VERTICAL STRIP TABLEAU GAMES

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ABSTRACT. We introduce two new tableau games. Instead of numbers moving along single insertion/deletion paths as they do in the Robinson-Schensted algorithm, evacuation, and the *jeu de taquin*, they move through vertical strips. We describe their relation to evacuation and the *jeu de taquin*, and also describe how a non-deterministic variation of them describes certain configuration problems in the invariant subspace lattice and, more generally, in semi-primary lattices.

This work is based on a portion of the author's Ph.D. thesis [5]; full details and similar analysis of other tableau games and configuration problems may be found there.

1. INTRODUCTION

In the well-known evacuation and *jeu de taquin* tableau games, a "hole" is placed in a tableau, and a number slides into it, leaving a hole in its stead. Another number slides into that hole, and another into the hole created by that number, according to certain rules, until a certain termination condition is reached, at which point the hole is removed from the tableau. The cells through which numbers move form a single path called a deletion path. In Sections 2 and 3, we introduce two new tableau games, $\mathcal{L}P$ and $\mathcal{R}P$, in which numbers in a tableau P percolate either leftward ($\mathcal{L}$) or rightward ($\mathcal{R}$) through the tableau as a vertical strip of holes percolates through in the opposite direction. We also introduce non-deterministic variations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ of these games that allow multiple options for where numbers and holes move at each step.

In Section 4, we next consider flags $f = (f_0 \lessdot \cdots \lessdot f_n)$ in the lattice of all subspaces of an n-dimensional vector space that are invariant under the action of a nilpotent transformation N; in the lattice of subgroups of an abelian p-group; and most generally, in semi-primary lattices. Here, $x \lessdot y$ means $x < y$ and nothing is between them, and $x \leq y$ means $x < y$ or $x = y$. In Section 5 we describe how these non-deterministic games describe the possible types of the flags $(Nf_0, \ldots, Nf_n)$ and $(Af_0, \ldots, Af_n)$, where Ax is the kernel of N on the space f_n/x, and analogous statements for the other lattices. The games $\mathcal{L}$ and $\mathcal{R}$ describe the dominant configuration in a topological sense if the vector space is over an algebraically closed field, and in an enumerative sense if it is over a finite field. This is similar to Steinberg's [4] relation between the relative position of two flags and the Robinson-Schensted algorithm, and to Hesselink [1] and van Leeuwen's [6] relation between flag cotypes and the evacuation algorithm.

Finally, in Section 6, we describe the relationship of these new games to evacuation and the *jeu de taquin*.

2. The leftward vertical strip game, $\mathcal{L}P$

Background on partitions, tableaux, and common tableau games may be found, for example, in Sagan [**3**]. A **partition** λ of a nonnegative integer n is a sequence of weakly decreasing nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose sum is n. We also write $|\lambda| = n$ or $\lambda \vdash n$. Two partitions are regarded as equal when they have the same sequence of nonzero entries, so if the sequence has length k, we implicitly set $\lambda_i = 0$ for $i > k$. The **Young diagram** of a partition λ is a pictorial representation as a left-justified array of $|\lambda|$ squares, with λ_i squares (or **cells**) on the ith row from the top. A **skew partition** is a pair of shapes, denoted λ/μ, such that $\mu_i \leq \lambda_i$ for all i, and its cells are the cells of λ not also in μ. We regard a **tableau** of shape λ to be a Young diagram of shape λ with a number (**entry**) placed in each cell in such a fashion that the numbers weakly increase from left to right in each row and top to bottom in each column. A **skew tableau** of shape λ/μ is similarly defined by filling the cells of the Young diagram of μ with the symbol $\bullet$, and the cells of λ/μ with numbers constrained as before. A tableau or skew tableau is **standard** when all the numeric entries are distinct. A skew tableau of shape λ/μ has **outer shape** λ and **inner shape** μ.

In the course of playing our games, we will consider tableaux with "holes," which are cells with an "entry" $\star$. We use this symbol for holes instead of the traditional $\bullet$ because we also require the traditional use of $\bullet$ for cells of the inner shape. A number k **slides** to a specified hole by placing k at that hole and a hole $\star$ at the former position of k. In previous tableau games, slides occurred between adjacent cells; while we do not require the cells to be adjacent, we do require that the resulting tableau still has all its numeric entries increasing from left to right and top to bottom.

