COMPLEXES OF NOT i-CONNECTED GRAPHS
EXTENDED ABSTRACT

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Abstract. Complexes of (not) connected graphs, hypergraphs and their homology appear in the construction of knot invariants given by V. Vassiliev [V1, V2, V3]. In this paper we study the complexes of not i-connected k-hypergraphs. In Section 3 we show that the complex of not 2-connected graphs has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$. This answers one of the questions raised by Vassiliev [V3] in connection with knot invariants. For this case the S_n-action on the homology of the complex is also determined. For not 2-connected k-hypergraphs we provide a formula for the generating function of the Euler characteristic. We also present partial results for some other cases. In particular, we show that the complex of not $(n-2)$-connected graphs is Alexander dual to the complex of partial matchings of the complete graph. The latter complexes and their homology are of interest in various parts of mathematics (see [BLVZ]). For not $(n-3)$-connected graphs we provide a formula for the generating function of the Euler characteristic.

Les complexes de graphes (non-connexes) et leur homologie paraîssent dans la construction des invariants de noeuds et de courbes planaires donnée par V. Vassiliev [V1,V2,V3]. Dans cet essai nous étudions les complexes de k-hypergraphes qui ne sont pas i fois connexes. Dans la section 3 nous montrons que le complexe des hypergraphes qui ne sont pas 2-fois connexes a le même type d'homotopie qu'un bouquet de $(n-2)!$ sphères de dimension $2n-5$. Ce résultat répond à une des questions de Vassiliev concernant les invariants de noeuds. Pour ce cas nous déterminons aussi l'action du group S_n sur l'homologie du complexe. Pour les k-hypergraphes qui ne sont pas 2-fois connexes nous donnons une formule de la caractéristique d'Euler. Nous présentons aussi des résultats partiels pour d'autres cas. En particulier, nous montrons que le complexe des graphes qui ne sont pas $(n-2)$-fois connexes est dual dans le sense d'Alexander au complexe des appariements partiels du graphe complet. Ce dernier complexe et son homologie ont un intérêt dans des parties différentes des mathématiques (voir [BLVZ]). Pour les graphes qui ne sont pas $(n-3)$-connexes nous donnons une formule pour la fonction génératrice de la caractéristique d'Euler.

1. Introduction

We study the homotopy type and homology of simplicial complexes whose simplices are the edge sets of not i-connected graphs and hypergraphs on n vertices. The case $i=1$ is already well understood (see Proposition 2.1), and here we begin the examination of the topological structure of such complexes for $i \geq 2$.

Although our point of view is mainly combinatorial, our original motivation for studying these complexes comes from the theory of Vassiliev invariants in knot theory. By determining the homotopy type of the complex of not 2-connected graphs on n vertices we answer a question posed by V. Vassiliev in [V3], where he presents a new approach to Vassiliev knot invariants using a filtration of the simplicial resolution of the space of not-knots as in [V2]. More precisely, he studies the space Σ of maps $f : S^1 \to \mathbb{R}^3$ such that $f(S^1)$ has multiple points or cusps. The simplicial resolution $\widetilde{\Sigma}$ of Σ is obtained roughly speaking as follows: singular knots are resolved by blowing up each r-fold self-intersection to an $(\binom{r}{2}-1)$-simplex, and similarly for the set of cusps. A suitable filtration (see [V3]) of $\widetilde{\Sigma}$, combinatorially defined in terms of these simplices, gives rise to a spectral sequence that contains the homology of the complex of not 2-connected graphs on n vertices as a basic ingredient.

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Our work continues the already fruitful interaction between the theory of Vassiliev invariants and questions in topological and homological combinatorics of graph complexes (see [V1]). The study of complexes of not i-connected graphs has intriguing combinatorial and algebraic aspects as well. For example, such aspects become apparent when considering the complex of not $(n-2)$-connected graphs on n vertices. In Section 7 this complex is shown to be Alexander dual to the complex of partial matchings of the complete graph on n vertices. These matching complexes, along with complexes of partial matchings of bipartite graphs, have previously been studied for other reasons, see [BLVZ]. In each case for which we calculate the Betti numbers, we detect nontrivial homology. For $(n-3)$-connected graphs (see Section 8) and for most complexes of not 2-connected hypergraphs (see Section 6) we have been unable to compute the Betti numbers explicitly, but we do determine the generating function of their reduced Euler characteristics. The homology is seen to be nontrivial in almost all of these cases.

Surprisingly, these non-vanishing phenomena are suggested by a result motivated by a conjecture in complexity theory. The conjecture states that complexes of graphs on n vertices having some non-trivial monotone graph property – like being not i-connected – are evasive (see for example [KSS]). Kahn, Saks & Sturtevant [KSS] showed that non-evasive complexes are contractible. In many naturally arising cases, including those examined here, the converse is true and evasive complexes in fact have non-vanishing reduced Euler characteristics.

In Section 4 we study the action of the symmetric group on the complex of not 2-connected graphs induced by its natural action on the vertices. This action induces a representation of S_n on the homology groups of the complex, which we determine. This representation coincides with a recently well studied representation which appears in the work of Robinson & Whitehouse [RW, Wh], Kontsevich [K], Getzler & Kapranov [GK], Mathieu [Ma], Hanlon & Stanley [H, HS] and Sundaram [Su].

2. Preliminaries

We now introduce the basic concepts used in this abstract. By a graph $G = (V(G), E(G))$ we mean a loopless graph without multiple edges on the vertex set $V(G)$ and with edge set $E(G) \subseteq \binom{V(G)}{2}$. Our standard vertex set will be the set $[n] := \{1, 2, \ldots, n\}$. A graph G is called *connected* if for any two distinct vertices $v, v' \in V(G)$ there is a path from v to v' in G, that is, a sequence of edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{l-1}, v_l\} \in E(G)$ such that $v = v_1$ and $v' = v_l$.

