# Inductively Free Arrangements and a Result of Headley

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**Abstract**

Let $\mathcal{S}_n$ be the arrangement of hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0, 1$ for $1 \le i < j \le n$, introduced by Shi [12]. Using deletion and restriction, we give a simple proof of a formula of Headley for the characteristic polynomial of $\mathcal{S}_n$. This method shows that $\mathcal{S}_n$ is inductively free, hence that its cone is free and verifies part of a conjecture of Edelman and Reiner. Moreover we classify the hyperplane arrangements between the cones of the braid arrangement and $\mathcal{S}_n$ which are free.

Soit $\mathcal{S}_n$ l'arrangement d'hyperplans de $\mathbb{R}^n$ associés aux équations $x_i - x_j = 0, 1$ pour $1 \le i < j \le n$, introduit par Shi [12]. En utilisant des éliminations et des restrictions nous donnons une démonstration simple d'une formule de Headly pour le polynôme charactéristique de $\mathcal{S}_n$. Cette méthode montre que $\mathcal{S}_n$ est inductivement libre, donc c'est un cône satisfaisant une partie d'une conjecture d'Edelman et Reiner. De plus nous classifions les arrangements libres d'hyperplans qui se trouvent entre les cônes de l'arrangement de tresses et $\mathcal{S}_n$.

## 1 The Shi arrangement

The braid arrangement $\mathcal{A}_n$ consists of the hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0$. It is the arrangement of reflecting hyperplanes of the Weyl group of type $A_{n-1}$. Let $\widehat{\mathcal{A}}_n$ be the arrangement of affine hyperplanes in $\mathbb{R}^n$ of the form

$$x_i - x_j = 0, 1 \text{ for } 1 \le i < j \le n.$$

This is called the *Shi arrangement* of type $A_{n-1}$. It was introduced by J.-Y. Shi [12] who was the first to prove, using techniques from combinatorial group theory, that $\widehat{\mathcal{A}}_n$ divides $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions. Assuming this, Headley [8, 9] went on to prove that the *characteristic polynomial* [10, §2.3] of $\widehat{\mathcal{A}}_n$ has the simple form $\chi(\widehat{\mathcal{A}}_n, q) = q(q-n)^{n-1}$. This is a stronger result because of Zaslavsky's theorem [19], which asserts that the number of regions into which a hyperplane arrangement $\mathcal{A}$ divides its ambient space $\mathbb{R}^n$ is $(-1)^n \chi(\mathcal{A}, -1)$.

A simple proof of Headley's formula, based on the "finite field method", was given in [1, 2]. In this section we give another elementary proof, based on the classical method of deletion and restriction [10, Definition 1.13]. In Section 2 we explain the motivation behind this proof and how it implies an even stronger result, namely that $\widehat{\mathcal{A}}_n$ is *inductively free* with certain exponents. In Sections 3 and 4 we classify all arrangements between $\mathcal{A}_n$ and $\mathcal{S}_n$ whose homogenizations, or *cones*, are free. This class of free arrangements appeared earlier in [1, 2]. Our present methods are similar to those employed in [6, 7].

*Headley's theorem.* Before we give the new proof of Headley's theorem, suitably generalized, we review some basic background and notation from [10]. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$, i.e. a finite collection of affine subspaces of $\mathbb{R}^n$ of codimension one. The characteristic polynomial of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x)\, q^{\dim x},$$

where $L_{\mathcal{A}}$ is the poset of all affine subspaces which can be written as intersections of some of the hyperplanes of $\mathcal{A}$, $\hat{0} = \mathbb{R}^n$ is the unique minimal element of $L_{\mathcal{A}}$ and $\mu$ stands for its Möbius function. The polynomial $\chi(\mathcal{A}, q)$ is a fundamental combinatorial and topological invariant of $\mathcal{A}$ and plays a significant role throughout the theory of hyperplane arrangements [10].

