Green and Gotzmann theorems for polynomial rings with restricted powers of the variables

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Abstract

Two of the fundamental theorems in finite set combinatorics are the theorems of Macaulay and Kruskal-Katona which characterize the h-vectors of multicomplexes and the f-vectors of simplicial complexes respectively. These theorems also characterize the Hilbert functions of quotients of polynomial rings and exterior algebras. Gotzmann proved a persistence theorem for vector spaces which are extremal in the sense of Macaulay. Aramova, Herzog, and Hibi proved a persistence theorem for vector spaces which are extremal in the sense of Kruskal-Katona. Let V be a vector space of homogeneous polynomials of degree d in general coordinates $x_1, \ldots, x_n$ and W be the vector space obtained from V by setting $x_n = 0$. Green proved that a bound similar to Macaulay's relates the codimensions of V and W.

In this paper we prove analogues of Green's result and the persistence theorems of Gotzmann and Aramova-Herzog-Hibi for strongly stable ideals in polynomial rings with restricted powers of the variables. Our results can be interpreted as results about h-vectors of multicomplexes with restricted multiplicities.

Résumé

Deux des théorèmes fontamentaux de la combinatoire des ensembles finis, le théorème de Macaulay et le théorème de Kruskal-Katona, fournissent une caractérisation des h-vecteurs des multicomplexes (resp. des f-vecteurs des complexes simpliciaux). Ces théorèmes caractérisent aussi les fonctions de Hilbert des quotients des anneaux polynomiaux et des algèbres extérieurs. Gotzmann a démontré un théorème de persistance pour les espaces vectoriels extrémaux au sens de Macaulay. Aramova, Herzog et Hibi ont démontré un théorème de persistance pour les espaces vectoriels extrémaux au sens de Kruskal-Katona. Soit V un espace vectoriel de polynômes homogènes de degré d en coordonnées générales $x_1, \ldots, x_n$, soit W l'espace vectoriel obtenu de V en posant $x_n = 0$. Green a demontré qu'une borne semblable à celle de Macauley donne une relation entre les codimensions de V et de W.

Dans ce travail nous démontrons des analogues de ce résultat de Green et des théorèmes de persistance de Gotzmann et de Aramova, Herzog et Hibi pour les idéaux fortement stables dans un anneau polynômial avec des puissances restreintes des variables. Nos résultats peuvent être interprétés comme des résultats sur les h-vecteurs des multicomplexes ayant des multiplicités restreintes.

1 Introduction

The extremal properties of Hilbert functions have been studied extensively. One of the main reasons for the fertility and appeal of this subject is that one can study Hilbert functions using methods and techniques from several mathematical areas: combinatorics, commutative algebra, and algebraic geometry. In [13] Macaulay characterized the Hilbert functions of quotients of polynomial rings, or equivalently, the h-vectors of multicomplexes [14, §2.2]. Given Macaulay's result, it is natural to ask whether vector spaces of forms of the same degree which achieve Macaulay's bound enjoy some other special properties. In [7] Gotzmann proved his remarkable Persistence Theorem which states that such extremal vector spaces in degree d generate extremal vector spaces in degree $d+1$. We will call such vector spaces *Gotzmann*. Structure results about Gotzmann vector spaces have been obtained in [3], [6], and [8]. Green [8] characterized the Hilbert functions of rings obtained by moding out quotients of polynomial rings with fixed Hilbert function by a general linear form. A result of Kruskal [12] and Katona [11] extended the study of the extremal properties of Hilbert functions to rings other than the polynomial rings. They characterized the f-vectors of simplicial complexes, or equivalently, the Hilbert functions of quotients of rings of the form $k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$. Since it does not make any difference if the variables commute or anticommute, this also characterizes the Hilbert functions of quotients of exterior algebras (see also [1]). Clements and Lindström [4] generalized both Macaulay's and Kruskal-Katona's results to rings of the form $R = k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$, where k is a field, $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \infty$, and $x_i^\infty = 0$. We will extend the definition of a Gotzmann vector space to extremal vector spaces in any such ring R. A vector space $V \subseteq R$ (resp. an ideal $I \subseteq R$) is called *strongly stable*, if V (resp. I) is generated by monomials and whenever $x_i m \in V$ (resp. $x_i m \in I$) for some monomial m, then $x_j m \in V$ (resp. $x_i m \in I$) for any $j \leq i$. In her dissertation 1995 Bigatti gave a new proof of Gotzmann Persistence Theorem for polynomial rings in characteristic 0. She proved the theorem for strongly stable vector spaces and used Gröbner basis theory to reduce the general case to that of strongly stable vector spaces. Aramova, Herzog, and Hibi [1] showed that with minor modifications Gröbner basis theory known from polynomial rings carries over to exterior algebras. They used an approach similar to Bigatti's to prove a Persistence Theorem for Gotzmann vector spaces in exterior algebras.