We begin with a standard skew tableau P on a subset of the numbers $1, \ldots, n$ (or more generally, a subset of some totally ordered index set), and form a sequence $P^{(n)}, \ldots, P^{(0)}$ of tableaux with holes according to the following rules. Set $P^{(n)} := P$. To transform $P^{(k)}$ to $P^{(k-1)}$ for $k = n, \ldots, 1$, if there is no k in $P^{(k)}$, let $P^{(k-1)} := P^{(k)}$, and otherwise do one of the following, depending on which game $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ is being played.

- **($\mathcal{L}$):** Slide k to the bottom leftmost $\star$ in a column strictly right of k, provided there is such a $\star$; if there is not, replace k by $\star$.
- **($\widetilde{\mathcal{L}}$):** If the cell right of k has $\star$, slide k into it; otherwise, choose either to slide k into the bottommost $\star$ in any column right of k, or to replace k by $\star$ without putting a new k anywhere.

Finally, replace all $\star$'s with $\bullet$'s in $P^{(0)}$ to obtain $\mathcal{L}P$ or $\widetilde{\mathcal{L}}P$.

Full details and proofs regarding this game may be found in [**5**, Ch. 8.1].

Example 2.1. We will compute all possible values of $\widetilde{\mathcal{L}}P$ for

$$P = P^{(8)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & 8 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

We begin with $k = 8$. There is no $\star$ in the tableau, so there is nowhere to slide 8, so 8 is replaced by $\star$.

$$P^{(7)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

There is no $\star$ in any column right of 7, so 7 is replaced by $\star$, and similarly, the same then happens with 6.

$$P^{(6)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \qquad P^{(5)} = \begin{array}{|c|c|c|} \hline 1 & 4 & \star \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Both columns right of 5 contain $\star$. For $\widetilde{\mathcal{L}}P$, there are choices as to how to proceed, which we'll examine later. To compute $\mathcal{L}P$, slide 5 to the lower $\star$ of column 2, obtaining

$$P^{(4)} = \begin{array}{|c|c|c|} \hline 1 & 4 & \star \\ \hline 2 & \star \\ \cline{1-2} 3 & 5 \\ \cline{1-2} \star \\ \cline{1-1} \end{array}$$

Next, the cell just right of 4 has $\star$, so slide 4 there.

$$P^{(3)} = \begin{array}{|c|c|c|} \hline 1 & \star & 4 \\ \hline 2 & \star \\ \cline{1-2} 3 & 5 \\ \cline{1-2} \star \\ \cline{1-1} \end{array}$$

The column right of 3 has two $\star$'s; slide 3 to the lower one.

$$P^{(2)} = \begin{array}{|c|c|c|} \hline 1 & \star & 4 \\ \hline 2 & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1} \end{array}$$

The column right of 2 has a $\star$, where we slide 2.

$$P^{(1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline \star & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1} \end{array}$$

There are no $\star$'s in columns right of 1, so 1 is replaced by $\star$.

$$P^{(0)} = \begin{array}{|c|c|c|} \hline \star & 2 & 4 \\ \hline \star & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1} \end{array}$$

Replace the $\star$'s by $\bullet$'s to obtain

$$\mathcal{L}P = \begin{array}{|c|c|c|} \hline \bullet & 2 & 4 \\ \hline \bullet & 3 \\ \cline{1-2} \bullet & 5 \\ \cline{1-2} \bullet \\ \cline{1-1} \end{array}$$

There are several places we could have made non-deterministic choices. The first was at $k = 5$; to compute $\mathcal{L}P$, we put 5 in the lower $\star$ of column 2, but to compute $\tilde{\mathcal{L}}P$, we can either do that, slide 5 to the $\star$ in column 3, or replace 5 by $\star$ without sliding 5 somewhere else. In the second case, we have

$$P^{(4)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & \star & \\ \cline{1-2} 3 & \star & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array}$$

There is no $\star$ in columns right of 4, so 4 is replaced by $\star$.

$$P^{(3)} = \begin{array}{|c|c|c|} \hline 1 & \star & 5 \\ \hline 2 & \star & \\ \cline{1-2} 3 & \star & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array}$$