Let $H \in \mathcal{A}$ be a distinguished hyperplane. The corresponding *deleted arrangement* is

$$\mathcal{A}' = \mathcal{A} - \{H\}.$$

The *restricted arrangement* to $H$ has $H$ as its ambient space and is given by

$$\mathcal{A}'' = \{H' \cap H \mid H' \in \mathcal{A}'\}.$$

The triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a *triple of arrangements.* The elementary Deletion-Restriction Theorem [10, Cor. 2.57] states that

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q). \tag{1}$$

We say that two hyperplane arrangements in $\mathbb{R}^n$ are *affinely equivalent* if there is an invertible affine endomorphism of $\mathbb{R}^n$ that maps the hyperplanes of one onto the hyperplanes of the other. The intersection poset and hence the characteristic polynomial are preserved under affine equivalence.

**Theorem 1.1** *For any integers $m \geq 0$ and $2 \leq k \leq n+1$, the arrangement*

$$\begin{aligned} x_1 - x_j &= 0, 1, \ldots, m \quad \text{for} \quad 2 \leq j < k, \\ x_1 - x_j &= 0, 1, \ldots, m+1 \quad \text{for} \quad k \leq j \leq n, \\ x_i - x_j &= 0, 1 \quad \text{for} \quad 2 \leq i < j \leq n \end{aligned} \tag{2}$$

*has characteristic polynomial $q\,(q-n-m+1)^{k-2}(q-n-m)^{n-k+1}$.*

*Proof.* We proceed by double induction on $n$ and $n-k$. The result is clear for $n=2$, so pick an $n \geq 3$. We first consider the case $m=0$ and $k=n+1$. The arrangement in question is

$$x_1 - x_j = 0 \text{ for } 2 \le j \le n,$$
$$x_i - x_j = 0, 1 \text{ for } 2 \le i < j \le n.$$

The arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i - x_j = 0, 1 \text{ for } 2 \le i < j \le n \tag{3}$$

has characteristic polynomial $q\chi(\widehat{\mathcal{A}}_{n-1}, q) = q^2(q-n+1)^{n-2}$. By (1), adding $r$ of the hyperplanes $x_1 - x_j = 0$ in any order produces an arrangement with characteristic polynomial $q(q-r)(q-n+1)^{n-2}$. Indeed, the restriction to the last hyperplane added at each step is affinely equivalent to (3). The case $r = n-1$ gives the desired result.

We can now assume $2 \le k \le n$, since the arrangement (2) having parameters $m \ge 1$ and $k = n+1$ coincides with (2) having parameters $m-1$ and $k = 2$. Consider the hyperplane $H$ of (2) with equation $x_1 - x_k = m+1$. Deletion of this hyperplane produces an arrangement which is of the same form as (2), with $k$ replaced by $k+1$. Restriction to $H$ produces an arrangement affinely equivalent to (2), with $n$ replaced by $n-1$ and $m$ replaced by $m+1$. To see this just set $x_k = x_1 - m - 1$ in the equations involving $x_k$. The equation $x_k - x_n = 1$, for example, becomes $x_1 - x_n = m+2$. The induction hypothesis for these two arrangements and (1) give the result for (2). □

For $m = 0$ and $k = 2$ we get Headley's theorem as a corollary.

**Corollary 1.2 (Headley)** *For all $n$, $\chi(\widehat{\mathcal{A}}_n, q) = q(q-n)^{n-1}$.*

The same argument can be given on the level of the number of regions $r(\mathcal{A})$ of $\mathcal{A}$ and yields a naive inductive proof of Shi's theorem. One just needs to use the recursion $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$ instead of (1). A simple bijective proof of Shi's theorem can be found in [3].

## 2 Inductive freeness

The class of inductively free arrangements $\mathcal{IF}$ is the smallest class of hyperplane arrangements which satisfies the following two conditions:

(*i*) The empty arrangement in $\mathbb{R}^n$ is in $\mathcal{IF}$ for all $n \ge 0$.