It is not hard to see that to prove Macaulay's, Green's, and Kruskal-Katona's theorems it is enough to consider strongly-stable vector spaces. Moreover, in the sense of Green's theorem, the last variable x_n is a general linear form for any strongly stable vector space.

In this paper we give generalizations of Green's theorem (in Theorem 2.1 (1)), Clements-Lindström theorem (in Theorem 2.1 (2)), and the persistence theorems of Gotzmann and Aramova-Herzog-Hibi (in Theorem 2.1 (3)) to strongly stable ideals in rings of the form

$$k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}), \tag{1}$$

where $2 \le a_i \le \infty$ for $1 \le i \le n$. (We are not assuming that $a_1 \le a_2 \le \dots \le a_n$.) The following example shows that the analog of the Gotzmann and Aramova-Herzog-Hibi Persistence theorems does not hold if V is not strongly stable.

Example 1.1. Let $S = k[x,y]/(x^3)$, V be the vector space spanned by y, and L the vector space spanned by x. Then $VS_1 = \operatorname{span}\{xy, y^2\}$ and $LS_1 = \operatorname{span}\{x^2, xy\}$, so $\dim V = \dim L$ and $\dim VS_1 = \dim LS_1$. However, for $n \ge 3$ we have $VS_{n-1} = S_n = \operatorname{span}\{x^2y^{n-2}, xy^{n-1}, y^n\}$ and $LS_{n-1} = \operatorname{span}\{x^2y^{n-2}, xy^{n-1}\}$, so $\dim VS_{n-1} \ne \dim LS_{n-1}$.

Specializing our proofs to the case of polynomial rings ($a_1 = a_2 = \dots = a_n = \infty$) one obtains new proofs of Macaulay's and Green's theorems. Since our proofs work for anticommuting, as well as for commuting indeterminates, we can also specialize to the case of exterior algebras ($a_1 = a_2 = \dots = a_n = 2$; anticommuting indeterminates) and obtain a new proof of Aramova-Herzog-Hibi Persistence Theorem.

2 Hilbert functions of strongly stable ideals

Let S be a ring of the form (1). We denote by S_d the vector space of homogeneous polynomials of degree d in S. Let $\tilde{S} = k[x_1,\dots,x_{n-1}]/(x_1^{a_1},\dots,x_{n-1}^{a_{n-1}}) \subset S$ and let $V \subseteq S_d$ be a vector space generated by monomials. Let $\tilde{V}$ be the vector space generated by the monomials in V which are not divisible by x_n and $V' = \{\frac{f}{x_n} : f \in V \text{ and } x_n | f\}$, so $V = \tilde{V} \oplus x_nV'$. If in addition V is strongly stable, then $x_iV' \subseteq V$ for any i, so $V'S_1 \subseteq V$. Then

$$VS_1 = \tilde{V}\tilde{S}_1 + \tilde{V}x_n + x_nV'S_1 \subseteq \tilde{V}\tilde{S}_1 + \tilde{V}x_n + x_nV = \tilde{V}\tilde{S}_1 \oplus x_nV \subseteq VS_1,$$

so $VS_1 = \tilde{V}\tilde{S}_1 \oplus x_nV$. The main result in this paper is:

Theorem 2.1. *Let $V, L \subseteq S_d$ be vector spaces such that V is strongly stable, L is generated by an initial lex-segment, and* $\dim V = \dim L$. *Then:*

1. $\dim \tilde{V} \geq \dim \tilde{L}$;
2. $\dim VS_1 \geq \dim LS_1$;
3. *If* $\dim VS_1 = \dim LS_1$, *then* $\dim VS_2 = \dim LS_2$.

In the proof of this theorem we use the following Theorem 2.2 about multicomplexes with restricted multiplicities.

If $C \subseteq S_d$ is a set of monomials and $m \in S$ is a monomial, we set $mC = \{mm' : m' \in C\}$ and $\phi(m) = \max\{i : x_1^i | m\}$. We also denote by $C^{(i)}$, $0 \leq i \leq a_1 - 1$, the set $C^{(i)} = \{\frac{m}{x_1^i} : m \in C \text{ and } \phi(m) = i\}$. We set $C' = \cup_{i=1}^{a_1-1} x_1^i C^{(i)} = \{m \in C : x_1 \,|\, m\}$ and $\Delta C = \{m \in S_{d-1} : m \text{ divides a monomial in } C\}$. (So $\Delta C = \emptyset$ when $d = 0$.) Then $C = \cup_{i=0}^{a_1-1} x_1^i C^{(i)} = C^{(0)} \cup C'$.