For a number $i \geq 1$ a graph G is called *i-connected* if for any j vertices $v_1, \ldots, v_j \in V(G)$, $j < i$, the graph G' that is obtained from G by deleting the vertices $v_1, \ldots, v_j$ and their adjacent edges is connected. Equivalently, if $|V(G)| > i$ then G is i-connected if and only if for every pair v, v' of not adjacent vertices there are at least i paths from v to v' that are pairwise disjoint except at their endpoints.

A graph which is not i-connected is also called *(i-1)-separable*, and a 1-separable (that is, not 2-connected) graph will often be called just separable. Of course, if $G = (V(G), E(G))$ is a graph that is not i-connected for some $i \geq 1$ then for any subset $E' \subseteq E(G)$ the graph $G' = (V(G), E')$ on the same vertex set is not i-connected either. Hence if we fix an n-element vertex set V and identify a graph with the set of its edges, then we may regard the set of not i-connected graphs on V as a simplicial complex.

Definition: Δ_n^i is the complex of not i-connected graphs on n vertices.

For a graph G and a vertex v we denote by $G - v$ the graph that is obtained from G by deleting the vertex v from its set of vertices and deleting all edges emerging from v from the set of edges. If v and w are two distinct vertices of G then we denote by vw the two-element set $\{v, w\}$, by $G \setminus vw$ the graph $(V(G), E(G) \setminus \{vw\})$, and by $G + vw$ the graph $(V(G), E(G) \cup \{vw\})$. A subset $V' \subseteq V(G)$ of the vertex-set of a graph $G = (V(G), E(G))$ is called a *cutset* if the graph obtained from G by deleting the vertices in V' and all adjacent edges is not connected. In particular, a graph is i-separable if and only if there is a cutset of cardinality i. A cutset of cardinality 1 is also called *cutpoint.*

More generally, one may consider complexes of not i-connected k-uniform hypergraphs. Recall that a *k-uniform hypergraph* on a vertex set V is a subset E of the set of k-element subsets $\binom{V}{k}$ of V. We will call the k-uniform hypergraphs *k-graphs* for short. Note that a 2-graph is just a graph. A k-graph is called *i-connected* if its underlying 2-graph is i-connected. The underlying 2-graph of a k-graph E is the graph on V whose edge set contains a k-clique on $\{v_1, \dots, v_k\}$ for each hyperedge $\{v_1, \dots, v_k\} \in E$.

Definition: $\Delta^i_{n,k}$ is the complex of all not i-connected k-graphs on n vertices.

Cutsets and cutpoints are defined analogously for k-graphs as they were for graphs. For the notation related to simplicial complexes and partially ordered sets – posets for short – used in this abstract, we refer the reader to Section 9.

Let us now review some known results. For $i = 1$ the complexes Δ^1_n and $\Delta^1_{n,k}$ are the complexes of disconnected graphs, resp., disconnected k-graphs. The topology of $\Delta^1_{n,k}$ is well understood up to homotopy type.

Proposition 2.1. *Let $n \geq 2$. Then*

(i) *The complex Δ^1_n is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $n-3$. In particular, $\widetilde{H}_i(\Delta^1_n) = 0$ for $i \neq n-3$ and $\widetilde{H}_{n-3}(\Delta^1_n) \cong \mathbb{Z}^{(n-1)!}$.*

(ii) *The complex $\Delta^1_{n,k}$ is homotopy equivalent to a wedge of spheres of dimensions $n-(k-2)\cdot t-3$, $1 \leq t \leq \frac{n}{k}$. In particular, the homology of $\Delta^i_{n,k}$ is free and concentrated in dimensions $n-(k-2)\cdot t-3$, $1 \leq t \leq \frac{n}{k}$.*

Part (i) follows from well-known properties of partition lattices (see [B, BWa, St2]) together with the crosscut theorem (see [B]). An alternative proof is provided in [V1]. Part (ii) was established by Björner and Welker in [BWe]. See Theorem 4.5 and Section 7.8 of [BWe] for exact numerical information on the homology of $\Delta^1_{n,k}$.

The character of the symmetric group for the representation on $\widetilde{H}_{n-3}(\Delta^1_n)$ was determined by Stanley in [St2] in terms of the character of S_n on the homology of the partition lattice. These two characters are equal by an equivariant version of the crosscut theorem. The character of the symmetric group on the homology of $\Delta^1_{n,k}$ was given by Sundaram & Wachs [SW].

Unless otherwise explicitly stated, all homology groups in this abstract have integer coefficients.

3. Homology and homotopy type of Δ^2_n

The results and computations presented in what follows will suggest that there is probably no uniform statement that covers the topology of all complexes Δ^i_n. This is consistent with the graph theoretical study of not i-connected graphs, where there is a good structure theory only when $i \leq 3$ (see for example Chapter 6 of Lovász' book [L], or the survey article by Oxley [O] and the references therein). The structure theory of not 2-connected graphs is particularly well understood.

The main theorem of this section gives a complete description of the homotopy type of Δ^2_n.

Theorem 3.1. *Let $n \geq 3$. Then Δ^2_n has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$.*

Remark: This result was circulated for several months as a conjecture. During that time, the Euler characteristic of Δ^2_n was calculated by Rodica Simion [Si]. The theorem was proved independently and simultaneously, almost to the day, by V. Turchin in Moscow, in a homology version [V3] that is equivalent to our result by some general arguments from homotopy theory.

For any natural number k, let B_k be the Boolean algebra on k elements (i.e., the lattice of subsets of a k-element set) and let Π_k be the lattice of partitions of a k-set into subsets, ordered by refinement. It is well-known that $\Delta(\overline{B_k})$ — being the barycentric subdivision of a simplex boundary — is homeomorphic to a $(k-2)$-sphere, and that $\Delta(\overline{\Pi_k}) \simeq \Delta^1_k$ has the homotopy type of a wedge of $(k-1)!$ spheres of dimension $k-3$ (see Proposition 2.1 (i) and its references). These facts imply the following.