(*ii*) If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements with $\mathcal{A}', \mathcal{A}'' \in \mathcal{IF}$ and $\chi(\mathcal{A}'', q)$ divides $\chi(\mathcal{A}', q)$ then $\mathcal{A} \in \mathcal{IF}$.

This is equivalent to [10, Definition 4.53], given only for central, i.e. linear arrangements. It follows immediately from (1) that if $\mathcal{A}$ is inductively free, then $\chi(\mathcal{A}, q)$ factors completely over the nonnegative integers. The roots of $\chi(\mathcal{A}, q)$ are called the *exponents* of $\mathcal{A}$. The multiset of exponents of $\mathcal{A}$ is denoted by $\exp\mathcal{A}$.

We write $\{a_1^{m_1}, a_2^{m_2}, \ldots, a_r^{m_r}\}$ for a multiset, where $m_1, m_2, \ldots, m_r$ denote multiplicities. The proof of Theorem 1.1 yields the following stronger result.

**Theorem 2.1** *For any integers $m \geq 0$ and $2 \leq k \leq n+1$, the arrangement (2) is inductively free with multiset of exponents $\{0^1, (n+m-1)^{k-2}, (n+m)^{n-k+1}\}$. In particular, the Shi arrangement $\widehat{\mathcal{A}}_n$ is inductively free with multiset of exponents $\{0^1, n^{n-1}\}$.* $\square$

Besides forming a class of arrangements whose characteristic polynomials can be easily controlled, the inductively free arrangements have interesting algebraic properties. If central, they are *free* arrangements by the Addition Theorem [10, Thm. 4.50]. In general, their cones are also inductively free and hence free. A central hyperplane arrangement is called free [17] [10, Ch. 4] if its *module of derivations* [10, Definition 1.19] is free as a module over a polynomial ring. Coning is the standard way to pass from any hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ to a central one. The cone $\mathbf{c}\mathcal{A}$ is the arrangement in $\mathbb{R}^{n+1}$ obtained by homogenizing each hyperplane

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + d = 0$$

of $\mathcal{A}$ to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + dx_0 = 0$$

and adding the hyperplane $x_0 = 0$. Here $x_0$ is the new coordinate attached to $\mathbb{R}^n$. The cone $\mathbf{c}\widehat{\mathcal{A}}_n$ of the Shi arrangement, for example, has hyperplanes

$$\begin{aligned} &x_i - x_j = 0 \ \text{ for } \ 1 \leq i < j \leq n, \\ &x_i - x_j - x_0 = 0 \ \text{ for } \ 1 \leq i < j \leq n, \\ &x_0 = 0. \end{aligned}$$

Theorem 2.1 implies that $\mathbf{c}\widehat{\mathcal{A}}_n$ is free. More generally, let $a \geq 1$ be an integer and consider the *extended Shi arrangement* $\widehat{\mathcal{A}}_n^{[-a+1,a]}$. It consists of the hyperplanes

$$x_i - x_j = -a+1, -a+2, \ldots, a \ \text{ for } \ 1 \leq i < j \leq n$$

and reduces to $\widehat{\mathcal{A}}_n$ for $a = 1$. It was conjectured by Edelman and Reiner that the cone $\mathbf{c}\widehat{\mathcal{A}}_n^{[-a+1,a]}$ is free. This is a special case for the root system $A_{n-1}$ of one half of Conjecture 3.3 in [7]. The weaker statement $\chi(\widehat{\mathcal{A}}_n^{[-a+1,a]}, q) = q(q-an)^{n-1}$ can easily be derived with the finite field method [2, Cor. 7.1.2]. Theorem 2.1 can be extended in this direction as follows.

