of $(1\,2\,3\cdots n)$ *into* k *terms, each a cycle of some length, with exactly* a_i *i-cycles for each* i.

Proof. By Theorem 3.2, there are exactly $\frac{n^{k-1}k!(n-1)!}{\prod_{i\geq 2} a_i!}$ doubly-labeled oriented cacti on n vertices and k cycles with exactly a_i i-cycles for each i (since there are $k!$ cycle labelings for each labeled cactus). But by Theorem 3.1, this gives the total number of ordered short factorizations of full cycles into the given cycle structure. Since there are $(n-1)!$ full cycles in S_n, exactly one in $(n-1)!$ of these factorizations gives a factorization of $(1\,2\cdots n)$, giving our result. ■

More generally, Goupil and Bédard [GB] have given an explicit expression for the number of ordered short factorizations of any n-cycle in S_n into two factors of given cycle structure. Goulden and Jackson [GJ2] subsequently gave an expression for the number of such factorizations into an arbitrary number of factors of given cycle structure, a more general result than that given by Corollary 3.3. In fact, they give a bijection with plane edge-rooted cacti, which are then enumerated using Lagrange inversion. Although our bijection also deals with cacti, a different class of cacti are considered. By restricting our attention only to factorizations into cycles (as opposed to arbitrary elements of the symmetric group), we are able to count the resulting cacti with a simple encoding, as opposed to the relatively complex computation required in [GJ2] for the general answer.

In fact, note that Theorem 3.1 can be extended to deal with this more general problem. Our proof of this theorem implies the following result:

Corollary 3.4. *Ordered short factorizations of all n-cycles in S_n into factors with cycle-structure $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively (where $\lambda_i = 1^{\alpha_i^1} 2^{\alpha_i^2} \cdots$ is a partition of n) are bijective with oriented labeled cacti with exactly α_i^j j-cycles labeled i such that no two adjacent cycles have the same label.*

While this result is in fact stronger than Theorem 3.1, it is not clear how to extend the encoding of Theorem 3.2 to give a purely bijective enumerative result for the general case. The relative simplicity of the general answer in [GJ2] gives some hope that such a bijective proof may indeed exist, however.

IV. Inequivalent Factorizations and Plane Trees.

Having enumerated ordered factorizations, we now turn our attention to the enumeration of equivalence classes of factorizations. As mentioned in the introduction, we have actually found two distinct approaches, only the more natural of which is presented here.

Longyear's derivation of the number of inequivalent short factorizations of $(1\,2\cdots n)$ into transpositions [Lo] depended heavily on the decomposition

$$(1\,2\cdots n) = (2\,3\cdots a)(b+1\;b+2\cdots n\;1)(1\,a)(a\;a+1\cdots b)$$

which holds for $1 < a \leq b \leq n$. From this decomposition, it is not entirely surprising that there are as many such factorizations as there are ternary trees: a ternary tree decomposes into a root vertex and three subtrees just as a factorization decomposes into a transposition and three factorizations of shorter cycles.

Her decomposition extends nicely to the general case under consideration here:

Lemma 4.1. *Any short factorization of $(1\,2\cdots n)$ may be written uniquely (up to equivalence) in the form*

$$[(2\,3\cdots a_1)][(b_1+1\;b_1+2\cdots a_2)]\cdots[(b_k+1\cdots n\;1)](1\;a_1\;a_2\cdots a_k)$$
$$[(a_1\;a_1+1\cdots b_1)]\cdots[(a_k\;a_k+1\cdots b_k)]$$

for some k and $1 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \leq n$ depending only on the factorization.

Proof. Suppose the rightmost factor which moves the element 1 is $\tau = (1\;a_1\;a_2\cdots a_k)$ (certainly this factor does not depend on the equivalent factorization chosen). Commute as many factors as possible to the right of τ, and let σ be the product of all factors to the right of τ in the resulting equivalent factorization. Of course, these terms form a *short* factorization of σ since the original factorization of the full cycle is short. Suppose that the cycle in the disjoint cycle form of σ which includes the element a_i is $(a_{i_0}\, a_{i_1} \cdots a_{i_{l_i}})$ where $l_i \geq 0$ and $i_0 < i_j$ for $1 \leq j \leq l_i$.