Lemma 3.2. $\Delta(\overline{B_k \times \Pi_k})$ *has the homotopy type of a wedge of $(k-1)!$ spheres of dimension $2k-3$.*

Thus, in order to prove Theorem 3.1 it suffices to demonstrate that Δ^2_n is homotopy equivalent to $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$. In order to state more precisely what we will prove, we make the following definitions.

Definition: For $x \in [n]$ and any graph G on $[n]$, $N_G(x)$ is the *neighborhood* of x in G, i.e. $N_G(x) = \{y \in [n] : \{x, y\} \in E(G)\}$, and $\pi(x, G)$ is the partition of the set $[n] \setminus \{x\}$ determined by the connected components of $G - x$.

Definition: $\phi : \overline{\mathcal{L}at(\Delta_n^2)} \to \overline{B_{n-1} \times \Pi_{n-1}}$ is the map of posets given by $G \mapsto (N_G(1), \pi(1, G))$, and $\phi^* : \Delta(\overline{\mathcal{L}at(\Delta_n^2)}) \to \Delta(\overline{B_{n-1} \times \Pi_{n-1}})$ is the simplicial map induced by ϕ.

Note that if G is a graph on $[n]$ such that $N_G(1) = \{2, \ldots, n\}$ and $G - 1$ is connected, then G is 2-connected. On the other hand, if $N_G(1) = \emptyset$ and $\pi(1, G) = 2|3|\ldots|n$ then G is the empty graph. Thus ϕ is well-defined. It is clear that ϕ is order preserving, so ϕ^* is well-defined. We can now state the key technical result, from which (in view of Lemma 3.2) Theorem 3.1 follows.

Lemma 3.3. *The simplicial map ϕ^* is a homotopy equivalence.*

To prove Lemma 3.3 we use Quillen's Fiber Lemma (see Proposition 9.1). In our situation this says that if for each $(S, \pi) \in \overline{B_{n-1} \times \Pi_{n-1}}$ the poset $\phi_{\leq}^{-1}(S, \pi) = \left\{G \in \overline{\mathcal{L}at(\Delta_n^2)} : \phi(G) \leq (S, \pi)\right\}$ has a contractible order complex, then ϕ^* is a homotopy equivalence. If $\pi \neq |2 \cdots n|$ then $\phi_{\leq}^{-1}((S, \pi))$ has a top element, namely the graph G such that $\{1, t\}$ is an edge of G for $t \in S$ and G induces the complete graph on each block of π. So assume that $\pi = |2 \cdots n|$. If $|S| \leq 1$ then there is also a top element in $\phi_{\leq}^{-1}((S, \pi))$, namely the graph G which induces a clique on $\{2, \ldots, n\}$ and has $N_G(1) = S$. If $S = \{2, \ldots, n\}$ then (S, π) does not lie in the proper part of $B_{n-1} \times \Pi_{n-1}$. In summary, it remains to consider the fibers $\phi_{\leq}^{-1}(S, \pi)$ for pairs (S, π) such that $\pi = |2 \cdots n|$ and $S \subsetneq \{2, \ldots, n\}$ with $2 \leq |S| \leq n - 2$. To handle these remaining cases, we make the following definitions.

Definition:

(1) For $2 \leq k \leq n - 1$, $\Delta(k) = \{G \in \Delta_n^2 : N_G(1) \subseteq \{2, \ldots, k\}\}$.
(2) For $3 \leq k \leq n - 1$, $\Delta(k - 1, k) = \{G \in \Delta(k - 1) : G + 1k \in \Delta(k)\}$.

Note that (by definition) if $xy \in E(G)$ then $G + xy = G$ and if $xy \notin E(G)$ then $G \setminus xy = G$. If $(S, \pi) = (\{2, \ldots, k\}, \hat{1})$ then $\Delta(k) = \phi_{\leq}^{-1}(S, \pi)$. Also, $\Delta(k - 1, k)$ consists of those graphs in $\Delta(k - 1)$ which do not become 2-connected when the edge $1k$ is added.

By the above discussion and the fact that the natural action of S_n on $\mathcal{L}at(\Delta_n^2)$ is order preserving, Lemma 3.3 follows immediately from the next lemma.

Lemma 3.4. *For $2 \leq k \leq n - 1$, $\Delta(k)$ is contractible.*

The proof of Lemma 3.4 proceeds by induction on k, the case $k = 2$ having been handled above. The inductive proof is therefore achieved by the combination of the following two lemmas.

Lemma 3.5. *Let $3 \leq k \leq n - 1$. If $\Delta(k - 1)$ and $\Delta(k - 1, k)$ are contractible, then so is $\Delta(k)$.*

Lemma 3.6. *For $3 \leq k \leq n - 1$, $\Delta(k - 1, k)$ is contractible.*

To prove Lemma 3.6 we use a special case of Forman's discrete Morse theory (see [F], and for this case also [Ch]). The following works for regular cell complexes, but we only need the simplicial case.

Definition: Let Σ be a simplicial complex.

(1) $D(\Sigma)$ is the digraph whose vertex set is Σ and whose edges are the edges in the Hasse diagram of $\mathcal{L}at(\Sigma) \setminus \{\hat{1}\}$, all directed downward.
(2) For any set X of edges in $D(\Sigma)$, $D_X(\Sigma)$ is the digraph obtained from $D(\Sigma)$ by reversing the direction of the edges in X, so these edges are directed upward while the remaining edges are directed downward.