**Theorem 2.2** *Fix an integer $a \geq 1$. For any integers $m \geq 0$ and $2 \leq k \leq n+1$, the arrangement*

$$\begin{aligned} &x_1 - x_j = -a+1, \ldots, m \ \text{ for } \ 2 \leq j < k, \\ &x_1 - x_j = -a+1, \ldots, m+1 \ \text{ for } \ k \leq j \leq n, \\ &x_i - x_j = -a+1, \ldots, a \ \text{ for } \ 2 \leq i < j \leq n \end{aligned}$$

*is inductively free with multiset of exponents $\{0^1, (an+m-a)^{k-2}, (an+m-a+1)^{n-k+1}\}$. In particular, the extended Shi arrangement $\widehat{\mathcal{A}}_n^{[-a+1,a]}$ is inductively free with multiset of exponents $\{0^1, (an)^{n-1}\}$ and hence its cone is free.* $\square$

## 3 More inductively free arrangements

Our motivation in what follows comes primarily from [1, 2]. In this work the characteristic polynomials of large classes of *deformations* [16, 11] of Coxeter arrangements were shown to factor completely over the nonnegative integers and the question of freeness of their cones was naturally raised [1, §7] [2, §8.4]. Headley's result, for instance, was generalized in several ways. We recall one such generalization next. We find it convenient to think of a simple graph $S$ on the vertex set $[n] = \{1, 2, \ldots, n\}$ as a directed graph. Each edge $ij$ is directed as $(j, i)$, i.e. from $j$ to $i$, if $i < j$. In other words, $S$ is a subset of the set

$$E_n = \{(j, i) \mid 1 \leq i < j \leq n\},$$

which is the edge set of the complete graph. Note that the arrangements between $\mathcal{A}_n$ and $\widehat{\mathcal{A}}_n$ correspond to simple graphs on the vertex set $[n]$. More precisely, each such arrangement is of the form

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for} \quad 1 \leq i < j \leq n, \\ x_i - x_j &= 1 \quad \text{for} \quad (j, i) \in S \end{aligned}$$

for some $S \subseteq E_n$. We denote this arrangement by $\widehat{\mathcal{A}}_{n,S}$. The Shi arrangement $\widehat{\mathcal{A}}_n$ corresponds to the complete graph $S = E_n$ and the braid arrangement $\mathcal{A}_n$ to the empty graph. The following theorem produces a family of arrangements between $\mathcal{A}_n$ and $\widehat{\mathcal{A}}_n$ whose characteristic polynomials have nonnegative integers as roots.

**Theorem 3.1** **([1, Thm. 3.4] [2, Thm. 6.2.2])** *Suppose that the graph $S \subseteq E_n$ has the following property: if $1 \leq i < j < k \leq n$ and $(j, i) \in S$ then $(k, i) \in S$. Then*

$$\chi(\widehat{\mathcal{A}}_{n,S}, q) = q \prod_{1 < j \leq n} (q - c_j),$$

*where $c_j = n + a_j - j + 1$ and $a_j = \#\{i < j \mid (j, i) \in S\}$ is the outdegree of $j$ in $S$, for $1 < j \leq n$.* $\square$

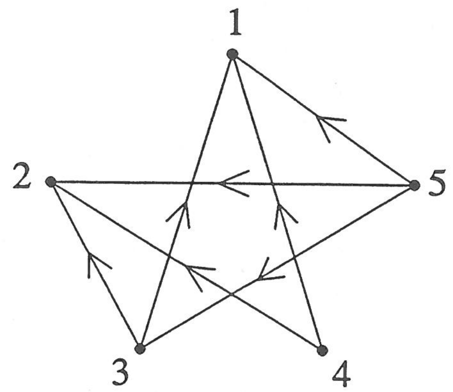


Figure 1: An example with $n = 5$

Figure 1 shows a graph satisfying the condition in Theorem 3.1. For this graph we have $a_2 = 0$, $a_3 = a_4 = 2$, $a_5 = 3$, so $c_2 = 4$, $c_3 = 5$, $c_4 = c_5 = 4$ and the corresponding characteristic polynomial is $q(q-4)^3(q-5)$. Here we generalize Theorem 2.1 and show that the arrangements in Theorem 3.1 are all inductively free. We will see in the next section that, up to a suitable permutation of the coordinates, these are all the arrangements between $\mathcal{A}_n$ and $\widehat{\mathcal{A}}_n$ with free cones.