Each of 3, 2, and 1 in turn have $\star$ just to their right, so we are successively forced to have

$$P^{(2)} = \begin{array}{|c|c|c|} \hline 1 & \star & 5 \\ \hline 2 & \star & \\ \cline{1-2} \star & 3 & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array} \qquad P^{(1)} = \begin{array}{|c|c|c|} \hline 1 & \star & 5 \\ \hline \star & 2 & \\ \cline{1-2} \star & 3 & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array} \qquad P^{(0)} = \begin{array}{|c|c|c|} \hline \star & 1 & 5 \\ \hline \star & 2 & \\ \cline{1-2} \star & 3 & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array}$$

Finally, replace the $\star$'s by $\bullet$'s to obtain another possible value of $\tilde{\mathcal{L}}P$.

We list all possible choices in Figure 1.

Example 2.2. Now consider the skew tableau

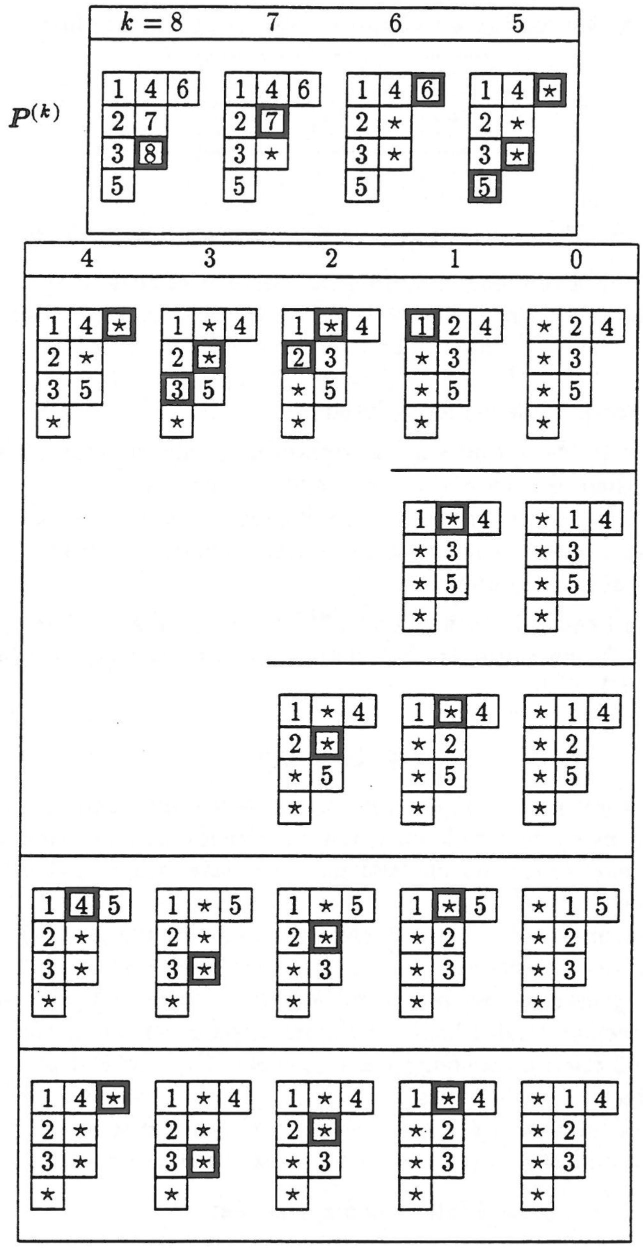
$$P = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 6 \\ \hline \bullet & 7 & \\ \cline{1-2} \bullet & 8 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

regarded as having entries that are a subset of $\{4, \dots, 8\}$ instead of $\{1, \dots, 8\}$. The movements of $k = 8, 7, 6$ are similar to the previous example, since the entries larger than 5 are in the same positions in both tableaux.

$$P^{(7)} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 6 \\ \hline \bullet & 7 & \\ \cline{1-2} \bullet & \star & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} \qquad P^{(6)} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 6 \\ \hline \bullet & \star & \\ \cline{1-2} \bullet & \star & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} \qquad P^{(5)} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & \star \\ \hline \bullet & \star & \\ \cline{1-2} \bullet & \star & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

There is no 5, so $P^{(4)} = P^{(5)}$. Next, 4 has the same position and possible holes as 5 in the previous example, so for $P^{(3)}$ we have three choices.