Then we can form a new short factorization of $(1\,2 \cdots n)$ of the form

$$(3.1) \qquad (1\,2 \cdots n) = \cdots (1\, a_1 \cdots a_k) \cdots (a_{i_0}\, a_{i_1} \cdots a_{i_{l_i}})$$

by replacing the factors forming $(a_{i_0} \cdots a_{i_{l_i}})$ with this cycle. Note that this cycle may in fact be commuted to the rightmost position; as a result of this we must have $a_{i_0} < a_{i_1} < \cdots < a_{i_{l_i}}$ by Lemma 2.3. The other factors in (3.1) besides $(a_{i_0}\, a_{i_1} \cdots a_{i_{l_i}})$ form in turn a short factorization of $(1\,2 \cdots n)(a_{i_{l_i}} \cdots a_{i_0})$, which is

$$(3.2) \qquad (1\,2 \cdots a_{i_0}\, a_{i_{l_i}} + 1 \cdots n)(a_{i_0} + 1 \cdots a_{i_1}) \cdots (a_{l_i - 1} + 1 \cdots a_{i_{l_i}}).$$

Since τ contributes only to the first cycle in (3.2) by Lemma 2.2, we must have $i_0 = i$. Furthermore, since $(a_{i_0}\, a_{i_1} \cdots a_{i_{l_i}})$ was a cycle in the disjoint cycle form of σ, all of the other cycles in (3.2) must be trivial. As a result, $a_{i_s} = a_i + s$ for all s, so the cycle of σ involving a_i is of the form $(a_i\, a_i + 1 \cdots b_i)$ for some b_i uniquely determined by the factorization.

From this, after commuting disjoint cycles of σ which commute with τ to the left of τ, we see that our factorization may be rearranged to give an equivalent factorization of the form

$$(1\,2 \cdots n) = \cdots (1\, a_1 \cdots a_k)[(a_1 \cdots b_1)] \cdots [(a_k \cdots b_k)].$$

But then we have

$$\begin{aligned}(1\,2 \cdots n)\{(1\, a_1 \cdots a_k)(a_1 \cdots b_1) \cdots (a_k \cdots b_k)\}^{-1} \\ = (2\,3 \cdots a_1)(b_1 + 1 \cdots a_2) \cdots (b_k + 1 \cdots n\, 1),\end{aligned}$$

so any factorization of $(1\,2 \cdots n)$ must indeed decompose in the manner stated in the lemma. ■

Having proven this lemma, we are in a position to give a direct bijective proof of the main theorem for this section.

Theorem 4.2. *There is a bijection between inequivalent short factorizations of $(1\,2\,3 \cdots n)$ into k cycles with a_i i-cycles for each i and rooted plane trees with k non-leaf vertices, all of odd degree with a_i of degree $2i - 1$.*

Proof.

Factorizations $\rightsquigarrow$ Trees. The null factorization of a null cycle corresponds to a tree consisting only of a root vertex.

For any other factorization, we may use Lemma 4.1 to decompose $(1\,2 \cdots n)$ uniquely in the form

$$\begin{aligned}[(2\,3 \cdots a_1)][(b_1 + 1\, b_1 + 2 \cdots a_2)] \cdots [(b_k + 1 \cdots n\, 1)](1\, a_1\, a_2 \cdots a_k) \\ [(a_1\, a_1 + 1 \cdots b_1)] \cdots [(a_k\, a_k + 1 \cdots b_k)]\end{aligned}$$