We need to extend the notation in Theorem 3.1. Suppose that $\mathcal{A}$ contains $\mathcal{A}_n$ and that it has hyperplanes of the form $x_i - x_j = s$, where $s \in \mathbb{Z}$ and $1 \le i < j \le n$. For $1 < j \le n$ let $a_j$ be the number of hypeplanes $x_i - x_j = s$ of $\mathcal{A}$ with $i < j$ and $s \neq 0$. Also let

$$c_j = n + a_j - j + 1.$$

We call these numbers the $a$ and $c$ *parameters* of $\mathcal{A}$ respectively. This notation agrees with the one in Theorem 3.1. A generalization of Theorem 3.1 [2, Thm. 6.2.10] produced a large class of arrangements, in the form described above, whose characteristic polynomials factor completely over the nonnegative integers. Thus, under assumptions, the characteristic polynomial of $\mathcal{A}$ equals $q(q-c_2)\cdots(q-c_n)$. The exponents proposed for the arrangements in Theorem 3.1 are exactly the corresponding $c$ parameters.

**Theorem 3.2** *Let $T$ be a graph on the vertex set $[2,n]$, i.e. $(j,i) \in T$ implies $2 \le i < j \le n$. Suppose that $T$ satisfies the condition in Theorem 3.1: if $2 \le i < j < k \le n$ and $(j,i) \in T$ then $(k,i) \in T$. Let $m, k$ be integers as in Theorem 3.1. The arrangement*

$$\begin{aligned} x_1 - x_j &= 0,1,\ldots,m \quad \text{for} \quad 2 \le j < k, \\ x_1 - x_j &= 0,1,\ldots,m+1 \quad \text{for} \quad k \le j \le n, \\ x_i - x_j &= 0 \quad \text{for} \quad 2 \le i < j \le n, \\ x_i - x_j &= 1 \quad \text{for} \quad (j,i) \in T \end{aligned} \tag{4}$$

*is inductively free with exponents $0, c_2, \ldots, c_n$, where $c_j$, $2 \le j \le n$ are its $c$ parameters.* □

The case $m = 0$ gives the result promised at the end of Section 2.

**Corollary 3.3** *Under the assumptions and notation of Theorem 3.1, the arrangement $\widehat{\mathcal{A}}_{n,S}$ is inductively free with exponents $0, c_2, \ldots, c_n$.* □

# 4 Free arrangements between $\mathbf{c}\mathcal{A}_n$ and $\mathbf{c}\widehat{\mathcal{A}}_n$

In this section we show that the cones of the arrangements of Theorem 3.1 are essentially the only free arrangements between $\mathbf{c}\mathcal{A}_n$ and $\mathbf{c}\widehat{\mathcal{A}}_n$.

We first recall two fundamental results in the theory of free arrangements. The Factorization Theorem of Terao [18] [10, Thm. 4.137] states that $\chi(\mathcal{A}, q)$ factors completely over the nonnegative integers for any free arrangement $\mathcal{A}$. The roots of $\chi(\mathcal{A}, q)$ are called the exponents of $\mathcal{A}$, as in the case of inductive freeness. Let $X$ be an element of the intersection poset $L_{\mathcal{A}}$. The *localization* $\mathcal{A}_X$ is the subarrangement of $\mathcal{A}$

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

The Localization Theorem [10, Thm. 4.37] asserts that any localization of a free arrangement is free. It can easily provide obstructions to freeness [20] and is therefore quite useful in classifying free arrangements [6, 7]. Lastly, we need to recall the very simple effect

$$\chi(\mathrm{c}\mathcal{A}, q) = (q-1)\ \chi(\mathcal{A}, q) \tag{5}$$

that coning has on the characteristic polynomial. Our main result can be stated as follows.