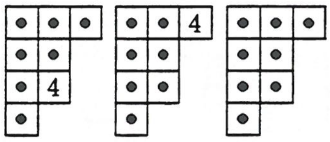
$$\begin{array}{|c|c|c|} \hline \bullet & \bullet & \star \\ \hline \bullet & \star & \\ \cline{1-2} \bullet & 4 & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline \bullet & \star & \\ \cline{1-2} \bullet & \star & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline \bullet & \bullet & \star \\ \hline \bullet & \star & \\ \cline{1-2} \bullet & \star & \\ \cline{1-2} \star & & \\ \cline{1-1} \end{array}$$



A missing tableau abbreviates the tableau above it. The possible transformations of $P^{(k)}$ to $P^{(k-1)}$ are indicated on $P^{(k)}$: either slide k to a $\boxed{\star}$ or change $\boxed{k}$ to $\boxed{\star}$. Finally, replace $\star$'s by $\bullet$'s in $P^{(0)}$ to obtain $\widetilde{\mathcal{L}P}$. The first game is $\mathcal{L}P$.

FIGURE 1. All possible non-deterministic leftward vertical strip games.

Replace all $\star$'s by $\bullet$'s to obtain the possible values of $\widetilde{\mathcal{L}}P$, the first of which is $\mathcal{L}P$.



3. The rightward vertical strip game, $\mathcal{R}P$

Again let P be a row and column strict tableau of skew shape on a subset of the numbers $1, \ldots, n$, with all entries distinct. We form a sequence of tableau $P^{(0)}, \ldots, P^{(n)}$. Set $P^{(0)} := P$. To transform $P^{(k-1)}$ to $P^{(k)}$ for $k = 1, \ldots, n$, if there is no k in $P^{(k-1)}$, let $P^{(k)} := P^{(k-1)}$, and otherwise do the following, depending on which game is being played.

($\mathcal{R}$): Slide k to the $\star$ that's upper rightmost in the columns strictly left of k, provided there is such a $\star$; if there is not, replace k by $\star$.

($\widetilde{\mathcal{R}}$): If the cell left of k has $\star$, slide k into it; otherwise, choose either to slide k into the topmost $\star$ in any column left of k, or to replace k by $\star$ without putting a new k anywhere.

Finally, delete all the cells with $\star$ from $P^{(n)}$ to obtain $\mathcal{R}P$ or $\widetilde{\mathcal{R}}P$.

See Figure 2 for an example. Full details and proofs regarding this game may be found in [**5**, Ch. 8.2].

4. Lattices

We consider some lattices that are known to have similar enumerative properties, and a uniform theory in which they may be considered simultaneously. We also introduce two new operators on lattices; they have wider applicability than we discuss here, but we only evaluate them for examples applicable here. Further details may be found in [**5**, Ch. 3]. Recall that a **lattice** is a partially ordered set in which each pair of elements x, y has a unique least upper bound $x \vee y$ (read "x join y") and unique greatest lower bound $x \wedge y$ (read "x meet y"). For our purposes, a **modular lattice** is a graded lattice with a minimum element $\hat{0}$, maximum element $\hat{1}$, and a rank function ρ satisfying $\rho(x \vee y) + \rho(x \wedge y) = \rho(x) + \rho(y)$. An **interval** $[x, y]$ is the sublattice of elements $\{\, z : x \leq z \leq y \,\}$ with the induced ordering, and the length of an interval is $\rho(x, y) = \rho(y) - \rho(x)$. We write $y \gtrdot x$ (read "y covers x") when $y > x$ and $\rho(y) = \rho(x) + 1$, and $y \geq\!\!\cdot\, x$ when $y \gtrdot x$ or $y = x$.

Definition 4.1. Let $\mathcal{L}$ be a lattice and $x \in \mathcal{L}$. Let

$$Ax = \bigvee_{y \gtrdot x} y$$

be the join of all elements covering x, or x if x is maximal, and let

$$Cx = \bigwedge_{y \lessdot x} y$$

be the meet of all elements covered by x, or x if x is minimal. If $x \leq z$ are both in $\mathcal{L}$, let