Before we can formulate the following lemma we have to recall some basic facts about collapsibility (see for example [B]). Given a simplicial complex Σ a cell $\sigma \in \Sigma$ is called *free* if σ is not maximal and is contained in a unique maximal cell of Σ. If σ is free in Σ then passing from Σ to the complex $\Sigma \setminus \{\tau : \tau \supseteq \sigma\}$ is called an *elementary collapse* of Σ. If we can obtain a single point by applying a sequence of elementary collapses to a complex Σ, then Σ is called *collapsible*. Since it is easily seen that an elementary collapse of Σ is a strong deformation retraction it follows that collapsible complexes are contractible.

Lemma 3.7. *Let Σ be a simplicial complex. If $D(\Sigma)$ contains a perfect matching M such that $D_M(\Sigma)$ is acyclic, then Σ is collapsible.*

We call a perfect matching of the type described in Lemma 3.7 an *acyclic perfect matching* of $D(\Sigma)$. In order to prove Lemma 3.6 we make some technical definitions.

Definition: Consider separable graphs on the vertex set $[n]$.

(1) We denote the set of cutpoints of such a graph G by $\mathrm{Cut}(G)$.
(2) For fixed $k \in \{3, \dots, n-1\}$, let
   (a) $I(k) := \{G \in \Delta(k-1,k) \mid N_G(1) = \emptyset\}$.
   (b) $J(k) := \{G \in \Delta(k-1,k) \mid N_G(1) \neq \emptyset \text{ and } \mathrm{Cut}(G + 1k) \neq \{1\}\}$.
   (c) $F(k) := \{G \in \Delta(k-1,k) \mid \mathrm{Cut}(G+1k) = \{1\}\}$.

Note that $\Delta(k-1,k)$ is the disjoint union of $I(k)$, $J(k)$ and $F(k)$, and that both $I(k)$ and $I(k) \cup J(k)$ are subcomplexes of $\Delta(k-1,k)$.

The following lemma implies Lemma 3.6, and therefore completes the proof of Theorem 3.1.

Lemma 3.8. *For any $k \in \{3, \dots, n-1\}$, $D(\Delta(k-1,k))$ admits an acyclic perfect matching.*

The proof is carried out in three steps. First we show that $D(I(k))$ admits an acyclic perfect matching, then that $D(I(k) \cup J(k))$ admits an acyclic perfect matching, and finally that $D(\Delta(k-1,k))$ admits an acyclic perfect matching.

4. The character for the action of S_n on $\tilde{H}_{2n-5}(\Delta_n^2)$

Given Theorem 3.1, it is natural to investigate the representation of the symmetric group S_n on the only non-zero homology group of Δ_n^2 induced by the obvious action. In this section we consider homology with complex coefficients, hence all representations are over $\mathbb{C}$.

Definition:

(i) We denote by ω_n^2 the character of S_n given by $g \mapsto Trace(g, \tilde{H}_{2n-5}(\Delta_n^2))$.
(ii) Let C_n be a cyclic subgroup of S_n generated by a full n-cycle. We denote by ℓie_n the character of S_n induced from the character on C_n which takes the value $e^{\frac{2\pi i}{n}}$ on a fixed generator. It is well known [Re, Chapter 8] that the character ℓie_n is the character of S_n on the multigraded piece of the free Lie-algebra generated by n variables.

For the rest of this section we let S_{n-1} be stabilizer of the point 1 in the natural action of S_n on the set $[n]$.

Theorem 4.1. *The character ω_n^2 is given by*

$$\omega_n^2 = \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} - \ell ie_n.$$

The proof of Theorem 4.1 uses the definition of induced characters and the following two lemmas.

Lemma 4.2. *If $g \in S_{n-1}$ then $\omega_n^2(g) = \ell ie_{n-1}(g)$.*

Since every element of S_n which has a fixed point is conjugate to an element of S_{n-1}, it remains to determine $\omega_n^2(g)$ for all fixed-point-free $g \in S_n$.

Definition: If $g \in S_n$ is fixed point free, then g^* is defined to be the element of S_{n+1} which fixes $n+1$ and acts as g does on $[n]$.

Lemma 4.3. *Let $g \in S_n$ be fixed-point-free. Then $\omega_n^2(g) = -\omega_{n+1}^2(g^*)$.*

By a result of Vassiliev the number of linearly independent knot invariants of bi-order $(n, n-1)$, modulo lower bi-order invariants, is bounded from above by the multiplicity of the trivial representation in the restriction of ω_n^2 to the cyclic group C_n generated by $(12\cdots n)$. See [V3] for all details. As an immediate corollary of Theorem 4.1 we obtain a formula for this multiplicity. We write $\langle \xi, 1 \rangle$ for the multiplicity of the trivial character in any character ξ of C_n.

Corollary 4.4.

$$\langle \omega_n^2 \downarrow_{C_n}^{S_n}, 1\rangle = (n-2)! - \frac{1}{n}\sum_{d|n} \mu(d)\phi(d)(\frac{n}{d}-1)!\, d^{\frac{n}{d}-1}$$

The values of $w_n = \langle \omega_n^2 \downarrow_{C_n}^{S_n}, 1\rangle$ for small n are given in the table below.

n	3	4	5	6	7	8	9	10	11
w_n	1	1	2	6	18	96	564	4,072	32,990

Table 1: Multiplicity w_n of the trivial character in $\omega_n^2 \downarrow_{C_n}^{S_n}$

The character ω_n^2 and the tensor product of ω_n^2 with the sign character have appeared recently in various different settings, first in the work of C. A. Robinson & S. Whitehouse [RW] and S. Whitehouse [Wh] on gamma-homology of algebras and later in work of E. Getzler & M. Kapranov [GK] on operads, O. Mathieu [Ma] on hyperplane arrangements and symplectic geometry, in the work of M. Kontsevich [K] on Lie algebras and symplectic geometry, and in the work of P. Hanlon [H], P. Hanlon & R.P. Stanley [HS] and S. Sundaram [Su] in a combinatorial and representation-theoretic context.