**Theorem 4.1** *Let $S \subseteq E_n$. The following are equivalent:*

*(i) $\widehat{\mathcal{A}}_{n,S}$ is inductively free.*

*(ii) $\mathrm{c}\widehat{\mathcal{A}}_{n,S}$ is free.*

*(iii) $S$ does not contain any of the two directed graphs in Figure 2 as induced subgraphs.*

*(iv) There is a permutation $w = w_1 w_2 \ldots w_n$ of $[n]$ such that*

$$w^{-1} \cdot S = \{(j, i) \mid (w_j, w_i) \in S\}$$

*is contained in $E_n$ and satisfies the condition in Theorem 3.1.*

*Proof*. The implication $(i) \Longrightarrow (ii)$ is clear and $(iv) \Longrightarrow (i)$ follows immediately from Corollary 3.3. We show the implications $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$.

Figure 2: Obstructions to freeness

Suppose that $(ii)$ holds. For $U \subseteq [n]$ let $S_U$ be the induced subgraph of $S$ on $U$. Note that the subspace $X_U$ defined by the equations $x_0 = 0$, $x_i = x_j$ for $i, j \in U$ is in the intersection poset $L_{\mathrm{c}\widehat{\mathcal{A}}_{n,S}}$ and that the localization of $\mathrm{c}\widehat{\mathcal{A}}_{n,S}$ on $X_U$ is affinely equivalent to $\mathrm{c}\widehat{\mathcal{A}}_{k,T}$, where $k = \#U$ and $T$ is isomorphic to $S_U$. By the Localization Theorem, these localizations are free. Hence to prove $(iii)$ it suffices to check that the arrangements $\mathrm{c}\widehat{\mathcal{A}}_{3,S_1}$ and $\mathrm{c}\widehat{\mathcal{A}}_{4,S_2}$ are not free, where $S_1$ is the path $\{(3,2),(2,1)\}$ and $S_4 = \{(2,1),(4,3)\}$. It follows from [2, Thm. 7.1.5] (see also [1, Thm. 5.6]) and can easily be checked otherwise that

$$\chi(\widehat{\mathcal{A}}_{3,S_1}, q) = q\ (q^2 - 5q + 7)$$

and

$$\chi(\widehat{\mathcal{A}}_{4,S_2}, q) = q\ (q-3)(q^2 - 5q + 7).$$

The Factorization Theorem and (5) imply that $\mathrm{c}\widehat{\mathcal{A}}_{3,S_1}$ and $\mathrm{c}\widehat{\mathcal{A}}_{4,S_2}$ are not free.

Finally suppose that (*iii*) holds. Equivalently, we require the following two conditions:

(I) For distinct indices $i, j, k$ with $1 \le i < j < k \le n$, $(k, j) \in S$ and $(j, i) \in S$ imply $(k, i) \in S$.

(II) For $1 \le i < j \le n$, $1 \le k < l \le n$ and $i, j, k, l$ distinct, $(j, i) \in S$ and $(l, k) \in S$ imply $(l, i) \in S$ or $(j, k) \in S$ or both.

We denote by $\mathrm{out}(w)$ the outdegree of a vertex $w$ of $S$ and let $w_1, w_2, \dots, w_n$ be any linear ordering of the vertices $1, 2, \dots, n$ of $S$ which satisfies $\mathrm{out}(w_i) \le \mathrm{out}(w_j)$ for $i < j$. First note that, by (I), $(w_j, w_i) \in S$ implies $\mathrm{out}(w_i) < \mathrm{out}(w_j)$ and hence $i < j$. This means that $w^{-1} \cdot S \subseteq E_n$, as claimed. To prove (*iv*) it remains to check the condition in Theorem 3.1. Let $1 \le i < j < l \le n$ with $(w_j, w_i) \in S$. We want to show that $(w_l, w_i) \in S$, so suppose the contrary. By (II), whenever $(w_l, w_k) \in S$ we have $(w_j, w_k) \in S$. Note also that, by (I), $(w_l, w_j)$ is not in $S$. It follows that $\mathrm{out}(w_j) > \mathrm{out}(w_l)$, contradicting the fact that $j < l$. □