$$A_z x = \bigvee_{y:\, x \lessdot y \leq z} y$$

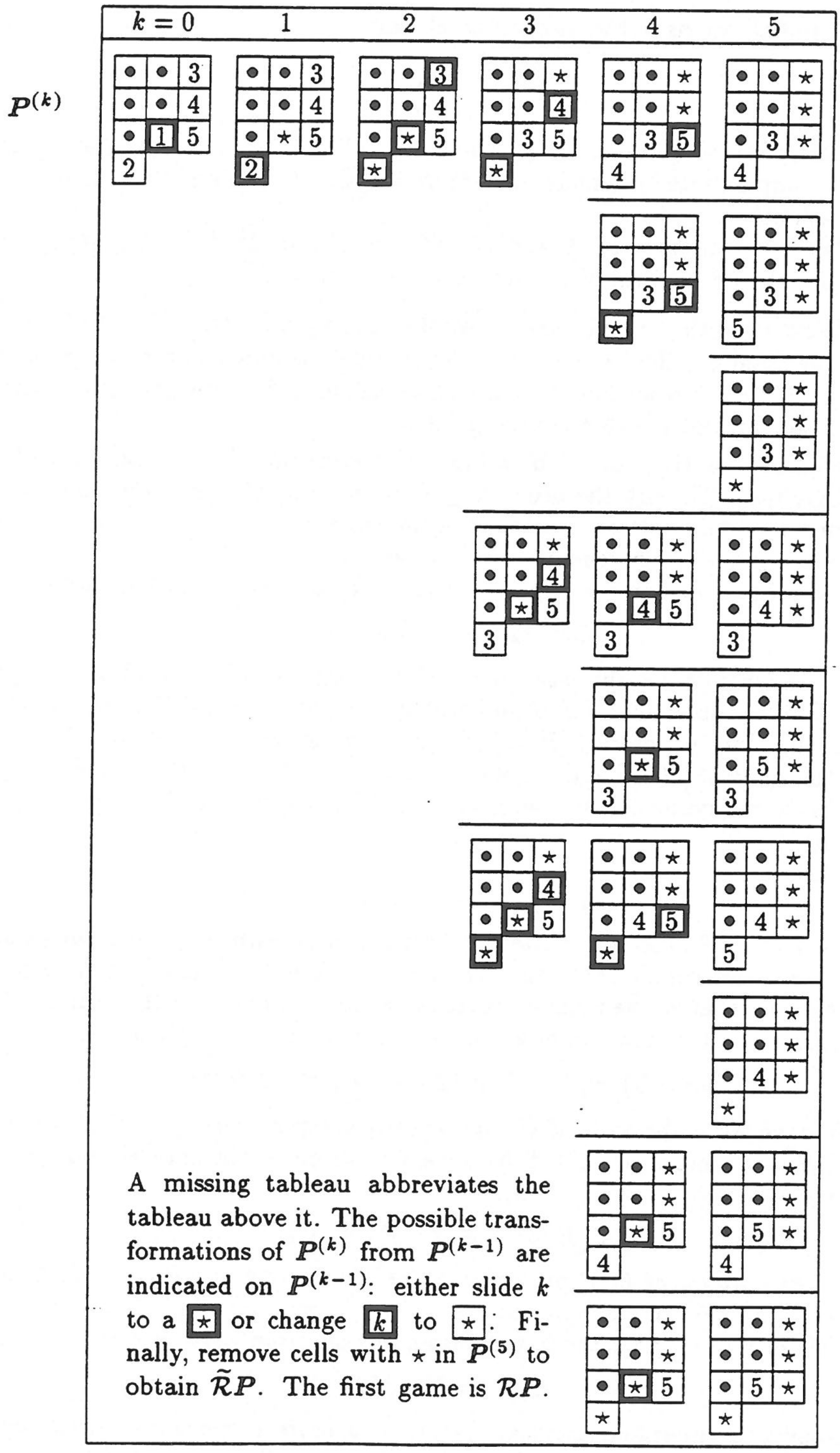


FIGURE 2. All possible non-deterministic rightward vertical strip games.

be the join of all atoms in the interval $[x, z]$, and

$$C_x z = \bigwedge_{y:\, x \lessdot y \leq z} y$$

be the meet of all coatoms in $[x, z]$. These are well defined in any discrete lattice where all complemented intervals have finite length. The operators A and C are dual to each other.

We also define iterates of A and C: $A_y^0 x = x$ and $A_y^{k+1} x = A_y(A_y^k x)$, and similarly for the unary form of A, and for both forms of C.