5. The lattice of block-closed graphs

In this section we will obtain information on the topology of $\Delta_{n,k}^2$ by producing a lattice $\Sigma_{n,k}$ such that $\Delta(\overline{\Sigma_{n,k}})$ is homotopy equivalent to $\Delta_{n,k}^2$ and examining the structure of $\Sigma_{n,k}$. We begin by recalling some elements of the well known structure theory of separable graphs, which appears e.g. in [L].

Definition: Let G be any graph. A *block* of G is a subset W of $V(G)$ such that the subgraph of G induced on W is 2-connected and the subgraph of G induced on any proper overset of W is separable. We will say that G is *block-closed* if the subgraph induced on each block is a clique.

Given a graph G, say that $e \cong e'$ for two of its edges e and e' if they both lie in some circuit of G. This is easily seen to be an equivalence relation on $E(G)$. If W is the set of vertices underlying an equivalence class then W is a block, and all the blocks correspond to equivalence classes of edges in this way. From this it is easy to derive the following basic facts about the "block decomposition" of G, see [L] for more details.

Proposition 5.1. *Let G be a graph. Then there exists a unique decomposition of $V(G)$ into blocks $W_1, \ldots, W_r$, and if $i \neq j$ we have $|W_i \cap W_j| \leq 1$. Moreover, if B_G is the graph with vertex set $\{w_1, \ldots, w_r\}$ such that $\{w_i, w_j\} \in E(B_G)$ if and only if $|W_i \cap W_j| = 1$, then B_G is a forest (that is, B_G contains no cycles).*

Note that if H is a k-graph with underlying graph G, then every block of G has size at least k or is a single vertex.

Definition: Let K be a k-graph with underlying graph G, and let $W_1, \ldots, W_r$ be the blocks of G. We define K^* to be the k-graph which induces the complete k-graph on each W_i and contains no other hyperedges. We also define $\Sigma_{n,k}$ to be the poset of all graphs on vertex set $[n]$ in which every block is either an isolated vertex or a clique of size at least k, ordered by inclusion.

The first part of the following lemma is immediate from the definition, and the second follows via a standard argument for closure operators.

Lemma 5.2. (i) *The map $K \mapsto K^*$ is a closure operator on $\mathcal{L}at(\Delta_{n,k}^2)$ whose image is isomorphic to $\Sigma_{n,k}$.*
(ii) *$\Sigma_{n,k}$ is a lattice.*

The meet operation in the lattice $\Sigma_{n,k}$ is intersection of edge-sets followed by deletion of the edges in all cliques of size smaller than k. Note that the elements of $\Sigma_{n,2}$ are the block-closed graphs, and that we have a tower of embeddings as subposets (not sublattices):

$$\Sigma_{n,k} \subseteq \cdots \subseteq \Sigma_{n,3} \subseteq \Sigma_{n,2}.$$

Hence, in view of the following result the topology of all the complexes $\Delta^2_{n,k}$ is encoded into the lattice $\Sigma_{n,2}$ of block-closed graphs.

Theorem 5.3. *The complexes $\Delta^2_{n,k}$ and $\Delta(\overline{\Sigma_{n,k}})$ are homotopy equivalent.*

We now investigate the structure of $\Sigma_{n,k}$. The next two lemmas follow immediately from the definition of $\Sigma_{n,k}$. We write $\hat{0}$ for the empty graph, which is the minimum element of $\Sigma_{n,k}$.

Lemma 5.4. *Let M be a coatom of $\Sigma_{n,k}$, that is, an element which is covered by $\hat{1}$. Then one of the following conditions holds.*

(i) *M is connected and has two blocks of size l, m with $k \le l \le m \le n-k+1$ and $l+m = n+1$. In this case, the interval $[\hat{0}, M]$ is isomorphic to $\Sigma_{l,k} \times \Sigma_{m,k}$.*

(ii) *M consists of an $(n-1)$-clique and an isolated vertex. In this case, $k > 2$ and the interval $[\hat{0}, M]$ is isomorphic to $\Sigma_{n-1,k}$.*

Lemma 5.5. *Let $G \in \overline{\Sigma}_{n,k}$ cover $H \in \Sigma_{n,k}$. Then one of the following conditions holds.*

(i) *$E(G) \setminus E(H)$ is a clique on k vertices.*

(ii) *$E(G) \setminus E(H)$ is a star (that is, a connected graph with at most one vertex of degree more than one), and the vertices of degree one in this star form a block in H.*

(iii) *$E(G) \setminus E(H)$ is a complete bipartite graph on parts A and B, and there is a vertex v such that $A \cup \{v\}$ and $B \cup \{v\}$ are blocks in H.*

Using Lemma 5.4 and induction, we prove the following result.

Theorem 5.6. (i) *$\Sigma_{n,2}$ is graded of rank $2n-3$.*

(ii) *$\Sigma_{n,3}$ is graded of rank $n-2$.*

(iii) *If $k > 3$ and $n < 2k-1$ then $\Sigma_{n,k}$ is graded of rank $n-k+1$.*

(iv) *If $k > 3$ and $n \ge 2k-1$ then the longest maximal chains in $\Sigma_{n,k}$ have length $n-k+1$. Also, the shortest maximal chains in $\Sigma_{n,k}$ have length $(n-2)-(k-3)\lfloor\frac{n-1}{k-1}\rfloor$.*

(v) *If $k > 3$ then $G \in \overline{\Sigma}_{n,k}$ is contained in a chain of length $n-k+1$ if and only if G consists of a clique of size $l \ge k$ and $n-l$ isolated vertices.*

The above results yield some nontrivial information about the topology of $\Delta^2_{n,k}$ when $k > 3$.