In contrast to the situation in [6], very few of the arrangements $\mathrm{c}\widehat{\mathcal{A}}_{3,S}$ of Theorem 4.1 are supersolvable. For the sake of completeness we state a precise result. The proof uses the Localization Theorem for supersolvable arrangements [15, Prop. 3.2].

**Theorem 4.2** *Let $S \subseteq E_n$. The arrangement $\mathrm{c}\widehat{\mathcal{A}}_{n,S}$ is supersolvable if and only if all the edges in $S$ have the same terminal vertex or they all have the same initial vertex.* □

# 5 Remarks and open problems

1. Classes of arrangements that correspond to pairs of graphs seem to be more complicated to analyze from the point of view of freeness. This is the case, for example, with the class of all subarrangements of the Coxeter arrangement of type $B_n$, as remarked in [6]. We have no obvious suggestion for what all subarrangements of $\widehat{\mathcal{A}}_n$ with free cones should look like.

The case of the arrangements between $\mathcal{A}_n$ and the *Catalan* arrangement

$$x_i - x_j = -1, 0, 1 \quad \text{for} \quad 1 \le i < j \le n \tag{6}$$

in $\mathbb{R}^n$ seems to deserve special mention. The following result, which extends Theorem 3.1 and is a special case of the more general [2, Thm. 6.2.10], suggests an explicit answer for this case.

**Theorem 5.1** **([1, Thm. 3.9] [2, Thm. 6.2.7])** *Suppose that the set*

$$G \subseteq \{(i, j) \mid i \ne j, \ \ 1 \le i, j \le n\}$$

*has the following properties:*

*(i) If $i, j < k$, $i \neq j$ and $(i,j) \in G$, then $(i,k) \in G$ or $(k,j) \in G$ or both.*

*(ii) If $i, j < k$, $i \neq j$ and $(i,k) \in G$, $(k,j) \in G$, then $(i,j) \in G$.*

*Then the characteristic polynomial of the arrangement*

$$\begin{array}{ll} x_i - x_j = 0 & \text{for} \quad 1 \leq i < j \leq n, \\ x_i - x_j = 1 & \text{for} \quad (j,i) \in G \end{array}$$

*factors as in Theorem 3.1, where $c_j$ are the corresponding c parameters.* □

Note that if $G \subseteq E_n$, the conditions in the previous theorem reduce to the one in Theorem 3.1. The arguments of Section 3 do not trivially extend to show inductive freeness of the arrangements in Theorem 5.1. Inductive freeness of the Catalan arrangement (6) was established by Edelman and Reiner (see the proof of [7, Thm. 3.2]).

2. The family of free arrangements in Theorem 4.1 contains simple counterexamples to Orlik's conjecture [10, p. 10, 155], which stated that the restriction of a free arrangement to any of its hyperplanes is free. This was first disproved by Edelman and Reiner [5]. The same authors provided infinitely many counterexamples in [6], including one of dimension 4 with 10 hyperplanes. A counterexample contained in the family of Theorem 4.1 is provided by $c\widehat{\mathcal{A}}_{4,S_3}$, where $S_3$ is shown in Figure 3. This arrangement is free by Corollary 3.3 and has rank 4 and 10 hyperplanes. The restriction of $c\widehat{\mathcal{A}}_{4,S_3}$ to the hyperplane $x_2 = x_4$ is affinely equivalent to $c\widehat{\mathcal{A}}_{3,S_1}$, corresponding to the first forbidden graph of Figure 2, and hence is not free. As Reiner has pointed out, $c\widehat{\mathcal{A}}_{4,S_3}$ is projectively equivalent to the minimum-dimensional counterexample given in [6].