Theorem 2.2. *Let $C, R \subseteq S_d$ be sets of monomials such that C is strongly stable and R is an initial rev-lex segment with $|R| = |C|$. Then*

1. $|C^{(0)}| \leq |R^{(0)}|$;
2. $|\Delta C| \geq |\Delta R|$;
3. *If* $|\Delta C| = |\Delta R|$, *then* $|C^{(0)}| = |R^{(0)}|$.

3 Proofs

To prove theorems 2.1 and 2.2 we will need two preliminary lemmas about the rev-lex order.

Lemma 3.1. *If $m_1 > m_2$ are two consecutive (with respect to the rev-lex order) monomials in S_d, then either $\phi(m_2) = \phi(m_1) - 1$, or $\phi(m_2) \geq \phi(m_1)$.*

Proof. Let $m_1 = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, so $\phi(m_1) = i_1$. Since m_1 is not the least monomial in S_d, it follows that there exists some $j \geq 2$, such that $i_j < a_j - 1$. Let u be the least such j. If $u = 2$, then $m_2 = x_1^{i_1-1} x_2^{i_2+1} x_3^{i_3} \cdots x_n^{i_n}$, so $\phi(m_1) = \phi(m_2) + 1$.

If $u > 2$, then $m_2 = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $r_u = i_u + 1$, $r_j = i_j$ for $j > u$, and for $1 \leq j \leq u - 1$ we define r_j inductively by $r_j = \min(\sum_{l=1}^{u-1} i_l - \sum_{l=1}^{j-1} r_l - 1, a_j - 1)$. In particular, $\phi(m_2) = r_1 \geq i_1 + i_2 - 1 = i_1 + a_2 - 2 \geq i_1 = \phi(m_1)$. $\square$

If $C \subseteq S_d$ is a strongly stable set of monomials, then $\Delta C^{(i)} \subseteq C^{(i+1)}$ for $0 \leq i \leq a_1 - 2$. The next lemma gives a necessary and sufficient condition to have $\Delta C^{(i)} = C^{(i+1)}$ when C is an initial rev-lex segment.

Lemma 3.2. *Let $R \subseteq S_d$ be an initial rev-lex segment and m be the least monomial in R. The following are equivalent:*

1. *$\phi(m) \leq r$;*
2. *$\Delta R^{(i)} = R^{(i+1)}$ for $r \leq i \leq a_1 - 2$.*

Proof. First we will prove the implication (1) $\Rightarrow$ (2). Let $s \geq r$. It follows by Lemma 3.1 that the least monomial m' in $\bigcup_{j=s}^{a_1-1} x_1^j R^{(j)}$ has $\phi(m') = s$. Then the least monomial in $\bigcup_{j=s}^{a_1-1} x_1^{j-s} R^{(j)}$ is $\frac{m'}{x_1^s}$ with $\phi(\frac{m'}{x_1^s}) = 0$. Moreover, $\bigcup_{j=s}^{a_1-1} x_1^{j-s} R^{(j)}$ is an initial rev-lex segment, which shows that it will be enough to prove only that $\Delta R^{(0)} = R^{(1)}$ in the case $r = 0$. Since R is strongly stable, we have that $\Delta R^{(0)} \subseteq R^{(1)}$, so it remains to prove that $\Delta R^{(0)} \supseteq R^{(1)}$. Let $m_1 = x_2^{i_2} \cdots x_n^{i_n} \in R^{(1)}$, so $x_1 m_1 \in R$. Since $x_1 m_1 > m$ and $m \in R^{(0)}$, it follows that there exists at least one j such that $i_j \leq a_j - 2$. Let u be the least such j. Then the element $m_2 = m_1 x_u$ is the largest monomial smaller than $x_1 m_1$ in S_d which is not divisible by x_1, so $m_2 \in R$. Since $m_1 \in \Delta\{m_2\}$, it follows that $R^{(1)} \subseteq \Delta R^{(0)}$.