Corollary 5.7. *Assume that $k > 3$.*

(i) *If $i > n-k-1$ then $\widetilde{H}_i(\Delta^2_{n,k}) = 0$.*

(ii) *$\widetilde{H}_{n-k-1}(\Delta^2_{n,k})$ is free of dimension $\binom{n-1}{k-1}$.*

(iii) *If $n < 2k-1$ then $\Delta^2_{n,k}$ has the homotopy type of a wedge of $\binom{n-1}{k-1}$ spheres of dimension $n-k-1$.*

(iv) *If $n = 2k-1$ then $\Delta^2_{n,k}$ has the homotopy type of a wedge of spheres. This wedge consists of $\binom{n-1}{k-1}$ $(n-k-1)-$spheres and $\frac{1}{2}n\binom{n-1}{k-1}$ 1-spheres.*

The posets $\Sigma_{n,k}$ are also useful in the case $k = 2$, and possibly in the case $k = 3$. An alternative proof of Theorem 3.1 is given by the following lemma.

Lemma 5.8. *Let $\phi : \overline{\mathcal{L}at(\Delta^2_n)} \to \overline{B_{n-1} \times \Pi_{n-1}}$ be as defined in Section 3. Then the restriction of ϕ to the subposet $\overline{\Sigma_{n,2}}$ is surjective and induces a homotopy equivalence between $\Delta(\overline{\Sigma_{n,2}})$ and $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$.*

If certain intervals in $\Sigma_{n,2}$ are well behaved, then the homology of $\Delta(\Sigma_{n,3})$ is concentrated in the highest possible dimension, as seen in the lemma below. Recall that a poset P is called *Cohen-Macaulay* if P is ranked and for every interval $[x, y]$ in P the reduced homology of $\Delta(x, y)$ is concentrated in dimension $rank(x, y) - 2$ (see [B]).

Lemma 5.9. *If $\Sigma_{n,2}$ is Cohen-Macaulay then the reduced homology of $\Delta^2_{n,3}$ is concentrated in dimension $n-4$.*

The homology of $\Delta^2_{n,3}$ has been computed for $4 \leq n \leq 7$. It is concentrated in dimension $n-4$.

$n\backslash i$	0	1	2	3
2	0	0	0	0
3	0	0	0	0
4	$\mathbb{Z}^3$	0	0	0
5	0	$\mathbb{Z}^{21}$	0	0
6	0	0	$\mathbb{Z}^{180}$	0
7	0	0	0	$\mathbb{Z}^{2010}$

Table 2: Homology groups $\widetilde{H}_i(\Delta^2_{n,3})$

6. The Euler characteristic of the complex $\Delta^2_{n,k}$

In Section 5 we were able to determine the homotopy type of $\Delta^2_{n,k}$ for $k > 3$ when $n \leq 2k-1$, but not for $k = 3$, nor for $k > 3$ and $n > 2k-1$. Indeed, in the case $k = 3$ we have no information on the topology of $\Delta^2_{n,k}$ unless n is very small, and in the case $k > 3$ and $n > 2k-1$ we are able to determine only the homology group $\widetilde{H}_{n-k-1}(\Delta^2_{n,k})$. In this section, we investigate the reduced Euler characteristic of $\Delta^2_{n,k}$. We will determine a formula for the exponential generating function

$$M_k(x) := \sum_{n=1}^{\infty} \tilde{\chi}(\Delta^2_{n,k})\frac{x^n}{n!},$$

for all $k \geq 2$. That formula is stated in the following theorem.

Theorem 6.1. *For $k \geq 2$, we have*

$$M'_k\left(x\frac{p_{k-1}(x)}{p_k(x)}\right) = \ln\left(\frac{p_{k-1}(x)}{p_k(x)}\right),$$

where $p_k(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}$.

Theorem 6.1 gives another proof that $\tilde{\chi}(\Delta^2_{n,2}) = -(n-2)!$. It also implies

$$M'_3(x) = \ln\left(\frac{-x(x-2)}{(x-1)+\sqrt{2-(x-1)^2}}\right),$$

which gives the sequence $0, 0, -1, 3, -21, 180, -2010, 27090, -430290, \ldots$ for $\tilde{\chi}(\Delta^2_{n,3})$, cf. Table 2. To obtain these corollaries set $y := x\frac{p_{k-1}(x)}{p_k(x)}$ and solve for x to get $x = \frac{1}{1-y}$ and $x = \frac{y-1+\sqrt{2-(y-1)^2}}{2-y}$, when $k = 2$ and 3 respectively. The proof of the theorem uses the posets $\Sigma_{n,k}$ defined in Section 5, induction on n and the exponential formula (Proposition 9.5).

7. $(n-2)$-connected graphs and matching complexes

Before we proceed to the consideration of $(n-2)$-connected graphs, we state some simple but useful facts about the general situation. What do maximal $(i-1)$-separable graphs on the n element set $[n]$ look like? Is is clear that each such graph is described by an $(i-1)$-set A and a partition $B \uplus C$ of $\{1, \ldots, n\} \setminus A$ into two non-empty blocks B, C. The corresponding maximal $(i-1)$-separable graph is the complete graph on $[n]$ with all edges connecting B and C removed.

Now let G be an $(n-2)$-connected graph on n vertices, so $G \notin \Delta^{n-2}_n$. Then by the above description of maximal $(n-3)$-separable graphs the induced subgraph on any three vertices must contain at least two edges. Thus the complementary graph (i.e., the graph containing precisely the edges that are not in G) is a matching. The graphs on n vertices that are matchings form a simplicial complex, that we denote by M_n. We conclude the following.