This coincides with the **Frattini element** of a complete lattice [7, p. 214], which is $C\hat{1}$ in our notation. However, we consider A and C as unary or binary operators, and the restriction that all complemented intervals have finite length yields certain properties not present in other complete lattices.

1. **Subgroup lattice.** Let G be a finite abelian group. The collection $L(G)$ of subgroups of G, with the order $H \leq K$ if and only if H is a subgroup of K, forms a modular lattice. The meet of two subgroups is their intersection, and the join is the group generated by the two.

   Any finite abelian p-group is isomorphic to a product of cyclic p-groups,

   $$\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

   Sort the λ_i into weakly decreasing order to form a partition called the **type of** G. Any quotient K/H of finite abelian p-groups is itself a finite abelian p-group, and the interval $[H, K]$ of $L(G)$ is isomorphic to $L(K/H)$. The type of the interval $[H, K]$ is the type of the group K/H, and this is less than or equal to the type of G in Young's lattice. The length of an interval of type μ is $|\mu|$.

   Let

   $$G = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

   The atoms of $L(G)$ are nonzero subgroups of G with no proper subgroups; they are isomorphic to $\mathbb{Z}/pZ$. The **socle** of G is the maximum elementary subgroup, that is, the unique subgroup isomorphic to one of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \cdots$. It is the join of all the atomic subgroups. Explicitly, it is

   $$\mathrm{socle}(G) = p^{\lambda_1 - 1}\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times p^{\lambda_k - 1}\mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

   We have $A\hat{0}$ is the socle of G, and for any subgroups $H \leq K$ of G, we have $A_K H$ is the image in $L(G)$ of the socle of K/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. Thus,

   $$A_K^r H = \{\, g \in H : p^r g \in K \,\}.$$

   The coatoms of $L(G)$ are those K for which $G/K \cong \mathbb{Z}/p\mathbb{Z}$. Their meet is $CG = pG$. For any subgroups $H \leq K$ of G, we have $C_H K$ is the image in $L(G)$ of pK/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. So

   $$C_H^r K = H + p^r K.$$

2. **Invariant subspace lattice.** Let V be a finite dimensional vector space, and N be a nilpotent transformation, that is, a linear transformation such that for all vectors $\vec{v} \in V$, we have $N^k \vec{v} = 0$ for sufficiently large k. A subspace W of V is N-invariant if and only if NW is a subspace of W. The collection $L(V, N)$ of all N-invariant subspaces of V, ordered by subspace

inclusion, forms a modular lattice. The meet of two N-invariant subspaces is their intersection, and the join is their span.

Because N is nilpotent, all the roots of its characteristic equation are 0. The Jordan canonical form of N is a matrix that is the block sum of blocks with the characteristic root 0 on the diagonal; 1 just above each entry on the main diagonal; and 0 everywhere else. The lengths of the blocks when sorted into weakly decreasing order form a partition that is an invariant of N, called its **Jordan type**. This is the type of the lattice $L(V, N)$.

If $W \leq X$ are two N-invariant subspaces of V, their quotient X/W has an action induced by N. The lattice $L(X/W, N)$ is isomorphic to the interval $[W, X]$ in $L(V, N)$. The type of the interval $[W, X]$ is the type of the lattice $L(X/W, N)$, and is a subpartition of the type of $L(V, N)$ in Young's lattice. The length of an interval of type μ is $|\mu|$.

The atoms of $L(V, N)$ are the nonzero N-invariant subspaces of V with no nonzero proper subspaces that are N-invariant. If $W \in L(V, N)$ and $W \neq 0$, then NW is a proper subspace of W because the chain $W \supseteq NW \supseteq N^2W \supseteq \cdots$ is 0 after a finite number of steps, and once a $\supseteq$ is $=$, so are all further ones to the right. Thus, if W is atomic, it is a subspace of the kernel of N. The atoms of $L(V, N)$ are 1-dimensional subspaces of V spanned by some vector from the kernel of N, and the join of all the atoms is the kernel of N.