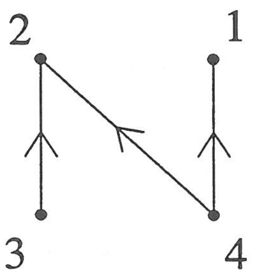


Figure 3: The graph $S_3$

Clearly, any arrangement $c\widehat{\mathcal{A}}_{n,S}$ such that $S$ contains an isomorphic copy of $S_3$ as an induced subgraph is a counterexample to Orlik's conjecture.

3. Curiously, the same directed graphs as in Theorems 4.1 and 4.2 have appeared in recent work of David Bailey [4, Ch. 7] and were shown to correspond to the free and supersolvable arrangements, respectively, in a *different* class. This class consists of certain discriminantal arrangements of zonotopes. There seems to be no obvious connection between the two kinds of results.

4. There are natural analogues of the extended Shi arrangements for the other irreducible crystallographic root systems [13, 7] and analogues of Headley's theorm [2,

Cor. 7.2.2]. These arrangements were conjectured to have free cones [7, Conjecture 3.3]. One approach to prove this conjecture is to find explicit bases for the modules of derivations in a uniform way. Such bases are not known even for $\mathbf{c}\widehat{\mathcal{A}}_n$. An indication that this task may be possible comes from recent work of Solomon and Terao [14].

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# References

[1] C.A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Advances in Math.* **122** (1996), 193–233.

[2] C.A. Athanasiadis, *Algebraic combinatorics of graph spectra, subspace arrangements and Tutte polynomials*, Ph.D. thesis, MIT, 1996.

[3] C.A. Athanasiadis and S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, MSRI Preprint # 1997-020.

[4] G.D. Bailey, *Tilings of zonotopes: Discriminantal arrangements, oriented matroids, and enumeration*, Ph.D. thesis, University of Minnesota, 1997.

[5] P.H. Edelman and V. Reiner, A counteraxample to Orlik's conjecture, *Proc. Amer. Math. Soc.* **118** (1993), 927–929.

[6] P.H. Edelman and V. Reiner, Free hyperplane arrangements between $A_{n-1}$ and $B_n$, *Math. Zeitschrift* **215** (1994), 347–365.

[7] P.H. Edelman and V. Reiner, Free arrangements and rhombic tilings, *Discrete Comput. Geom.* **15** (1996), 307–340.

[8] P. Headley, On reduced words in affine Weyl groups, in *Proc. "Formal Power Series and Algebraic Combinatorics (FPSAC) 1994"* (L.J. Billera, C. Greene, R. Simion, R. Stanley, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc. **24** (1996), pp. 225–232.

[9] P. Headley, *Reduced expressions in infinite Coxeter groups*, Ph.D. thesis, University of Michigan, 1994.

[10] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren 300, Springer-Verlag, New York, NY, 1992.

[11] A. Postnikov and R.P. Stanley, Deformations of Coxeter hyperplane arrangements, Preprint, 1997.

[12] J.-Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics, no. 1179, Springer-Verlag, Berlin/Heidelberg/New York, 1986.

[13] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

[14] L. Solomon and H. Terao, The double Coxeter arrangement, Preprint, 1997.

[15] R.P. Stanley, Modular elements of geometric lattices, *Algebra Universalis* **2** (1972), 197–217.

[16] R.P. Stanley, Hyperplane arrangements, interval orders and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

[17] H. Terao, Arrangements of hyperplanes and their freeness I, II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 293–320.

[18] H. Terao, Generalized exponents of a free arrangement of hyperplanes and the Shepherd-Todd-Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.

[19] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154, (1975).

[20] G.M. Ziegler, Some almost exceptional arrangements, *Advances in Math.* **101** (1993), 50–58.