Proposition 7.1. *The matching complex M_n is Alexander dual (in the sense of Proposition 9.4) to the complex Δ^{n-2}_n. In particular, there is an isomorphism*

$$\widetilde{H}_i(M_n) \cong \widetilde{H}^{\binom{n}{2}-i-3}(\Delta^{n-2}_n).$$

The matching complexes M_n have attracted attention for various reasons. In [BLVZ, Theorem 4.1] the matching complex M_n is shown to be topologically $(\lfloor\frac{n+1}{3}\rfloor - 2)$-connected, which implies that $\widetilde{H}_i(M_n) = 0$, for $i = 0, \ldots, \lfloor\frac{n+1}{3}\rfloor - 2$. We thus get the following corollary.

Corollary 7.2. *The cohomology of Δ_n^{n-2} vanishes in dimensions $i \geq \binom{n}{2} - \lfloor\frac{n+1}{3}\rfloor - 1$.*

The following table shows what we know about the homology groups $\widetilde{H}_i(M_n)$, based on the results of [BLVZ] for $n \leq 6$ and $n = 8$, and our own computations.

$n\backslash i$	0	1	2	3	4	5
2	0	0	0	0	0	0
3	$\mathbb{Z}^2$	0	0	0	0	0
4	$\mathbb{Z}^2$	0	0	0	0	0
5	0	$\mathbb{Z}^6$	0	0	0	0
6	0	$\mathbb{Z}^{16}$	0	0	0	0
7	0	torsion[1]	$\mathbb{Z}^{20}$	0	0	0
8	0	0	$\mathbb{Z}^{132}$	0	0	0
9	0	0	$\mathbb{Z}^{42} \oplus$ torsion[2]	$\mathbb{Z}^{70}$	0	0
10	0	0	torsion[3]	$\mathbb{Z}^{1216}$	0	0
11	0	0	0	$\mathbb{Z}^{1188} \oplus$ torsion[4]	$\mathbb{Z}^{252}$	0
12	0	0	0	torsion[5]	$\mathbb{Z}^{12440}$	0

Table 3: Homology groups $\widetilde{H}_i(M_n)$ of matching complexes

We see that the complexes Δ_n^{n-2} can have torsion, and that this phenomenon begins with Δ_7^5. The rational homology of M_n has been determined in [RR], and the ranks of the torsion-free parts of $\widetilde{H}_i(M_n)$ given in Table 3 agree with the results given in that paper.

8. $(n-3)$-CONNECTED GRAPHS

Theorem 8.1. *The exponential generating function $F_n^{n-3}(x) = \sum_{n\geq 0} \widetilde{\chi}((\Delta_n^{n-3})^*)\frac{x^n}{n!}$ of the Euler-characteristic of the Alexander Dual of Δ_n^{n-3} is given by:*

$$F_n^{n-3}(x) = x - \frac{\exp(\frac{x}{2(1+x)}) + x - \frac{1}{4}x^2 - \frac{1}{8}x^4}{\sqrt{1+x}}.$$

The exponential generating function of the Euler characteristic of Δ_n^{n-3} is then the sum of real and imaginary part of $-F_n^3(ix)$.

The complex Δ_n^{n-3} has maximal simplexes of dimensions $n-1$ and $n-2$ only. It is also easily collapsible to a pure complex of dimension $n-2$. A Maple computation (see below) shows that neither the Euler characteristic of $(\Delta_n^{n-3})^*$ nor the Euler characteristic of Δ_n^{n-3} alternate in sign, so the pure complexes are certainly not all Cohen-Macaulay. The calculation shows that

$$F_n^{n-3}(x) = -1 - x + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \frac{1}{20}x^6 - \frac{1}{27}x^7 - \frac{1}{224}x^8 - \frac{1}{480}x^9 + O(x^{10}).$$

We have also studied the slightly larger complex of graphs which are the disjoint union of cycles and paths of any length (i.e., graphs with maximum vertex degree at most 2). This is also a reasonable generalization of the matching complex, which is the complex of all graphs with maximum vertex degree at most 1. The Euler characteristic of the corresponding Alexander dual has almost the same generating function as for $(\Delta_n^{n-3})^*$. That generating function is

[1] There is $\mathbb{Z}_3$-torsion of rank 1. No $\mathbb{Z}_p$-torsion for $p = 2$, $5 \leq p \leq 17$.
[2] There is $\mathbb{Z}_3$-torsion of rank 8. No $\mathbb{Z}_p$-torsion for $p = 2$, $5 \leq p \leq 17$.
[3] There is $\mathbb{Z}_3$-torsion of rank 1. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.
[4] There is $\mathbb{Z}_3$-torsion of rank 35. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.
[5] There is $\mathbb{Z}_3$-torsion of rank 56. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.

$$x - \frac{\exp(\frac{x}{2(1+x)}) + x - \frac{1}{4}x^2}{\sqrt{1+x}} =$$

$$-1 - x + \frac{1}{8}x^4 - \frac{3}{40}x^5 + \frac{1}{20}x^6 - \frac{1}{28}x^7 - \frac{17}{896}x^8 - \frac{7}{1920}x^9 - \frac{23}{2400}x^{10} + O(x^{11}).$$

The maximal simplices in the cycle and path complex have dimension $n-1$ or $n-2$, and the complex can be collapsed to a pure $(n-2)$-dimensional complex. The generating function for the Euler characteristic shows that these collapsed complexes are not all Cohen-Macaulay.

9. Notation and Tools

In this short section we will summarize the main tools that we use in the study of the complexes $\Delta^i_{n,k}$. We refer the reader to the survey paper [B] for more details and references.