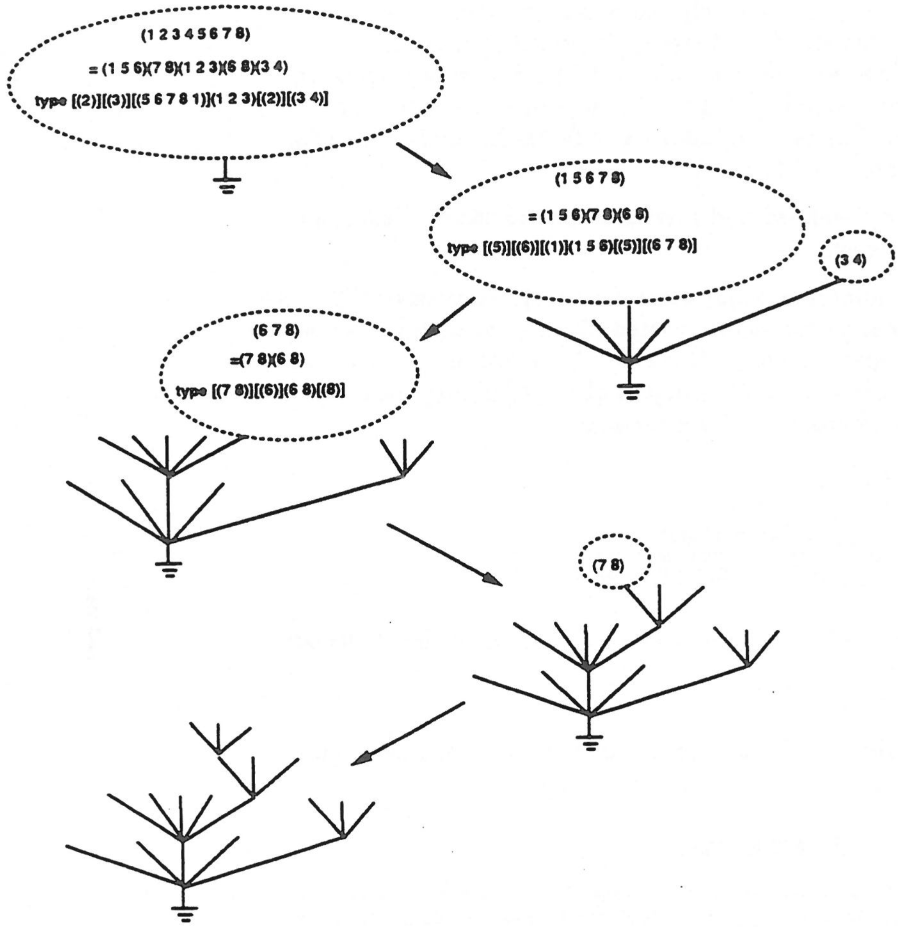

FIGURE 4.1. The tree corresponding to $(1\,2\,3\,4\,5\,6\,7\,8) = (1\,5\,6)(7\,8)(1\,2\,3)(6\,8)(3\,4)$

This corresponds to the tree with root degree $2k+1$, whose subtrees correspond, from left to right, to the trees determined by the factorizations $[(2\,3 \cdots a_1)], \ldots, [(a_k\; a_k + 1 \cdots b_k)]$ respectively.

This procedure is demonstrated in Figure 4.1.

Trees ⇝ Factorizations. A tree consisting only of a root vertex corresponds to the null factorization of a null cycle.

For any other tree, suppose the root vertex has degree $2k+1$ and the $2k+1$ subtrees, read from left to right, have $2l_1 - 1, 2l_2 - 1, \ldots, 2l_{2k+1} - 1$ leaf vertices respectively.

Then we correspondingly decompose $(12 \cdots n)$ as

$$[(2\,3 \cdots a_1)][(b_1 + 1\; b_1 + 2 \cdots a_2)] \cdots [(b_k + 1 \cdots n\; 1)](1\; a_1\; a_2 \cdots a_k)$$
$$[(a_1\; a_1 + 1 \cdots b_1)] \cdots [(a_k\; a_k + 1 \cdots b_k)]$$

where the first $k+1$ cycles have lengths $l_1, l_2, \ldots\; l_{k+1}$ respectively and the last k cycles have lengths $l_{k+2}, \ldots, l_{2k}, l_{2k+1}$ respectively (note that the values of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are determined uniquely from these cycle length restrictions).

To complete the factorization, we now determine a factorization of each of the $2k+1$ subcycles in the factorization above as given by the respective subtree of the given tree. Note that the term $(1\,a_1\,a_2 \cdots a_k)$ above is a factor in the factorization we seek, not a cycle to be decomposed further.

The maps above are each well-defined and inverse to one another. Hence they give a bijective proof of our theorem. ■

Theorem 4.2 shows that to enumerate inequivalent short factorizations of $(1\,2 \cdots n)$ into smaller cycles it is sufficient to enumerate rooted plane trees with a specified number of internal vertices of given degrees. However, this question has been answered in full generality by Erdélyi and Etherington [EE]. Applying their result gives the following answer to our problem of this section:

Corollary 4.3. *There are exactly*

$$\frac{(\sum_{i\geq 1}(2i-1)a_i)!}{(1+\sum_{i\geq 2}(2i-2)a_i)!\prod_{i\geq 1} a_i!}$$

inequivalent short factorizations of $(1\,2 \cdots n)$ *into smaller cycles such that there are exactly* a_i *i-cycles for each* i.

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