Now we will prove the implication (2) $\Rightarrow$ (1). Suppose that (1) is not satisfied, so $m = x_1^s m_1$, where $s > r$ and $m_1 \in R^{(s)}$. Since by assumption $R^{(s)} = \Delta R^{(s-1)}$, it follows that there exists $m_2 \in R^{(s-1)}$, such that $m_2 = x_i m_1$ for some $i \geq 2$. Then $R \ni x_1^{s-1} m_2 = x_1^{s-1} x_i m_1 < x_1^s m_1 = m$, which contradicts the fact that m is the least element in R. $\square$

Note that the conclusion of Lemma 3.2 is not true for arbitrary strongly stable sets. Take for example C to be the smallest strongly stable subset of S_4 containing $x_1 x_2 x_3^2$ and $x_2^3 x_4$. The least element of C is $x_2^3 x_4$ with $\phi(x_2^3 x_4) = 0$. However, $x_2 x_3^2 \in C^{(1)} \setminus \Delta C^{(0)}$, so $\Delta C^{(0)} \subsetneq C^{(1)}$.

Proof of Theorem 2.2. We give a proof by induction on the number of variables. When $n = 1$ the theorem is obvious. Now assume that the theorem is true for $n-1$ variables.

First we will prove that $|C^{(0)}| \leq |R^{(0)}|$. Assume that on the contrary $|C^{(0)}| > |R^{(0)}|$ and let $r = \min\{\phi(m) : m \in R\}$. If p is the least element of S_d, then $p = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_n = \min(d, a_n - 1)$ and for $1 \leq i \leq n-1$, $\alpha_i = \min(d - \sum_{j=i+1}^n \alpha_j, a_i - 1)$. This shows that $\phi(m) \geq \phi(p)$ for any $m \in S_d$, so by Lemma 3.1 it follows that there exists an initial rev-lex segment $\tilde{R} \subseteq S_d$ such that $\tilde{R} \supseteq R$, the least element q in $\tilde{R}$ has $\phi(q) = r = \min\{\phi(m) : m \in \tilde{R}\}$, and $|\tilde{R}^{(r)}| \leq |R^{(r)}| + 1$. By Lemma 3.2 we have that

$\Delta \tilde{R}^{(i)} = \tilde{R}^{(i+1)}$ for $r \leq i \leq a_1 - 2$. If $r = 0$, then $|C^{(0)}| \geq |R^{(0)}| + 1 \geq |\tilde{R}^{(0)}|$. If $r > 0$, then $\tilde{R}^{(0)} = R^{(0)} = 0$, so in both cases $|C^{(0)}| \geq |\tilde{R}^{(0)}|$. Since $C^{(0)}$ is a strongly stable set of monomials in $k[x_2, \ldots, x_n]_d$ and $R^{(0)}$ is an initial rev-lex segment in $k[x_2, \ldots, x_n]_d$ we can apply the induction hypothesis and conclude that $|\Delta C^{(0)}| \geq |\Delta \tilde{R}^{(0)}|$, so $|C^{(1)}| \geq |\Delta C^{(0)}| \geq |\Delta \tilde{R}^{(0)}| = |\tilde{R}^{(1)}| \geq |R^{(1)}|$. Using the induction hypothesis again for $C^{(1)}$ and $\tilde{R}^{(1)}$ we see that $|C^{(2)}| \geq |\Delta C^{(1)}| \geq |\Delta \tilde{R}^{(1)}| = |\tilde{R}^{(2)}| \geq |R^{(2)}|$. Repeating this argument we get that $|C^{(i)}| \geq |\tilde{R}^{(i)}| \geq |R^{(i)}|$ for $1 \leq i \leq a_1 - 1$. Then $|C'| = \sum_{i=1}^{a_1-1} |C^{(i)}| \geq \sum_{i=1}^{a_1-1} |R^{(i)}| = |R'|$. However, $|R'| = |R| - |R^{(0)}| > |C| - |C^{(0)}| = |C'|$, which is a contradiction. This proves that $|C^{(0)}| \leq |R^{(0)}|$ (and hence that $|C'| \geq |R'|$).