Let P be a finite partially ordered set – *poset* for short. If P has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$, we denote by $\overline{P}$ the *proper part* of P, that is the poset obtained by removing from P the elements $\hat{0}$ and $\hat{1}$. By $\Delta(P)$ we denote the simplicial complex of all chains in P. The complex $\Delta(P)$ is called the *order complex* of P. By convention we include the empty set $\emptyset$ in every simplicial complex. For any simplicial complex Δ, $\mathcal{L}at(\Delta)$ will denote the poset of faces of Δ, ordered by inclusion and enlarged by an additional greatest element $\hat{1}$. Then the order complex $\Delta(\overline{\mathcal{L}at(\Delta)})$ of the proper part of $\mathcal{L}at(\Delta)$ is homeomorphic to Δ. Indeed, $\Delta(\overline{\mathcal{L}at(\Delta)})$ is the barycentric subdivision of Δ.

For a poset P and $p \in P$ we denote by $P_{\leq p}$ the subposet $\{p' \mid p' \in P;\ p' \leq p\}$. The posets $P_{\geq p}$, $P_{<p}$ and $P_{>p}$ are analogously defined. For $p \leq p'$ in P we denote by $[p,p']$ the closed interval $P_{\geq p} \cap P_{\leq p'}$ in P, and by (p,p') the open interval $P_{>p} \cap P_{<p'}$.

For a poset P we denote by μ_P the $\mathbb{Z}$-valued function defined recursively on the intervals of P by $\mu_P(x,x) = 1$ and $\mu_P(x,y) = -\sum_{x \leq z < y} \mu_P(x,z)$ if $x < y$.

By a map $f : P \to Q$ of posets we always mean a poset homomorphism (i.e., $x \leq y$ implies $f(x) \leq f(y)$). For an element $q \in Q$ we denote by $f^{-1}_{\leq}(q)$ the preimage of $Q_{\leq q}$ under f. Analogously defined is $f^{-1}_{\geq}(q)$.

Proposition 9.1 (Quillen Fiber Lemma [Q]). *Let $f : P \to Q$ be a map of posets. If $\Delta(f^{-1}_{\leq}(q))$ is contractible for all $q \in Q$ then $\Delta(P)$ and $\Delta(Q)$ are homotopy equivalent.*

A map $f : P \to P$ from a poset to itself is called a *closure operator* if $f(x) \geq x$ and $f(f(x)) = f(x)$ for all $x \in P$. The Quillen Fiber Lemma immediately implies the fact that closure operators preserve the homotopy type.

Corollary 9.2 (Closure Lemma). *Let $f : P \to P$ be a closure operator on the partially ordered set P. Then $\Delta(P)$ and $\Delta(f(P))$ are homotopy equivalent.*

If the poset P is a lattice (i.e., suprema, denoted by "$\vee$", and infima, denoted by "$\wedge$", exist) then there is another tool for computing the homotopy type. Note that if P is a finite lattice then there is a least element $\hat{0}$ and a largest element $\hat{1}$ in P. For an arbitrary element $p \in P$ we say that $a \in P$ is a complement of p if $p \wedge a = \hat{0}$ and $p \vee a = \hat{1}$.

Proposition 9.3 (Homotopy Complementation Formula [BWa]). (i) *Let P be a poset and $A \subseteq P$ an antichain. Assume $\Delta(P \setminus A)$ is contractible. Then $\Delta(P)$ is homotopy equivalent to*

$$\bigvee_{x \in A} \Sigma\Big(\Delta(P_{<x}) * \Delta(P_{>x})\Big).$$

(ii) *Let P be a lattice and let Co be the set of complements of some element $p \neq \hat{0}, \hat{1}$. Then $\Delta(P \setminus Co)$ is contractible.*

In the formulation of the proposition $\bigvee$ denotes the wedge product, Σ denotes the suspension and $*$ denotes the join of topological spaces.

Our next tool is the combinatorial version of a standard duality theorem from algebraic topology.

Proposition 9.4 (Combinatorial Alexander Duality). *Let Δ be a finite simplicial complex on vertex set V and define*

$$\Delta^* = \{B \subseteq V \mid V \setminus B \notin \Delta\}.$$

Then

$$\widetilde{H}_i(\Delta) \cong \widetilde{H}^{|V|-i-3}(\Delta^*).$$

This is derived as follows. The usual Alexander duality theorem (see e.g. Munkres [Mu]) says that

$$\widetilde{H}_i(A) \cong \widetilde{H}^{n-i-1}(S^n \setminus A)$$

for any compact subset A of the n-sphere S^n. In our situation, let $P = 2^V \setminus \{\emptyset, V\}$. This truncated Boolean algebra is the proper part of the face lattice of the boundary complex of a simplex, so $\Delta(P) \cong S^{|V|-2}$. Now let A be the realization of $\Delta(\overline{\mathcal{L}at(\Delta)})$ as a subspace of $\Delta(P)$. It is easy to see that $\Delta(P \setminus \overline{\mathcal{L}at(\Delta)})$ is a strong deformation retract of $S^{|V|-2} \setminus A$, and since $P \setminus \overline{\mathcal{L}at(\Delta)} \cong \overline{\mathcal{L}at(\Delta^*)}$ the result follows.

Finally, we recall a result from enumerative combinatorics. For a number sequence $(a_n)_{n\geq 0}$ the formal power series $\sum_{n\geq 0} a_n \frac{x^n}{n!}$ is called its *exponential generating function.*

Proposition 9.5 (Exponential formula). *Suppose that two functions $a, b : \mathbb{N} \longrightarrow \mathbb{Z}$ are given such that*

$$b(n) = \sum_{S_1|\cdots|S_t \in \Pi_n} a(|S_1|)\cdots a(|S_t|), \quad n \geq 1,$$

where the sum ranges over all set-partitions of $[n]$ and $a(0) = 0$, $b(0) = 1$. Then the exponential generating functions $A(x) := \sum_{n=0}^{\infty} \frac{a(n)x^n}{n!}$ and $B(x) := \sum_{n=0}^{\infty} \frac{b(n)x^n}{n!}$ satisfy

$$B(x) = e^{A(x)}.$$

For the proof see [St1, St3].

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