Let $W \leq X$ be N-invariant subspaces of V. Then $A_X W$ is the image in $L(V, N)$ of the kernel of N on X/W, and

$$A_X^r W = \{ \vec{v} \in W \,:\, N^r \vec{v} \in X \}.$$

The coatoms of $L(V, N)$ are maximal proper subspaces of V. Their meet is $CV = NV$. For N-invariant subspaces $W \leq X$ of V, we have $C_W X = W + NX$ is the image in $L(V, N)$ of NX/W under the natural isomorphism from $L(X/W)$ to $[W, X]$, and

$$C_W^r X = W + N^r X.$$

3. **Semi-primary Lattices.** Both $L(G)$ and $L(V, N)$ are instances of **semi-primary lattices** [2, 5]. A semi-primary lattice is a modular lattice of finite length in which if x is a join-irreducible then $[\hat{0}, x]$ is a chain, and if x is a meet-irreducible then $[x, \hat{1}]$ is a chain. Closed intervals in a semi-primary lattice are themselves semi-primary lattices. The type of a semi-primary lattice is obtained by writing decomposing $\hat{1}$ into join-irreducibles $\hat{1} = x_1 \vee \ldots \vee x_k$ in such a fashion that $\rho(\hat{1}) = \rho(x_1) + \cdots + \rho(x_k)$ and sorting $\rho(x_1), \ldots, \rho(x_k)$ into weakly decreasing order to form a partition; though the decomposition is generally not unique, the partition is an invariant of $\hat{1}$ called the type of the lattice. The type of an interval is its type regarded as a lattice.

We call a semi-primary lattice q-regular when all its intervals of type $(1, 1)$ have exactly $q + 1$ atoms. The subgroup lattice $L(G)$ for an abelian p-group is p-regular, and the invariant subspace lattice over a finite field of order q is q-regular. Over an infinite field, we use topological properties instead of enumerative ones.

Theorem 4.2. *In each case above, we have*

$$\begin{aligned}(\text{type}[x, y])' &= \left(\rho(x, A_y x), \rho(A_y x, A_y^2 x), \rho(A_y^2 x, A_y^3 x), \ldots\right) \\ &= \left(\rho(C_x y, y), \rho(C_x^2 y, C_x y), \rho(C_x^3 y, C_x^2 y), \ldots\right).\end{aligned}$$

Here, the **conjugate** λ' of a partition λ is obtained by reflecting the Young diagram across its main diagonal, so that λ'_i is the number of cells in the ith column of λ.

A **flag** is an indexed sequence of elements of a lattice in weakly increasing order, $f = (f_0, f_1, \ldots, f_n)$ where $f_0 \leq f_1 \leq \cdots \leq f_n$. It is multisaturated when all $\leq$ are cover relations $\lessdot$. The type of a flag f is the chain of partitions $\text{ftype}_x f = (\text{type}[x, f_0], \cdots, \text{type}[x, f_n])$, where $x = f_0$ is assumed if x is omitted. It is expressed as a skew tableau by filling the cells of the diagram $\text{type}[x, f_i] \,/\, \text{type}[x, f_{i-1}]$ with the entry i for $i = 1, \ldots, n$, and the cells of the inner shape $\text{type}[x, f_0]$ with $\bullet$.

We define $A_y f = (A_y f_0, \cdots, A_y f_n)$ with $y = f_n$ if y is omitted, and $C_x f = (C_x f_0, \cdots, C_x f_n)$ with $x = f_0$ if x is omitted.

5. Operations on flags

The relation of these flag operators to the new tableau games was determined in [5, Thm. 8.9].

Theorem 5.1. *For multisaturated flags in a semi-primary lattice,* $\text{ftype}_{\hat{0}} Af = \tilde{\mathcal{L}}(\text{ftype}_{\hat{0}} f)$ *and* $\text{ftype}_{\hat{0}} Cf = \tilde{\mathcal{R}}(\text{ftype}_{\hat{0}} f)$*, for some possible evaluation of these non-deterministic games.*

Now consider a q-regular semi-primary lattice of type λ. Let P be a skew tableau of outer shape λ.