Next we prove (2). As C is strongly stable, it follows that $\Delta C^{(i-1)} \subseteq C^{(i)}$ for $1 \leq i \leq a_1 - 1$. Hence $\Delta C = \cup_{i=1}^{a_1-1} x_1^{i-1} C^{(i)} \cup x_1^{a_1-1} \Delta C^{(a_1-1)}$, so $|\Delta C| = |C'| + |\Delta C^{(a_1-1)}|$. Similarly $|\Delta R| = |R'| + |\Delta R^{(a_1-1)}|$. Since we already know that $|C'| \geq |R'|$, it will be enough to prove that $|\Delta C^{(a_1-1)}| \geq |\Delta R^{(a_1-1)}|$. By the induction hypothesis this will in turn follow if $|C^{(a_1-1)}| \geq |R^{(a_1-1)}|$. Assume on the contrary that $|C^{(a_1-1)}| < |R^{(a_1-1)}|$. Since $|C'| \geq |R'|$ it follows that there exists a $t \geq 1$ such that $|C^{(t)}| > |R^{(t)}|$. Applying Lemmas 3.1 and 3.2 again we see as before that there exists an initial rev-lex segment $\tilde{R} \supseteq R$ with the properties that $|\tilde{R}^{(t)}| \leq |R^{(t)}| + 1$ and $\Delta \tilde{R}^{(i)} = \tilde{R}^{(i+1)}$ for $t \leq i \leq a_1 - 2$. Then $|C^{(t)}| \geq |\tilde{R}^{(t)}|$ and by the induction hypothesis we conclude as in the proof of part (1) that $|C^{(i)}| \geq |\tilde{R}^{(i)}| \geq |R^{(i)}|$ for $r \leq i \leq a_1 - 1$. But this contradicts our assumption that $|C^{(a_1-1)}| < |R^{(a_1-1)}|$, so $|C^{(a_1-1)}| \geq |R^{(a_1-1)}|$, which proves (2).

Finally, we prove (3). We have that $|C'| + |\Delta C^{(a_1-1)}| = |\Delta C| = |\Delta R| = |R'| + |\Delta R^{(a_1-1)}|$. Since $|C'| \geq |R'|$ and $|\Delta C^{(a_1-1)}| \geq |\Delta R^{(a_1-1)}|$, it follows that $|C'| = |R'|$. Thus $|C^{(0)}| = |C| - |C'| = |R| - |R'| = |R^{(0)}|$, which proves (3). □

Proof of Theorem 2.1. Let $C, R \subseteq S_d$ be the unique sets of monomials such that the image of C (resp. R) in S_d/V (resp. S_d/L) forms a basis of S_d/V (resp. S_d/L). It is easily seen that if we reverse the order of the variables, $x_n < x_{n-1} < \cdots < x_1$, then C becomes strongly stable and R becomes an initial rev-lex segment. Therefore (1) follows from Theorem 2.2 (1).

Since (2) and (3) are easily seen to be true when $n = 1$, we can use induction to prove them. So let $n > 1$ and assume we have already proved (2) and (3) for $n - 1$. We have that $\tilde{V}$ is strongly stable, $\tilde{L}$ is an initial lex-segment in $\tilde{S}_d$, and $\dim \tilde{V} \geq \dim \tilde{L}$. Then the induction hypothesis implies that $\dim \tilde{V}\tilde{S}_1 \geq \dim \tilde{L}\tilde{S}_1$. Thus $\dim VS_1 = \dim \tilde{V}\tilde{S}_1 + \dim V \geq \dim \tilde{L}\tilde{S}_1 + \dim L = \dim LS_1$, which proves (2).

To prove (3), note that if $\tilde{K} \subseteq \tilde{S}_d$ is a vector space generated by an initial lex-segment such that $\dim \tilde{K} = \dim \tilde{V}$, then $\tilde{K} \supseteq \tilde{L}$, so $\dim \tilde{V}\tilde{S}_1 \geq \dim \tilde{K}\tilde{S}_1 \geq \dim \tilde{L}\tilde{S}_1$. This implies that $\dim \tilde{V}\tilde{S}_1 = \dim \tilde{K}\tilde{S}_1 = \dim \tilde{L}\tilde{S}_1$, so by the induction hypothesis $\dim \tilde{V}\tilde{S}_2 = \dim \tilde{K}\tilde{S}_2$. Since $\tilde{K}\tilde{S}_1$ and $\tilde{L}\tilde{S}_1$ are both generated by initial lex-segments in

$\tilde{S}_{d+1}$, it also follows that $\tilde{K}\tilde{S}_1 = \tilde{L}\tilde{S}_1$, so $\tilde{K}\tilde{S}_2 = \tilde{L}\tilde{S}_2$. Therefore $\dim \tilde{V}\tilde{S}_2 = \dim \tilde{L}\tilde{S}_2$. We have that $VS_2 = \tilde{V}\tilde{S}_2 \oplus x_n VS_1$ and $LS_2 = \tilde{L}\tilde{S}_2 \oplus x_n LS_1$, so

$$\dim VS_2 = \dim \tilde{V}\tilde{S}_2 + \dim VS_1 = \dim \tilde{L}\tilde{S}_2 + \dim LS_1 = \dim LS_2,$$

which proves (3). □

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