Fix a particular non-deterministic value of $Q = \tilde{\mathcal{R}}P$ of outer shape μ. The number of flags f for which $\text{ftype}_{\hat{0}} f = P$ *and* $\text{ftype}_{\hat{0}} Cf = Q$ *is a particular polynomial in q depending on P and Q, and for a given P, the degree is maximized uniquely by $Q = \mathcal{R}P$.*

Fix $Q = \tilde{\mathcal{L}}P$. The number of flags g for which $\text{ftype}_{\hat{0}} f = P$ *and* $\text{ftype}_{\hat{0}} Af = Q$ *is a particular polynomial in q depending on P and Q, and for a given P, the degree is maximized uniquely by $Q = \mathcal{L}P$.*

We say that the **generic** value of $\text{ftype}_{\hat{0}} Cf$ is $\mathcal{R}(\text{ftype}_{\hat{0}} f)$, meaning that over all q-regular semi-primary lattices as $q \to \infty$, all but a fraction $O(1/q)$ of flags satisfy $\text{ftype}_{\hat{0}} Cf = \mathcal{R}(\text{ftype}_{\hat{0}} f)$ (note that the remaining $O(1/q)$ of the flags have the other possible values $\tilde{\mathcal{R}}(\text{ftype}_{\hat{0}} f)$). Similarly, the generic value of $\text{ftype}_{\hat{0}} Af$ is $\mathcal{L}(\text{ftype}_{\hat{0}} f)$. This theorem has a topological counterpart, which motivates this terminology. For the lattice $L(V, N)$ over an algebraically closed field, in the Zariski topology, the number of flags being a certain polynomial is replaced by the dimension of the space of such flags being the degree of that polynomial, and "generic" means that the set of flags f of type P with $\text{ftype}_{\hat{0}} Cf = \mathcal{R}P$ (alternately, with $\text{ftype}_{\hat{0}} Af = \mathcal{L}(\text{ftype}_{\hat{0}} f)$) is dense in the set of all flags of type P.

Let CP be obtained from the tableau P by deleting its first column. Since $\text{type}[f_0, Af_0]$ is the first column of the type of each $\text{type}[f_0, Af_i]$, on computing $\text{ftype}\, Af$ instead of $\text{ftype}_{\hat{0}} Af$ we obtain the following.

Corollary 5.2. *For multisaturated flags in a semi-primary lattice,* $\text{ftype}\, Af = C\tilde{\mathcal{L}}(\text{ftype}\, f)$ *for some non-deterministic evaluation, and generically,* $\text{ftype}\, Af = C\mathcal{L}(\text{ftype}\, f)$.

6. Relation of vertical strip games to evacuation and *jeu de taquin*

We now give the relationship between the vertical strip games and two well-known games: the *jeu de taquin*, $j(P)$, and Schützenberger's evacuation algorithm, ev P, both of which may be found, for example, in [3]. The proofs are in [5, Ch. 8.3].

Theorem 6.1. *Let P be a standard tableau of shape $\lambda \vdash n$ on entries $1, \ldots, n$. Form a tableau Q as follows: for each $r > 0$, take the entries that vanish in the game $\mathcal{L}^{r-1}P \to \mathcal{L}^r P$, complement them by subtracting each from $n+1$, and place the complements in column r of Q in increasing order from top to bottom. Then $Q = \operatorname{ev} P$.*

Example 6.2.

$$P = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 7 & \\ 3 & 8 & \\ 5 & & \end{array} \qquad \mathcal{L}P = \begin{array}{ccc} \bullet & 2 & 4 \\ \bullet & 3 & \\ \bullet & 5 & \\ \bullet & & \end{array} \qquad \mathcal{L}^2P = \begin{array}{ccc} \bullet & \bullet & 3 \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array} \qquad \mathcal{L}^3P = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

The entries that disappear in the first step are 1,6,7,8; in the second step, 2,4,5; and in the third step, 3. Subtracting them from 9, the first column of ev P has 8,3,2,1; the second, 7,5,4; and the third, 6. So

$$\operatorname{ev} P = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & 7 & \\ 8 & & \end{array}.$$

Theorem 6.3. *Let P be a skew tableau on distinct entries. Form a Young tableau Q whose entries in the rth column are the entries that vanish in the game $\mathcal{R}^{r-1}P \to \mathcal{R}^r P$. Then $Q = j(P)$.*

Example 6.4.

$$P = \begin{array}{ccc} \bullet & \bullet & 3 \\ \bullet & \bullet & 4 \\ \bullet & 1 & 5 \\ 2 & & \\ 6 & & \end{array} \qquad \mathcal{R}P = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & 3 \\ 4 & \end{array} \qquad \mathcal{R}^2P = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} \qquad j(P) = \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \\ 6 & \end{array}$$

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